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ON THE FREQUENCY OF TITCHMARSH'S PHENOMENON FOR $\zeta(s)$-VII

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1. Introduction

In this paper we study small intervals I for t (of length $H \geq \exp\exp(e)$) contained in $[T, 2T]$ for which

$$\max_{t\in I} \left|\zeta(1+it)\right| \geq e^{\gamma}(\log\log H - \log\log\log H - \varrho), \tag{1}$$

where γ is the Euler's constant and ϱ a certain real constant which is effective. All our constants including the 0-constants are effective. The Greek letter θ will denote the least upper bound of the real parts of the zeros of $\zeta(s)$. In the present state of knowledge we do not know whether $\theta < 1$ or not. Only one of our results depends on the hypothesis $\theta < 1$ (in place of the more drastic Riemann hypothesis which asserts that $\theta = 1/2$) and in this case it is only for convenience that we assume that θ is effective. The n^{th} iterated logarithm $\log_n T$ is defined inductively $\log_1 T = \log T$, and $\log_{n+1} T = \log(\log_n T)$. Similarly the n^{th} iterated exponential is defined by $\exp_1(T) = \exp(T)$ and $\exp_{n+1}(T) = \exp\bigl(\exp_n(T)\bigr)$. Our first result is that the inequality (1) holds for all H satisfying

$$T \geq H \geq C_1 \log_4 T \tag{2}$$

where $C_1 \geq 1$ is a certain constant. Our next result is that *if the hypothesis* $\theta < 1$ *is true*, then (1) holds for all H satisfying

$$T \geq H \geq C_2 \log_5 T \tag{3}$$

where $C_2 \geq 1$ is a certain constant. *We assume throughout that* $T \geq C_3$ *and* $H \geq C_4$ *where* C_3 *and* C_4 *are certain positive constants.* Let now $H < C_1 \log_4 T$. Consider a set of disjoint intervals I contained in $[T, 2T]$ for which (1) is false. Our third result asserts that the number of such disjoint intervals does not exceed TX_1^{-1} where $X_1 = \exp_4(\beta H)$, where $\beta > 0$ is a constant. Again let $H < C_2 \log_5 T$. Consider a set of disjoint intervals I contained in $[T, 2T]$ for which (1) is false. Our (fourth and) final result asserts that the number of such disjoint intervals does not exceed TX_2^{-1} where $X_2 = \exp_5(\beta' H)$ where $\beta' > 0$ is a constant. Similar results can be proved for $\left|\zeta(1+it)\right|^{-1}$. We have only to replace e^{γ} by $6e^{\gamma}/\pi^2$. These results can be generalized suitably to ζ and L-functions of algebraic number fields and so on.

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2. Titchmarsh series and a main theorem

The study referred to in the introduction is based (apart from other ideas) on the following Theorem B due essentially to the author [7] (see also [8] and [3]). There the author proved the following two Theorems A and B. We begin with the following definition.

Titchmarsh series. Let $A \geq 1$ be a constant. Let $1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ where $1/A \leq \lambda_{n+1} - \lambda_n \leq A$. Let $1 = a_1, a_2, a_3, \ldots$ be a sequence of complex numbers, possibly depending on a parameter H (≥ 10) such that $|a_n| \leq (\lambda_n H)^A$. Put $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ where $s = \sigma + it$. Then $F(s)$ is analytic in $\sigma \geq A+2$. $F(s)$ is called a *Titchmarsh series* if there exists a constant $A \geq 1$ with the above properties and further a system of infinite rectangles $R(T, T+H)$ defined by $\{\sigma \geq 0, T \leq t \leq T+H\}$ where $10 \leq H \leq T$ and T (which may be related to H) tends to infinity and $F(s)$ admits an analytic continuation into these rectangles and the maximum of $|F(s)|$ taken over $R(T, T+H)$ does not exceed $\exp_2(H/80A)$.

Remark. It suffices for all our purposes to assume that $|F(s)|$ is continuous in $R(T, T+H)$ and that $F(s)$ is analytic in $\{\sigma > 0, T \leq t \leq T+H\}$ besides the other properties.

Theorem A. *We have*

$$\frac{1}{H} \int_L |F(it)|\, dt > C_A$$

where $C_A > 0$ depends only on A and L is the side $\{\sigma = 0, T \leq t \leq T+H\}$ of $R(T, T+H)$.

Theorem B. *We have*

$$\frac{1}{H} \int_L |F(it)|^2 dt > C_A \sum_{\lambda_n \leq X} |a_n|^2 \left(1 - \frac{\log \lambda_n}{\log H} + \frac{1}{\log_2 H}\right),$$

where $X = 2 + D_A H$, and $C_A > 0$, $D_A > 0$ depend only on A.

Remark. If $\lambda_n = n$ then it was shown in [3] that X can be taken to be $H/200$. The essential point in that paper was that the tapering factor multiplying $|a_n|^2$ was improved. The bound on $|F(s)|$ was relaxed to $\exp_2(H/80A)$. (This was known to the author for quite some time.) However, for our applications Theorem B is enough and the improvement in the tapering factor does not seem to have any extra advantage for the purposes of the present paper.

From Theorem B we deduce (in the rest of this section) our main theorem.

Main Theorem. *Let I be an interval contained in $[T, 2T]$ and of length H and let the maximum of $|\zeta(\sigma + it)|$ taken over the rectangle $\{\sigma \geq 1, t \in I\}$ not exceed $\exp_2(H/100)$. Then*

$$\max_{t \in I} |\zeta(1+it)| \geq e^{\gamma}(\log_2 H - \log_3 H - \varrho), \tag{4}$$

where γ is Euler's constant and ϱ is a certain real constant.

We do this in a series of lemmas. The deduction can be done in a somewhat similar fashion as in [2] or [10] although we follow the latter. First of all we take $F(s) = \big(\zeta(1+s)\big)^k$ and set k equal to the greatest integer not exceeding $\log H/5\log_2 H$. We verify that $F(s)$ is a Titchmarsh series with $\lambda_n = n$ and $a_1 = 1$. Now

$$a_n = \frac{d_k(n)}{n} < n\sum_{m=1}^{\infty}\frac{d_k(m)}{m^2} = n\big(\zeta(2)\big)^k < nH$$

since $k < \log H$ and $\zeta(2) < e$. Under the conditions of the main theorem, the maximum of $|F(s)|$ in the relevant rectangle does not exceed $\exp\{(\log H)\exp(H/100)\} \le \exp_2(H/80)$, provided that $\log H \le \exp(H/400)$ which is certainly true if $(H/400)^2 \ge 2H$, i.e., if $H \ge 320000$. Hence we can take $A = 1$. Thus we have the following

Lemma 1. *Under the hypothesis of the main theorem, with k chosen as the greatest integer not exceeding $\log H/5\log_2 H$, we have*

$$\frac{1}{H}\int_{t\in I}|\zeta(1+it)|^{2k}dt > C_5\sum_{n\le H/200}\left(\frac{d_k(n)}{n}\right)^2\left(1-\frac{\log n}{\log H}+\frac{1}{\log_2 H}\right)$$

and so

$$\max_{t\in I}|\zeta(1+it)| \ge \left(\frac{C_5}{\log_2 H}\right)^{1/2k} Q, \tag{5}$$

where $Q = \max_{n\le H/200}\big(d_k(n)/n\big)^{1/k}$ and $C_5 > 0$ is a constant.

Lemma 2. *We have*

$$\left(\frac{C_5}{\log_2 H}\right)^{1/2k} = 1 + O\left(\frac{\log_2 k}{k}\right).$$

Proof. The lemma follows from the definition of k.

Lemma 3. *The quantity $d_k(n)/n$, which is defined on prime powers by*

$$\frac{d_k(1)}{1} = 1 \text{ and } \frac{d_k(p^m)}{p^m} = \frac{k(k+1)\cdots(k+m-1)}{m!p^m}.$$

is a multiplicative function of n.

Proof. The lemma follows from the definition of $d_k(n)$ as the coefficient of n^{-s} in $\big(\zeta(s)\big)^k$ and the Euler product for $\zeta(s)$.

Lemma 4. *For* $m \geq 0$ *we have*

$$\frac{d_k(p^{m+1})}{p^{m+1}} = \left(\frac{d_k(p^m)}{p^m}\right)\left(\frac{k+m}{(m+1)p}\right) < \frac{3}{4}\left(\frac{d_k(p^m)}{p^m}\right), \tag{6}$$

provided $4k + 4m < 3mp + 3p$, *i.e.,* $m > (4k-3p)/(3p-4)$.

Proof. This follows directly from Lemma 3.

Lemma 5. *The inequality* (6) *holds if* $p \leq k$ *and* $m \geq [m_0 + 1]$, *where* $m_0 = (4k-3p)/(3p-4)$. *We also have*

$$m_0 + 1 < \frac{4k}{p}. \tag{7}$$

Proof. We have

$$m_0 + 1 = \frac{4k-4}{3p-4} = \frac{4k-4}{p+2p-4} \leq \frac{4k-4}{p} < \frac{4k}{p}$$

and hence the lemma follows.

Lemma 6. *We have*

$$\left(1 - \frac{1}{p}\right)^{-k} < (m_0 + 5) \max_{m \leq m_0 + 1} \left(\frac{d_k(p^m)}{p^m}\right). \tag{8}$$

Proof. Let $\nu = [m_0 + 1]$. Then the LHS equals

$$\sum_{m=0}^{\nu} \frac{d_k(p^m)}{p^m} + \sum_{m=\nu+1}^{\infty} \frac{d_k(p^m)}{p^m}.$$

Here the first sum does not exceed $(m_0 + 2)$ times the maximum in question. The second sum is by (6) less than $((3/4) + (3/4)^2 + (3/4)^3 + \cdots)$ times the maximum in question. This proves the lemma.

Lemma 7. *Let* $p \leq k$. *If* m *denotes the integer (to avoid a complicated notation) not exceeding* $m_0 + 1$ *for which the maximum of* $(d_k(p^m))p^{-m}$ *is attained, we have*

$$\frac{d_k(p^m)}{p^m} \geq \frac{p}{8k}\left(1 - \frac{1}{p}\right)^{-k}. \tag{9}$$

Proof. This lemma follows from (7) and (8) since $4 + (4k/p) \leq (8k)/p$.

Lemma 8. *With k as in Lemma 1, we have, for $H \geq H_0$,*

$$\prod_{p \leq k} p^m \leq \frac{H}{200}.$$

Proof. Since $m \leq m_0 + 1 \leq 4k/p$ it suffices to check that

$$\prod_{p \leq k} p^{4k/p} \leq \frac{H}{200}, \quad \text{i.e.,} \quad \sum_{p \leq k} \frac{\log p}{p} \leq \log H - \log(200).$$

The last statement follows since (by prime number theorem) $\sum_{p \leq k} \log p/p$ is asymptotic to $\log k$ as k tends to infinity.

Lemma 9. *We have, for Q defined in Lemma 1, the lower bound given by*

$$Q \geq e^{\gamma}\big(\log k + O(1)\big). \tag{10}$$

Proof. By (9) it suffices to check that

$$\left(\prod_{p \leq k} \left(\frac{p}{8k}\right)^{1/k}\right) \prod_{p \leq k} \left(1 - \frac{1}{p}\right)^{-1} \geq e^{\gamma}\big(\log k + O(1)\big). \tag{11}$$

It is well-known that the second product in (11) is $\geq e^{\gamma}\big(\log k + O(1)\big)$ (see (3.15.2) of [11] for a weaker result which is not hard to improve; see also p. 81 of Prachar's Primzahlverteilung, Springer-Verlag, 1959). The logarithm of the first product is

$$\frac{1}{k} \sum_{p \leq k} (\log p - \log k - \log 8) = O\left(\frac{1}{\log k}\right)$$

on using the prime number theorem in the forms

$$\sum_{p \leq k} \log p = k + O\left(\frac{k}{\log k}\right) \quad \text{and} \quad \sum_{p \leq k} \log k = k + O\Big(\frac{k}{\log k}\Big).$$

Hence (11) follows. This completes the proof of Lemma 9.

Lemmas 2 and 9 complete the proof of the main theorem.

3. First application of the main theorem

Theorem 1. *Let I be an interval of length H contained in $[T, 2T]$. Let $T \geq H \geq C_1 \log_4 T$. Then*

$$\max_{t \in I} \left|\zeta(1+it)\right| \geq e^{\gamma}(\log_2 H - \log_3 H - \varrho),$$

where γ is Euler's constant and ϱ is a certain real constant.

We prove this by a series of lemmas.

Lemma 1. *Divide I into three equal parts each of length $H/3$. Denote the middle interval by I_2 and the others by I_1 and I_3. Then we have either*

$$\max_{t \in I_2} \left|\zeta(1+it)\right| \geq e^{\gamma}\big(\log_2 H - \log_3 H - O(1)\big), \tag{12}$$

or

$$\max_{\sigma > 1, t \in I_2} \left|\zeta(\sigma+it)\right| \geq \exp_2(H/300). \tag{13}$$

Proof. The lemma follows from the main theorem.

Lemma 2. *If the maximum in (13) occurs for $1 < \sigma \leq 1+\delta$, where $\delta = \big(\exp_3(H/400)\big)^{-1}$, then (12) holds.*

Proof. In this case we have for some t_0 in I_2 the inequality

$$\begin{aligned} \left|\zeta(1+it_0)\right| &\geq \left|\zeta(\sigma+it_0)\right| - \delta \max_{1 \leq u \leq 1+\delta} \left|\zeta'(u+it_0)\right| \\ &\geq \exp_2(H/300) - \delta(C_6 \log^2 t_0) \geq \tfrac{1}{2}\exp_2(H/300), \end{aligned}$$

provided $\exp_3(H/400) \geq C_6(\log t_0)^2$. Hence the lemma is proved.

Lemma 3. *For any complex number z with $\left|\mathrm{Re}(z)\right| \leq 1/4$, we have the inequality*

$$\left|\exp\left((\sin z)^2\right)\right| << \exp\left(-\exp\left|\mathrm{Im}(z)\right|\right) \tag{14}$$

where the constant implied by the Vinogradov symbol $<<$ is absolute.

Proof. Let $z = x+iy$ where x and y are real and $i = \sqrt{-1}$. Now

$$\mathrm{Re}\big(\sin z)^2\big) = \mathrm{Re}\left(\tfrac{1}{2}\big(1 - \tfrac{1}{2}(e^{2iz} + e^{-2iz})\big)\right) = \tfrac{1}{4}\big(2 - e^{-2y}\cos(2x) - e^{2y}\cos(2x)\big).$$

Note that in $|x| \leq 1/4$, $\cos(2x)$ is positive and is greater than or equal to $\cos(1/2) \geq \cos(\pi/6 = \sqrt{3}/2$. Hence

$$\mathrm{Re}\big(\sin z)^2\big) \leq -\frac{\sqrt{3}}{8}\left(e^{-2y} + e^{2y}\right) + \tfrac{1}{2},$$

and the lemma follows.

Lemma 4. *Let B be any positive constant. Then for any complex number z with $|\mathrm{Re}(z)| \le B/4$, we have the inequality*

$$\left|\exp\left(\left(\sin\frac{z}{B}\right)^2\right)\right| << \exp\left(-\exp\left(|\mathrm{Im}(z)|/B\right)\right) \tag{15}$$

where the constant implied by the Vinograd symbol $<<$ is absolute.

Proof. This is a corollary to Lemma 3 obtained by replacing z by z/B.

Lemma 5. *Let the maximum in (13) be attained for $\sigma = \sigma_0 \ge 1 + \delta$ and $t = t_0$ where t_0 is in I_2. Then the assertion of Theorem 1 holds.*

Proof. Put $s_0 = \sigma_0 + it$. We can certainly assume that $\sigma_0 < 2 - 0.01$. Let R be the rectangle formed by the vertical line segments $\sigma = 1$, $\sigma = 2$ and t in I and the horizontal line segments connecting the upper and lower extremities of these vertical line segments. Let D be the boundary of this rectangle in the anti-clockwise direction. Then by Cauchy's theorem we get

$$\frac{1}{2\pi i}\int_D \frac{\zeta(s)}{s - s_0}\exp\left(\left(\sin(s - s_0)/B\right)^2\right) ds = \zeta(s_0). \tag{16}$$

Here B is any positive constant. We can fix $B = 4$ for our purpose. The integral along $\sigma = 2$, $t \in I$ is $O(1)$. The integral along $\sigma = 1$, $t \in I$ is $O(M \log \delta^{-1})$, where $M =$ maximum of $|\zeta(1 + it)|$ as t varies over I. The horizontal line segments H_1 and H_2 contribute

$$O\left(\left(\exp_2(H/3B)\right)^{-1}\left(\int_{H_1} |\zeta(s)|\, d\sigma + \int_{H_2} |\zeta(s)|\, d\sigma\right)\right).$$

We have fixed $B = 4$. Since $\zeta(s) = O\big(1/(\sigma - 1)\big)$ and also $\zeta(s) = O(\log T)$, the integrals over H_1 and H_2 are

$$O\left((\log T)(\log T)^{-1} + \int_{1+(\log T)^{-1}}^{2} \frac{d\sigma}{\sigma - 1}\right) = O(\log_2 T).$$

Thus

$$\exp_2(H/300) = O\left(M \log\frac{1}{\delta} + (\log_2 T)\big(\exp_2(H/12)\big)^{-1}\right).$$

From this and our choice of δ our assertion is proved if we make $\exp_2(H/12)$ greater than $\log_2 T$. This proves the lemma and hence Theorem 1 is completely proved.

4. Second application of the main theorem

In Section 3 we saw that the proof worked because

$$\max_{\sigma\geq 1, t\in I}\left|\zeta(\sigma+it)\right| = O(\log T) \quad \text{and} \quad \max_{\sigma\geq 1, t\in I}\left|\zeta'(\sigma+it)\right| = O\big((\log T)^2\big).$$

By the Riemann hypothesis the corresponding estimates are $O(\log_2 T)$ and $O\big((\log_2 T)^2\big)$. The method of proving these estimates are via $\log\zeta(s)$. An examination of the proof of these results shows that it is enough to assume that $\theta < 1$. Hence we record:

Theorem 2. *Let I be an interval of length H contained in $[T, 2T]$. Let $T \geq H \geq C_2 \log_5 T$. Then*

$$\max_{t\in I}\left|\zeta(1+it)\right| \geq e^{\gamma}(\log_2 H - \log_3 H - \varrho),$$

where γ is Euler's constant and ϱ is a certain real constant.

5. Third application of the main theorem

Theorem 3. *Let $H \leq C_1 \log_4 T$. Consider disjoint intervals I, contained in $[T, 2T]$, all of length H. Put $X = \exp_4(\alpha H)$ where α is a certain positive constant satisfying $\alpha \leq C_1^{-1}/2$. Then, except possibly for $O(TX^{-1/2})$ intervals I, we have*

$$\max_{t\in I}\left|\zeta(1+it)\right| \geq e^{\gamma}(\log_2 H - \log_3 H - \varrho),$$

where γ is Euler's constant and ϱ is a certain real constant.

We prove this theorem by a few lemmas.

Lemma 1. *Let $a = 0.1$, $s = \sigma + it$ where $T \leq t \leq 2T$ and $1 + a \geq \sigma \geq 1 - a$. Then*

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(w)=2} \zeta(s+w) X^w \exp(w^2)\frac{dw}{w} = \sum_{n=1}^{\infty} \Delta\left(\frac{x}{n}\right) n^{-s}, \tag{17}$$

where for $u > 0$ we have

$$\Delta(u) = \frac{1}{2\pi i}\int_{\mathrm{Re}(w)=2} u^w \exp(w^2)\frac{dw}{w}. \tag{18}$$

The proof is trivial.

Lemma 2. *We have*

$$\Delta(u) = O(u^5) \qquad \text{for } 0 < u \leq 1 \tag{19}$$

and

$$\Delta(u) = 1 + O(u^{-5}) \qquad \text{for } u \geq 1. \tag{20}$$

Proof. To prove (19) we move the line of integration in (18) to $\mathrm{Re}(w) = 5$ and to prove (20) we move it to $\mathrm{Re}(w) = -5$.

Lemma 3. *Let*

$$\zeta(s) = \sum_{n=1}^{\infty} \Delta\left(\frac{X}{n}\right) n^{-s} + E(s, X). \tag{21}$$

Then we have

$$\int_{1-a}^{1+a} \int_{T}^{2T} \left|E(s, X)\right|^2 d\sigma\, dt = O(TX^{-1/2}). \tag{22}$$

Proof. In the left hand side of (17) we move the line of integration to $\mathrm{Re}(w) = 13/20 - \sigma$ and note that $\exp_2(e) \leq X \leq T$. Using

$$\int_{T/2}^{3T/2} \left|\zeta\left(\frac{13}{20} + it\right)\right|^2 dt = O(T),$$

we complete the proof of the lemma.

Lemma 4. *We have*

$$\sum_{n=1}^{\infty} \Delta\left(\frac{X}{n}\right) n^{-s} = \sum_{n \leq X} n^{-s} + G(s, X) \tag{23}$$

where

$$G(s, X) = \sum_{n \leq X} \left(\Delta\left(\frac{X}{n}\right) - 1\right) n^{-s} + \sum_{n > X} \Delta\left(\frac{X}{n}\right) n^{-s} \tag{24}$$

$$= O(X^{1-\sigma}). \tag{25}$$

Proof. Follows from Lemma 2.

Lemma 5. *The number of intervals I for which*

$$\max_{1+a/2\geq\sigma\geq 1-a/2,\, t\in I}\left|E(s,X)\right| \geq 1 \tag{26}$$

is

$$O(TX^{-1/2}). \tag{27}$$

Proof. The quantity $\left|E(s,X)\right|$ is not greater than its mean value over a disc with centre s and radius $a/2$. The lemma now follows from (22).

Lemma 6. *In the region defined by* $\big(1-(\log X)^{-1} \leq \sigma \leq 2, t \in I\big)$ *we have*

$$\zeta(s) = O(\log X) \tag{28}$$

and also in $(1 \leq \sigma \leq 2, t \in I)$

$$\zeta'(s) = O\big((\log X)^2\big), \tag{29}$$

except possibly for $O(TX^{-1/2})$ *intervals* I.

Proof. The equation (28) follows from Lemma 4 and 5 on noting (21). To prove (29) we may apply Cauchy's theorem to $\zeta(s)(s-z_0)^{-2}$ where z_0 lies in $(\sigma \geq 1, t \in I)$. We integrate over a circle with centre z_0 and radius $(\log X)^{-1}$. Rough estimates now give (29). Another proof consists in differentiating (21), (23) and (24) partially with respect to s. (In the second proof we have also to establish (22) with $E(s,X)$ replaced by $\partial E(s,X)/\partial s$.) Thus Lemma 6 is proved.

From Lemma 6, Theorem 3 follows in the same way as Theorem 1 was derived in Section 3 from the estimates $\zeta(s) = O(\log t)$ and $\zeta'(s) = O\big((\log t)^2\big)$ in $(\sigma \geq 1, t \geq 2)$. Thus Theorem 3 is completely proved, and by choosing a smaller constant β in place of α our third assertion in the introduction follows.

6. Fourth application of the main theorem

Theorem 4. *Let $H \leq C_2 \log_5 T$. Consider disjoint intervals I, contained in $[T, 2T]$, all of length H. Put $Y = \exp_5(\alpha' H)$ where α' is a certain positive constant satisfying $\alpha' \leq C_2^{-1}/2$. Then except possibly for $O(TY^{-1/3})$ intervals I, we have*

$$\max_{t\in I}\left|\zeta(1+it)\right| \geq e^{\gamma}(\log_2 H - \log_3 H - \varrho),$$

where γ is Euler's constant and ϱ is a certain real constant.

We prove this theorem by a series of lemmas. Note that $\exp_3(e) \leq Y \leq T$.

Lemma 1. *Let $a = 0.1$. The number of intervals I for which*

$$\max_{1+a/2\geq\sigma\geq 1-a/2, t\in I} |\zeta(s)| >> \gamma^{a/2} \tag{30}$$

is $O(TY^{-1/2})$. Here the constant implied by the Vinogradov symbol $>>$ is a certain positive constant. Let I' denote the intervals I (above) with intervals of length $\log Y$ annexed at each end. Then the total length of the intervals I' is $O(TY^{-1/2}\log Y)$.

Proof. The proof is similar to that of Lemmas 1 to 5 of the previous section. We have only to replace X by Y.

Lemma 2. *Let us consider the zeros $\varrho_0 = \beta_0 + i\gamma_0$ of $\zeta(s)$ with $T \leq \gamma_0 \leq 2T$ and $\beta_0 \geq 1 - a = 0.9$. Let ε be a small positive constant. With each such zero ϱ_0, we associate the rectangle $R(\varrho_0)$ consisting of complex numbers $z = x + it$ satisfying $1 \geq x \geq 1 - a$ and $|\gamma_0 - t| \leq T^{\varepsilon}$. If $H(\varrho_0)$ denotes the height of $R(\varrho_0)$ then*

$$\sum_{\varrho_0} H(\varrho_0) = O(N(9/10, 2T)T^{\varepsilon}) \tag{31}$$

where $N(9/10, 2T)$ denotes the number of zeros of $\zeta(s)$ with a real part $\geq 9/10$ and imaginary part lying between 0 and $2T$.

The proof is trivial.

Lemma 3. *From the interval $[T, 2T]$ we omit the intervals I' of Lemma 1 of total length $O(TY^{-1/2}\log Y)$ and also the t-intervals counted in Lemma 2 of total length $O\big(N(9/10, 2T)T^{\varepsilon}\big)$. Then the maximum number of intervals I which have at least one point in common with these t-intervals is*

$$O(TY^{-1/2}H^{-1}\log Y) + O\big(N(9/10, 2T)H^{-1}T^{\varepsilon}\big). \tag{32}$$

Proof. We have only to annex on either side of the excluded intervals t-intervals of length H. We then exclude the maximum possible number of t-intervals I which are wholly contained in the union of extended intervals.

Lemma 4. *We have*

$$N(9/10, 2T) = O\big(T^{1/3}(\log T)^{50000}\big) \tag{33}$$

and so the maximum possible number of intervals I (which are excluded) is

$$O(TY^{-1/2}H^{-1}\log Y). \tag{34}$$

Proof. Using only the mean square result regarding $|\zeta(1/2 + it)|$ we can prove the result $N(\sigma, T) = O\big(T^{\lambda(1-\sigma)}(\log T)^{50000}\big)$ where $\lambda = 4/(3 - 2\sigma)$. (See [6] and the references therein.) This is the simplest non-trivial density result.

(The method of obtaining such results can be traced to many authors. See [11] and [4].) This result gives the lemma. We can choose $\varepsilon = 1/4$.

Lemma 5. *Let I be a t-interval which is not excluded by Lemmas 3 and 4. (We will prove the expected lower bound for the maximum of $|\zeta(1+it)|$ taken over such intervals.) Then for any point t_1 belonging to I, the rectangle $S(t_1)$ defined by*

$$S(t_1) = \{\sigma + it \,|\, 0.9 \leq \sigma \leq 2, |t - t_1| \leq \log Y\} \tag{35}$$

is free from zeros of $\zeta(s)$ and also $|\zeta(s)| \leq Y$ there.

Proof. Follows from Lemmas 1 to 4.

Lemma 6. *Let C_6 be a large positive constant. Then in the rectangle $U(t_1)$ defined by*

$$U(t_1) = \{\sigma + it \mid 0.95 \leq \sigma \leq 2,\ |t - t_1| \leq \log Y - C_6\} \tag{36}$$

we have $\log \zeta(s) = O(\log Y)$.

Proof. The lemma follows by a suitable application of the Borel–Caratheodory theorem (see p. 282 of [11]).

Lemma 7. *Let s by any point of the rectangle $V(t_1)$ defined by*

$$V(t_1) = \{\sigma + it \mid 0.975 \leq \sigma \leq 2,\ |t - t_1| \leq \tfrac{1}{2}\log Y - C_6\}. \tag{37}$$

Let $V = (\log Y)^{100}$ and let

$$\log \zeta(w) = \sum_{n=2}^{\infty} a_n n^{-w} \tag{38}$$

in $\mathrm{Re}(w) \geq 2$. Then for s in $V(t_1)$, we have

$$\frac{1}{2\pi i} \int_{\mathrm{Re}(w)=2} \log \zeta(s+w) V^w \exp(w^2) \frac{dw}{w} = \sum_{n=2}^{\infty} \Delta\left(\frac{V}{n}\right) a_n n^{-s}, \tag{39}$$

where $\Delta(u)$ for $u > 0$ is defined as in (18).

The proof is trivial.

Lemma 8. *We have uniformly for all s in $V(t_1)$*

$$\sum_{n=2}^{\infty} \Delta\left(\frac{V}{n}\right) a_n n^{-s} = \log \zeta(s) + o(1). \tag{40}$$

Proof. Let us consider the integral in (39). The contribution from $|\mathrm{Im}(w)| \geq (\log Y)/4$ is $o(1)$ since $|\exp(w^2)| << \exp(-|\mathrm{Im}(w)|^2)$ where the constant implied by the Vinogradov symbol is absolute if $|\mathrm{Re}(w)|$ does not exceed an absolute constant. We deform the rest of the contour as follows. $\mathrm{Im}(w) = -(\log Y)/4$ in the direction of $\mathrm{Re}(w)$ decreasing from 2 to $0.95 - \sigma$; then the vertical line $\mathrm{Re}(w) = 0.95 - \sigma$ in the direction of $\mathrm{Im}(w)$ increasing and then $\mathrm{Im}(w) = (\log Y)/4$ in the direction of $\mathrm{Re}(w)$ increasing from $0.95 - \sigma$ to 2. Using Lemma 6 it is easily seen that the integrals along the deformed contour contribute $o(1)$. The pole $w = 0$ contributes $\log \zeta(s)$. Thus the lemma is completely proved.

Lemma 9. *For s in $V^*(t_1)$, defined by*

$$V^*(t_1) = \{\sigma + it \mid 1 - (\log V)^{-1} \le \sigma \le 2,\ |t - t_1| \le \tfrac{1}{2}\log Y - C_6\}, \tag{41}$$

we have the estimates $\zeta(s) = O(\log V)$ and, in the part $\sigma \ge 1$ of $V^*(t_1)$, $\zeta'(s) = O\big((\log V)^2\big)$ with $V = (\log Y)^{100}$.

Proof. Let $\sigma^* = 1 - (\log V)^{-1}$. Then in the region $V^*(t_1)$ we have

$$\begin{aligned} \big|\log\zeta(s)\big| &\le \sum_{n=2}^{\infty} \Delta\left(\frac{V}{n}\right) a_n n^{-\sigma^*} + O(1) \le \sum_p \Delta\left(\frac{V}{p}\right) p^{-\sigma^*} + O(1) \\ &\le \sum_{p \le V} p^{-\sigma^*} + O(1) \le \sum_{p\le V} p^{-1} + O(1) \le \log_2 V + O(1). \end{aligned}$$

Here the first three inequalities follow as in Lemma 4 of the previous section. The fourth follows since for $p \le V$, $p^{-\sigma^*} = p^{-1} + O(p^{-1}\log p/\log V)$ and since $\sum_{p\le V}\log p/p = O(\log V)$ by prime number theorem. The last inequality follows by prime number theorem. The first estimate of the lemma follows since $\log\big|\zeta(s)\big| \le \big|\log\zeta(s)\big|$. The second follows from the first by applying Cauchy's theorem. Thus Lemma 9 is completely proved.

Theorem 4 follows from Lemma 9 (just as we derived Theorem 1 from the main theorem). Our fourth assertion in the introduction follows from Theorem 4.

Concluding remarks

The arguments of the previous section resemble to some extent the definition and treatment of the Huxley–Hooley contour in [5]. We can work out the results corresponding to the previous sections for $1/2 < \sigma < 1$ and also for $\sigma = 1/2$. Thus

$$\max_{t\in I}\big|\zeta(\sigma + it)\big| > \exp\left(\frac{C_7(\log H)^{1-\sigma}}{\log_2 H}\right) \tag{i}$$

holds with the exception of at most $O\big(T\big(\exp_2(\beta'' H)\big)^{-1}\big)$ intervals where β'' is a positive constant, provided $C_4 \le H \le C_8 \log_2 T$ (C_8 being any positive constant and β'' is allowed to depend on C_8).

On $\sigma = 1/2$ we get

$$\max_{t\in I}\big|\zeta(\tfrac{1}{2} + it)\big| > \exp\left(\frac{3}{4}\left(\frac{\log H}{\log_2 H}\right)^{1/2}\right) \tag{ii}$$

(some positive constant in place of $3/4$ comes out by [2], but $3/4$ comes out by using a result in [1]) with exceptions nearly the same as before but with an extra restriction $H \ge \log_3 T$.

The improvement of these results seems to be difficult. (For some kernel functions used in the present paper see [9] and the reference list there, especially the reference number 3.)

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