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CONVERGENCE PROPERTIES FOR THE TIME-DEPENDENT SCHRÖDINGER EQUATION

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Abstract. Consider the solution to the generalized Schrödinger equation $Pu = i\partial u/\partial t$ in the halfspace $\{(x,t) \in \mathbf{R}^n \times \mathbf{R}; t > 0\}$, with initial values $u(x,0) = f(x)$. Here P is an elliptic operator in the x variables with constant coefficients. Assume that f belongs to the Sobolev space H_s. When $P = \Delta$, it is known that $s > 1/2$ implies that u converges to f along almost all vertical lines. We extend this result to an arbitrary P and sharpen it by replacing "almost all" by "quasiall". The values of u must then be made precise in a certain way. A related maximal function estimate is proved.

By means of a counterexample, it is shown that the vertical lines cannot be widened into convergence regions. However, for quasiall boundary points $(x,0)$, we prove that $u \to f$ along almost all lines through $(x,0)$.

1. Introduction and results

For f belonging to the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ set

$$(1.1) \qquad u(x,t) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{it|\xi|^2} \hat{f}(\xi)\, d\xi, \qquad x \in \mathbf{R}^n,\ t \in \mathbf{R},$$

where the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-i\xi\cdot x} f(x)\, dx.$$

The function u is then a solution to the Schrödinger equation $\Delta u = i\partial u/\partial t$. We set

$$(1.2) \qquad u^*(x) = \sup_{0<t<1} \left|u(x,t)\right|, \qquad x \in \mathbf{R}^n,$$

and also introduce Sobolev spaces $H_s = H_s(\mathbf{R}^n)$, $s \in \mathbf{R}$, by defining the norm

$$\|f\|_{H_s} = \left(\int_{\mathbf{R}^n} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi \right)^{1/2}.$$

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It is then known that the estimate

$$(1.3)\qquad \left(\int_B \left|u^*(x)\right|^2\right)^{1/2} \le C_B \|f\|_{H_s}, \qquad f\in\mathcal{S},$$

holds for all balls B in $\mathbf{R}^n$ if $s \ge n/4$ and if $s > 1/2$ (see L. Carleson [1], B.E.J. Dahlberg and C.E. Kenig [4], C.E. Kenig and A. Ruiz [5], P. Sjölin [6], and L. Vega [7]). In particular it was proved in [6] that (1.3) holds for $s > 1/2$, and this result was applied to study the existence almost everywhere of $\lim_{t\to 0} u(x,t)$ for solutions u to the Schrödinger equation.

We shall here extend these results from [6] in several ways. First we replace Δ by an elliptic operator $P = -p(D)$, where $D = (D_1,\dots,D_n)$ and $D_k = -i\partial/\partial x_k$. The polynomial p is real and elliptic, i.e., its pricipal part does not vanish in $\mathbf{R}^n \setminus \{0\}$. Its degree m is at least 2. Then if $f\in\mathcal{S}(\mathbf{R}^n)$, the function

$$(1.4)\qquad u(x,t) = (2\pi)^{-n}\int e^{ix\cdot\xi} e^{itp(\xi)}\hat f(\xi)\,d\xi, \qquad x\in\mathbf{R}^n,\ t\in\mathbf{R},$$

solves the Cauchy problem $Pu = i\partial u/\partial t$, $u(\cdot,0)=f$. With this u, we use again (1.2) to define u^*. We then have the following extension of (1.3).

Theorem 1. *If $s > 1/2$, then*

$$\|u^*\|_{L^2(B)} \le C_B \|f\|_{H_s}, \qquad f\in\mathcal{S},$$

for any ball B in $\mathbf{R}^n$.

This inequality is related to the convergence properties of u at the boundary, when $f\in H_s$. Improving the known almost everywhere convergence results, we shall obtain convergence along quasievery vertical line. The capacities to be used are those of Sobolev spaces. They are defined for $s>0$ by

$$C_s(E) = \inf\left\{\|g\|_2^2 ; 0\le g\in L^2(\mathbf{R}^n),\ G_s * g \ge 1 \text{ on } E\right\}, \qquad E\subset\mathbf{R}^n.$$

Here G_s is the Bessel kernel, $\hat G_s(\xi) = \left(1+|\xi|^2\right)^{-s/2}$. By C_s-q.e. we mean everywhere except on a set of C_s-capacity 0, and similarly for C_s-q.a. When $s > n/2$, only the empty set has C_s-capacity 0.

A function $f\in H_s$ can be written as $f = G_s * g$ with $g\in L^2$, and conversely. At C_s-q.a. points x, this convolution is well defined in the sense that $G_s * |g|(x) < \infty$. One can recover these well-defined values of f, knowing f almost everywhere. Indeed, it is easily seen that the means of f in small balls centered at x converge to $G_s * g(x)$ if $G_s * |g|(x) < \infty$.

We now describe how to make the solution u precise by defining it at sufficiently many points. Let $f\in H_s$. For every t, (1.4) defines $u(\cdot,t)$ as an $L^2(\mathbf{R}^n)$

function, because of Plancherel's theorem. This gives a measurable, a.e. defined function u in $\mathbf{R}^n \times \mathbf{R}$. With a point (x,t) as center, we let $B_{x,t}(\delta)$ be the ball in $\mathbf{R}^{n+1}$ of radius $\delta > 0$, and

$$B'_{x,t}(\delta) = \{(x',t); |x'-x| < \delta\}$$

the horizontal disc. Define the value $u(x,t)$ as the limit as $\delta \to 0$ of the mean value of u in either $B_{x,t}(\delta)$ or $B'_{x,t}(\delta)$, at all points (x,t) where this limit exists. We shall speak of the ball and the disc method. Notice in particular that the disc method for $t=0$ gives us back the C_s-q.e. defined values of f.

Theorem 2. *Let $s > 1/2$ and take $f \in H_s$. Define u by means of (1.4), and make u precise by the ball or the disc method. If $0 < \varrho < s - 1/2$, the following holds for C_ϱ-q.a. x: The function u is defined at every point of the vertical line $\{x\} \times \mathbf{R}$, its restriction to the line is continuous, and its value at $(x,0)$ is $f(x)$.*

We remark that instead of balls $B_{x,t}(\delta)$, it is possible to use half-balls $B_{x,t}(\delta) \cap \{(x',t'); t' > t\}$. This is more natural at $t=0$ if one is interested in u for $t > 0$ only.

For solutions to initial-value problems in a halfspace $\mathbf{R}^n \times \mathbf{R}_+$ given by kernels like the Poisson or heat kernel, one has convergence in an approach region at almost all boundary points. This means that there exists a strictly increasing function $\gamma\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that the solution $u(y,t)$ tends to the boundary value at $(x,0)$ as $(y,t) \to (x,0)$ and $|y-x| < \gamma(t)$, for a.a. $x \in \mathbf{R}^n$. For our problem, however, there is no such convergence region, except trivially when $f \in H_s$ and $s > n/2$. (In that case, f is continuous and u is a continuous extension of f.) The following counterexample is for the standard Schrödinger equation $\Delta u = i\partial u/\partial t$.

Theorem 3. *Assume that $\gamma\colon \mathbf{R}_+ \to \mathbf{R}_+$ is a strictly increasing function. Let u and f be related by (1.1). Then there exists an $f \in H_{n/2}(\mathbf{R}^n)$ such that u is continuous in $\{(x,t); t > 0\}$ and*

$$\limsup_{\substack{(y,t)\to(x,0)\\ |y-x|<\gamma(t),\, t>0}} |u(y,t)| = +\infty \tag{1.5}$$

for all $x \in \mathbf{R}^n$.

This means that near the vertical line through every boundary point $(x,0)$ there can be bad points accumulating at $(x,0)$, at which u takes values far from $f(x)$. However, the bad points are sparse at most boundary points, in the sense that most lines through $(x,0)$ do not intersect them. This is the content of our last result.

Theorem 4. *For $f \in H_s$, $s > 1/2$, let u be given by (1.4) and made precise as described above. Let $0 < \varrho < s - 1/2$. Then for C_ϱ-q.a. $x \in \mathbf{R}^n$, the restriction of u to the line $t \to (x + \alpha t, t)$ is continuous for a.a. $\alpha \in \mathbf{R}^n$.*

This of course implies convergence to $f(x)$ along almost all lines through $(x,0)$, since we know from Theorem 2 that $u(x,0)=f(x)$.

We prove Theorems 2 and 4 by first showing that u is locally in a mixed Sobolev space. This can also be seen by the method of Constantin and Saut [2], [3].

2. Proofs for vertical approach

Proof of Theorem 1. We shall follow the idea in the proof of Theorem 1 in [6]. Choose real functions $\varphi_0 \in C_0^\infty(\mathbf{R}^n)$ and $\psi_0 \in C_0^\infty(\mathbf{R})$. Instead of u we shall consider

$$Sf(x,t)=\varphi_0(x)\psi_0(t)u(x,t). \tag{2.1}$$

We shall first prove that

$$\|Sf\|_{L^2(\mathbf{R}^{n+1})} \le C\,\|f\|_{H_{-s}}, \qquad f\in\mathcal{S}, \tag{2.2}$$

where $s=(m-1)/2$. One finds that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} \left|Sf(x,t)\right|^2 dx\,dt = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \hat\varphi(\eta-\xi)\hat\psi\big(p(\eta)-p(\xi)\big)\hat f(\xi)\overline{\hat f(\eta)}\,d\xi\,d\eta$$

where $\varphi=\varphi_0^2$, $\psi=\psi_0^2$. We set

$$K(\xi,\eta)=\big(1+|\xi|\big)^s\big(1+|\eta|\big)^s\hat\varphi(\eta-\xi)\hat\psi\big(p(\eta)-p(\xi)\big).$$

Arguing as in [6], we see that to prove (2.2) it suffices to prove that

$$\int_{\mathbf{R}^n}\left|K(\xi,\eta)\right|d\eta \le C, \qquad \xi\in\mathbf{R}^n. \tag{2.3}$$

The case $|\xi|\le 2$ in (2.3) is easy since the $\hat\varphi$ factor makes K rapidly decreasing in η. Now assume that $|\xi|>2$. It is clear that

$$\big(1+|\xi|\big)^s\big(1+|\eta|\big)^s \le C|\xi|^{2s}+C|\eta-\xi|^{2s},$$

and hence

$$\begin{aligned}\int\left|K(\xi,\eta)\right|d\eta \le C|\xi|^{2s}\int\left|\hat\varphi(\eta-\xi)\right|\left|\hat\psi\big(p(\eta)-p(\xi)\big)\right|d\eta+\\ +C\int|\eta-\xi|^{2s}\left|\hat\varphi(\eta-\xi)\right|\left|\hat\psi\big(p(\eta)-p(\xi)\big)\right|d\eta.\end{aligned}$$

The last integral is bounded because of the $\hat{\varphi}$ factor, and (2.3) follows if we can prove that

$$\int |\hat{\varphi}(\eta-\xi)| \left|\hat{\psi}(p(\eta)-p(\xi))\right| d\eta \le C|\xi|^{-2s}. \tag{2.4}$$

We need only deal with large $|\xi|$, and since $\hat{\varphi} \in \mathcal{S}$ it suffices to prove that

$$\int_{B_\xi} |\hat{\varphi}(\eta-\xi)| \left|\hat{\psi}(p(\eta)-p(\xi))\right| d\eta \le C|\xi|^{-2s} \tag{2.5}$$

where $B_\xi = B(\xi\,;\,c_0|\xi|) = \{\eta\,;\,|\eta-\xi| < c_0|\xi|\}$ and $c_0 > 0$.

To estimate $p(\eta)-p(\xi)$ in B_ξ, we fix ξ and consider $\operatorname{grad} p$. Let m be the degree of p and p_m its principal part. Since $\operatorname{grad} p_m$ is homogeneous of degree $m-1$, the ellipticity of p implies that $\operatorname{grad} p_m \ne 0$ in $\mathbf{R}^n \setminus \{0\}$. With $v = \left|\operatorname{grad} p_m(\xi)\right|^{-1} \operatorname{grad} p_m(\xi)$, one can therefore choose c_0 and $c > 0$ so that $v \cdot \operatorname{grad} p_m > c|\xi|^{m-1}$ in B_ξ. The constants c_0 and c do not depend on ξ. Since $\operatorname{grad}(p-p_m)$ is of degree at most $m-2$, it follows that

$$v \cdot \operatorname{grad} p > c|\xi|^{m-1} \text{ in } B_\xi$$

for large ξ, with a new c. We replace η by coordinates (s, η') defined by

$$\eta = \xi + sv + \eta', \qquad s \in \mathbf{R}, \quad \eta' \perp v.$$

With $p = p(\eta) = p(s,\eta')$, this gives $|\partial p/\partial s| \ge c|\xi|^{m-1}$ in B_ξ. For each η', there exists an $s_0 \in \mathbf{R}$ such that

$$\left|p(\eta)-p(\xi)\right| \ge c|s-s_0||\xi|^{m-1} \quad \text{in} \quad B_\xi,$$

so that

$$\left|\hat{\psi}(p(\eta)-p(\xi))\right| \le C\big(1+|s-s_0||\xi|^{m-1}\big)^{-N}$$

for any N. Also

$$\left|\hat{\varphi}(\eta-\xi)\right| \le C\big(1+|\eta'|\big)^{-N}.$$

Integrating in the new coordinates, we obtain (2.5) from these two estimates. Now (2.3) and (2.2) follow.

Setting

$$\|Sf\|^2_{L^2(H_r)} = \int_{\mathbf{R}^n} \|Sf(x,\cdot)\|^2_{H_r(\mathbf{R})}\, dx,$$

we can write (2.2) as

$$\|Sf\|_{L^2(H_0)} \le C\,\|f\|_{H_{(1-m)/2}}.$$

An estimate for $\partial Sf/\partial t$ can be obtained in a similar way, cf. [6]. One finds that

$$\|Sf\|_{L^2(H_1)} \le C\,\|f\|_{H_{(1+m)/2}}.$$

Interpolation yields

$$\|Sf\|_{L^2(H_{1/2+\delta})} \le C\,\|f\|_{H_s},$$

where $\delta = \delta(s) > 0$ for $s > 1/2$. But the supremum norm in $\mathbf{R}$ is dominated by the $H_{1/2+\delta}(\mathbf{R})$ norm when $\delta > 0$. Since φ_0 and ψ_0 are arbitrary, Theorem 1 follows.

To prepare for the next proof we introduce mixed Sobolev spaces $H_{\varrho,r}$ for $\varrho, r \ge 0$. Define

$$H_{\varrho,r} = H_{\varrho,r}(\mathbf{R}^n \times \mathbf{R}) = (G_\varrho \otimes G_r) * L^2(\mathbf{R}^{n+1}),$$

where G_ϱ and G_r are Bessel kernels in $\mathbf{R}^n$ and $\mathbf{R}$, respectively. The norm in $H_{\varrho,r}$ is the obvious one. Notice that $H_{0,r} = L^2(H_r)$. We start by establishing some properties of $H_{\varrho,r}$, assuming $r > 1/2$.

Let $*_1$ and $*_2$ denote convolution in x and in t, respectively. If $v \in H_{\varrho,r}$, we can write

$$v = (G_\varrho \otimes G_r) * g = G_\varrho *_1 (G_r *_2 g) \tag{2.6}$$

with $g \in L^2(\mathbf{R}^{n+1})$. For $r > 1/2$ one has $G_r \in L^2(\mathbf{R})$, so that for each t

$$\big|(G_r *_2 g)(x,t)\big| \le \|G_r\|_{L^2(\mathbf{R})}\,\|g(x,\cdot)\|_{L^2(\mathbf{R})}.$$

The right-hand side here is in $L^2(\mathbf{R}^n)$ as a function of x. But then (2.6) says that $x \to v(x,t)$ is in $H_\varrho(\mathbf{R}^n)$ for each t. This means that we have a continuous restriction map R_t: $H_{\varrho,r} \to H_\varrho(\mathbf{R}^n)$ to each horisontal hyperplane $\mathbf{R}^n \times \{t\}$.

Interchanging the variables, we write $v = G_r *_2 (G_\varrho *_1 g)$. The function $t \to v(x,t)$ will belong to $H_r(\mathbf{R})$ if and only if $t \to G_\varrho *_1 g(x,t)$ is in $L^2(\mathbf{R})$. By Minkowski's inequality,

$$\|G_\varrho *_1 g(x,\cdot)\|_{L^2(\mathbf{R})} \le \big(G_\varrho * \|g\|_{L^2(dt)}\big)(x). \tag{2.7}$$

Here

$$\|g\|_{L^2(dt)}(x) = \left(\int \big|g(x,t)\big|^2\,dt\right)^{1/2}$$

is a function in $L^2(\mathbf{R}^n)$. But then the right-hand side of (2.7) is in $H_\varrho(\mathbf{R}^n)$, hence finite for C_ϱ-q.a. x. We conclude that $t \to v(x,t)$ is in $H_r(\mathbf{R})$, and hence continuous, for C_ϱ-q.a. x.

We shall say that the value $v(x,t)$ is well defined if

$$(G_\varrho \otimes G_r) * |g|(x,t) < \infty. \tag{2.8}$$

What we have just seen implies that this happens for $(x,t) \in E \times \mathbf{R}$, where the complement of $E \subset \mathbf{R}^n$ is of C_ϱ-capacity 0.

We claim that (2.8) implies

$$v(x,t) = \lim_{\delta \to 0} \frac{1}{|B_{x,t}(\delta)|} \int_{B_{x,t}(\delta)} v(x',t')\,dx'\,dt' \tag{2.9}$$

and similarly for the means in $B'_{x,t}(\delta)$. Indeed, set $\chi_\delta = |B(\delta)|^{-1}\chi_{B(\delta)}$ with $B(\delta) = B_{0,0}(\delta)$. The mean in (2.9) is then $\chi_\delta * (G_\varrho \otimes G_r) * g(x,t)$. Clearly, $\chi_\delta * (G_\varrho \otimes G_r)$ converges pointwise to $(G_\varrho \otimes G_r)$ as $\delta \to 0$. Inscribing $B(\delta)$ in a product of an n-dimensional ball and an interval, we obtain a majorization

$$\chi_\delta * (G_\varrho \otimes G_r) \le C G_\varrho \otimes G_r.$$

Now (2.8) implies (2.9) via dominated convergence. For $B'_{x,t}(\delta)$ we need only use the fact that $x \to v(x,t)$ is in $H_\varrho(\mathbf{R}^n)$.

Proof of Theorem 2. Let $f \in H_s$. We write Sf for the function obtained when we define u by means of (1.4) and then multiply by $\varphi_0(x)\psi_0(t)$. Since φ_0 and ψ_0 are arbitrary, we can replace u by Sf in the whole proof.

With $f \in \mathcal{S}$, we first argue as in the preceding proof, using instead of Sf its first-order derivatives with respect to x. This will produce either an extra ξ factor or a differentiation of $\varphi_0(x)$ in the integral expression for Sf. For $f \in \mathcal{S}$ we get

$$\|\operatorname{grad}_x Sf\|_{L^2(H_0)} \le C\,\|f\|_{H_{(3-m)/2}}$$

and thus

$$\|Sf\|_{H_{1,0}} \le C\,\|f\|_{H_{(3-m)/2}}.$$

If we differentiate also with respect to t, the result will be

$$\|Sf\|_{H_{1,1}} \le C\,\|f\|_{H_{(3+m)/2}}.$$

This can be combined with our previous estimates in $H_{0,0} = L^2(H_0)$ and $H_{0,1} = L^2(H_1)$. Interpolating one index at a time, we conclude

$$\|Sf\|_{H_{\varrho,r}} \le C\,\|f\|_{H_{\varrho+1/2+m(r-1/2)}}$$

for $0 \le \varrho,\ r \le 1$ and $f \in \mathcal{S}$. By means of higher order derivatives, this can actually be extended to arbitrary $\varrho, r \ge 0$. Given $s > 1/2$ and $0 < \varrho < s - 1/2$, we can choose $r > 1/2$ so that $s = \varrho + 1/2 + m(r - 1/2)$. Extending S, we get a continuous linear map $\bar{S}$: $H_s(\mathbf{R}^n) \to H_{\varrho,r}(\mathbf{R}^n \times \mathbf{R})$.

Let $f \in H_s$. Then $\bar{S}f$ is a convolution $(G_\varrho \otimes G_r) * g$, $g \in L^2$. On C_ϱ-q.a. vertical lines, this convolution is well defined, with a continuous restriction. It remains to see that its values there coincide with those obtained when we make Sf precise. For the ball method, it is enough to verify that $\bar{S}f$ and Sf agree a.e. in $\mathbf{R}^{n+1}$, because of the properties of $H_{\varrho,r}$ discussed above. But $\bar{S}f$ and Sf define the same function in $L^2(\mathbf{R}^{n+1})$, since we get two coinciding continuous maps $H_s \to L^2(\mathbf{R}^{n+1})$. To deal with the disc method, observe that (1.4) gives for any fixed t a continuous map $H_s(\mathbf{R}^n) \to H_s(\mathbf{R}^n)$. Multiplying by $\varphi_0(x)\psi_0(t)$, we conclude that the restriction of Sf to $\mathbf{R}^n \times \{t\}$ defines a continuous map $H_s \to H_s$. This last map agrees with $R_t \circ \bar{S}$: $H_s \to H_\varrho$ on $\mathcal{S}$ and thus everywhere. It follows that all the well-defined values of $\bar{S}f$ are obtained when Sf is made precise by means of discs.

It only remains to see that the values of f, or rather $\psi_0(0)\varphi_0 f$, are recovered C_ϱ-q.e. in the hyperplane $t = 0$ when Sf is made precise. Both methods produce the same well-defined values of $\bar{S}f$. But since $\psi_0(0)\varphi_0 f$ is obviously recovered if discs are used, the proof is complete.

3. Proof for wider approach

Proof of Theorem 3. We shall first define sequences $(R_j)_1^\infty$ and $(R'_j)_1^\infty$ such that $2 = R_1 < R'_1 < R_2 < R'_2 < R_3 < R'_3 < \cdots$ and points $(x_j, t_j) \in \mathbf{R}^n \times \mathbf{R}_+$. We set $S_j = \{\xi \in \mathbf{R}^n ; R_j < |\xi| < R'_j\}$,

$$\hat{f}(\xi) = |\xi|^{-n}(\log|\xi|)^{-3/4} e^{-x_j \cdot \xi} e^{-it_j|\xi|^2}, \qquad \xi \in S_j,$$

and $\hat{f}(\xi) = 0$ otherwise. It is then clear that $f \in H_{n/2}$. Our idea is to make $|u|$ large at the points (x_j, t_j). Also set $\delta_k = \gamma(1/k)/\sqrt{n}$, $k = 1, 2, 3, \ldots$ We let x_1, x_2, …, x_{n_1} denote all points x in $B(0;1) = \{x \in \mathbf{R}^n ; |x| < 1\}$ such that $x/\delta_2 \in \mathbf{Z}^n$, x_{n_1+1}, …, x_{n_2} all points in in $B(0;2)$ such that $x/\delta_3 \in \mathbf{Z}^n$, and generally x_{n_k+1}, …, $x_{n_{k+1}}$ all points in $B(0; k+1)$ such that $x/\delta_{k+2} \in \mathbf{Z}^n$. Then choose $(t_j)_1^\infty$ such that $1 > t_1 > t_2 > t_3 > \cdots > 0$ and such that

$$\frac{1}{k+1} > t_j > \frac{1}{k+2}$$

for $n_k + 1 \le j \le n_{k+1}$, $k = 0, 1, 2, \ldots$ ($n_0 = 0$). Note that the points (x_j, t_j) accumulate at each boundary point $(x, 0)$, even if only (x_j, t_j) with $|x_j - x| < \gamma(t_j)$ are considered. To define $(R_j)_1^\infty$ and $(R'_j)_1^\infty$ we first choose $R_1 = 2$ and $R'_1 = 3$. Given R_1, R'_1, …, R_{j-1}, R'_{j-1} we then choose $R_j > R'_{j-1}$ such that for $k < j$ one has

$$R_j^2 \ge C 2^j / |t_k - t_j| \tag{3.1}$$

and

$$|t_k - t_j|R_j > |x_k - x_j| + 1. \tag{3.2}$$

Also set $R'_j = R_j^K$ where K is large.

Now let

$$u_m(x,t) = (2\pi)^{-n} \int_{|\xi|<\mathbf{R}'_m} e^{ix\cdot\xi} e^{it|\xi|^2} \hat{f}(\xi)\, d\xi.$$

Then $u_m(\cdot, t) \to u(\cdot, t)$ in $L^2(\mathbf{R}^n)$ for each t, and

$$u_m(x,t) = \sum_{j=1}^{m} (2\pi)^{-n} \int_{S_j} e^{i(x-x_j)\cdot\xi} e^{i(t-t_j)|\xi|^2} |\xi|^{-n} \big(\log|\xi|\big)^{-3/4} d\xi = \sum_{j=1}^{m} A_j(x,t).$$

We first observe that

$$\begin{aligned} \left| \sum_{j=1}^{k-1} A_j(x,t) \right| &\le \int_{2\le|\xi|\le R'_{k-1}} |\xi|^{-n}\big(\log|\xi|\big)^{-3/4} d\xi \\ &= C \int_2^{R'_{k-1}} r^{-1} (\log r)^{-3/4}\, dr \le C(\log R'_{k-1})^{1/4} \le C(\log R_k)^{1/4} \end{aligned}$$

for all (x,t). We also have

$$\begin{aligned} A_k(x_k, t_k) &= (2\pi)^{-n} \int_{S_k} |\xi|^{-n}\big(\log|\xi|\big)^{-3/4} d\xi \\ &= C \int_{R_k}^{R'_k} r^{-1}(\log r)^{-3/4}\, dr = C\big((\log R'_k)^{1/4} - (\log R_k)^{1/4}\big) \\ &> c(\log R'_k)^{1/4}, \qquad c > 0. \end{aligned}$$

For $j > k \ge 2$ one finds that

$$A_j(x_k, t_k) = (2\pi)^{-n} \int_{S^{n-1}} dS(\xi') \int_{R_j}^{R'_j} r^{-1} (\log r)^{-3/4} e^{iF(r)}\, dr,$$

where

$$F(r) = (x_k - x_j)\cdot \xi' r + (t_k - t_j) r^2.$$

It follows that

$$F'(r) = (x_k - x_j)\cdot\xi' + 2(t_k - t_j) r$$

and

$$F''(r) = 2(t_k - t_j). \tag{3.3}$$

Using (3.2) we conclude that

$$|F'(r)| \geq |t_k - t_j| r \geq |t_k - t_j| R_j, \qquad R_j < r < R'_j, \tag{3.4}$$

and an integration by parts gives

$$\int_{R_j}^{R'_j} \frac{1}{r(\log r)^{3/4}} e^{iF(r)}\, dr = \int_{R_j}^{R'_j} \frac{1}{r(\log r)^{3/4} iF'(r)} iF'(r) e^{iF(r)}\, dr$$
$$= \left[\frac{1}{r(\log r)^{3/4} iF'(r)} e^{iF(r)} \right]_{R_j}^{R'_j} - \int_{R_j}^{R'_j} \frac{d}{dr} \left(\frac{1}{ir(\log r)^{3/4} F'(r)} \right) e^{iF(r)}\, dr = A - B.$$

Invoking (3.4) and (3.1), one obtains

$$|A| \leq \frac{C}{|t_k - t_j| R_j^2} \leq C 2^{-j}$$

and according to (3.4) and (3.3) we also have

$$\left| \frac{d}{dr} \left(\frac{1}{ir(\log r)^{3/4} F'(r)} \right) \right| \leq C \frac{1}{r^2 |F'|} + C \frac{|F''|}{r |F'|^2}$$
$$\leq C \frac{1}{r^3 |t_k - t_j|} + C \frac{|t_k - t_j|}{r^3 |t_k - t_j|^2} = C \frac{1}{|t_k - t_j| r^3}$$

and hence

$$|B| \leq \frac{C}{|t_k - t_j| R_j^2} \leq C 2^{-j}.$$

We conclude that for $j > k$

$$|A_j(x_k, t_k)| \leq C 2^{-j}. \tag{3.5}$$

It follows that

$$|u_m(x_k, t_k)| \geq c(\log R'_k)^{1/4} - C(\log R_k)^{1/4} - C \sum_{k+1}^{m} 2^{-j} \geq c(\log R'_k)^{1/4}, \tag{3.6}$$

when $m > k$ and K is sufficiently large.

To see that u is continuous in $\{t > 0\}$, take a compact set $L \subset \{(x,t)\,;\, t > 0\}$. Since the sequence (R_j) is very rapidly increasing, there exists a $j_0 < \infty$ such that (3.1) and (3.2) hold for $j > j_0$ with (x_k, t_k) replaced by any $(x,t) \in L$. But then one can also take $(x,t) \in L$ instead of (x_k, t_k) in (3.5), $j > j_0$. Hence, the

u_m converge locally uniformly in $\{t > 0\}$. Since each u_m is continuous, so is u in $\{t > 0\}$. From (3.6) we conclude that

$$\left|u(x_k, t_k)\right| \geq c(\log R'_k)^{1/4} \to +\infty$$

as $k \to +\infty$. This implies (1.5), and Theorem 3 is proved.

Before the last proof, we must introduce more mixed Sobolev spaces. Fix a large ball $B \subset \mathbf{R}^n$. Define a space

$$H_{\varrho,r,0} = H_{\varrho,r,0}(\mathbf{R}^n \times \mathbf{R} \times B) = (G_\varrho \otimes G_r) *_{1,2} L^2(\mathbf{R}^n \times \mathbf{R} \times B),$$

with the obvious norm. By $*_{1,2}$ we mean convolution in $\mathbf{R}^n \times \mathbf{R}$. The variables will be denoted $x \in \mathbf{R}^n$, $t \in \mathbf{R}$, $\alpha \in B$.

Let $v = (G_\varrho \otimes G_r) *_{1,2} g \in H_{\varrho,r,0}$ with $r > 1/2$. For C_ϱ-q.a. x, we claim that for a.a. $\alpha \in B$ the value $v(x, t, \alpha)$ is well defined for all $t \in \mathbf{R}$ and depends continuously on t. As before, "well defined" means that the convolution integral is absolutely convergent. We argue as when discussing $H_{\varrho,r}$ in Section 2. Write $v = G_r *_2 (G_\varrho *_1 g)$. We need only verify that for C_ϱ-q.a. x the inner convolution here is in $L^2(dt)$ for a.a. $\alpha \in B$. But

$$\|G_\varrho *_1 g(x, \cdot, \cdot)\|_{L^2(\mathbf{R}\times B)} \leq G_\varrho * \|g\|_{L^2(dt\, d\alpha)}(x),$$

and this last quantity is finite for C_ϱ-q.a. x. The claim follows.

Proof of Theorem 4. For $f \in \mathcal{S}$ we write

$$S'f(x, t, \alpha) = Sf(x + \alpha t, t) \tag{3.7}$$

with Sf as before. To deduce an a priori estimate for $S'f$, we consider one α at a time and argue as in Section 2. The only difference is that $p(\xi)$ will be replaced by $p(\xi) + \alpha \cdot \xi$. The result is

$$\|S'f\|_{H_{\varrho,r,0}} \leq C\, \|f\|_{H_s}, \qquad f \in \mathcal{S}.$$

Here ϱ and r are as before and $C = C_B$. This gives a continuous extension $\bar{S}'$: $H_s \to H_{\varrho,r,0}$.

We now examine how equality (3.7) extends to $\bar{S}'f$. Let $f \in H_s$ and take $f_j \in \mathcal{S}$ with $f_j \to f$ in H_s. Then $S'f_j \to \bar{S}'f$ in $H_{\varrho,r,0}$. The $H_{\varrho,r,0}$ norm is given by

$$\|v\|^2_{H_{\varrho,r,0}} = \int_B \|v(\cdot, \cdot, \alpha)\|^2_{H_{\varrho,r}}\, d\alpha.$$

Convergence $v_j \to v$ in $H_{\varrho,r,0}$ therefore implies that $v_j(\cdot, \cdot, \alpha) \to v(\cdot, \cdot, \alpha)$ in $H_{\varrho,r}$ for a.a. α, at least for a subsequence. Restricting to $\mathbf{R}^n \times \{t\}$, we get that

$v_j(\cdot,t,\alpha) \to v(\cdot,t,\alpha)$ in H_ϱ for all t, for a.a. α. On the other hand, $Sf_j(\cdot+t\alpha,t) \to Sf(\cdot+t\alpha,t)$ in H_s because of (1.4). For a.a. α, we conclude that for all t

$$\bar{S}'f(x,t,\alpha) = Sf(x+t\alpha,t), \qquad \text{a.a.} \quad x. \tag{3.8}$$

When $f \in H_s$, we have

$$\bar{S}'f = (G_\varrho \otimes G_r) *_{1,2} g \in H_{\varrho,r,0}.$$

The property of $H_{\varrho,r,0}$ deduced before the proof implies that for most x and α, the value $\bar{S}'f(x,t,\alpha)$ is well defined for all t and depends continuously on t. Here "most" is taken in the sense of Theorem 4.

It remains to see that if

$$(G_\varrho \otimes G_r) *_{1,2} |g|(x,t,\alpha) < \infty, \tag{3.9}$$

then the value $(G_\varrho \otimes G_r) *_{1,2} g(x,t,\alpha)$ is obtained when Sf is made precise at the point $(x+t\alpha,t)$. Disregarding those α in a null set, we can assume that (3.8) holds. Notice that α can be kept fixed, since only the restriction $g(\cdot,\cdot,\alpha)$ is used. We know that (3.9) implies that the value of $(G_\varrho \otimes G_r) *_{1,2} g$ at (x,t,α) is the limit as $\delta \to 0$ of the mean of the same function in the disc $B'_{x,t}(\delta) \times \{\alpha\}$. But this mean equals the mean of Sf in $B'_{x+t\alpha,t}(\delta)$, because of (3.8). This settles the case of the disc method.

For the ball method, we see from (3.8) that the mean of Sf in $B_{x+t\alpha,t}(\delta)$ equals the mean of $\bar{S}'f$ in a set $E^\alpha_{x,t}(\delta) \times \{\alpha\}$. Here $E^\alpha_{x,t}(\delta)$ is defined by

$$(x',t') \in E^\alpha_{x,t}(\delta) \qquad \Leftrightarrow \qquad (x'+t'\alpha,t') \in B_{x+t\alpha,t}(\delta).$$

But (3.9) implies that the means of $(G_\varrho \otimes G_r) *_{1,2} g$ in $E^\alpha_{x,t}(\delta) \times \{\alpha\}$ tend to the value of the same function at (x,t,α). This is because $E^\alpha_{x,t}(\delta)$ is contained in the ball $B_{x,t}\left(\sqrt{2}(1+|\alpha|)\delta\right)$, and its volume is comparable to that of this ball. The dominated convergence argument used for $H_{\varrho,r}$ now carries over. This takes care of the ball method and ends the proof of Theorem 4.

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