Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 14, 1989, 47–55

SETS OF ZERO ELLIPTIC HARMONIC MEASURES

O. Martio

1. Introduction

An elliptic partial differential equation $\nabla \cdot A(x, \nabla u(x)) = 0$ in a domain G with $|A(x,h)| \approx |h|^{p-1}$ produces a solution ω called an A-harmonic measure. For $p \neq 2$, ω is non-additive and hence does not define a measure in the Borel sets of ∂G as the classical harmonic measure induced by the Laplace operator $A(x,h) = h$ does. The most interesting problem associated with ω is to determine the class of subsets E of ∂G such that $\omega(E) = 0$. This class depends on A. For example, in the plane unit disk B there is a linear elliptic operator $A(x,h) \approx h$ which induces ω such that $\omega(E) > 0$ for some compact set $E \subset \partial B$ whose linear measure is zero. Such an operator A can be constructed using quasiconformal mappings, see [GLM 2] and [CFK]. Hence ω essentially differs from the ordinary plane harmonic measure induced by the Laplace operator. Contrary to this example we show in this paper that there exists a reasonable class of subsets E of ∂G such that $\omega(E) = 0$ for all operators A. Clearly ∂G must be sufficiently thick for this purpose. For compact subsets E of ∂G our main result, Theorem 3.1, is formulated in terms of certain metric conditions of E with respect to ∂G. Here the quasihyperbolic distance [GP] is useful. Surprisingly, for $G = B$, the unit ball of R^n, Theorem 3.1 shows that there are compact sets $E \subset \partial B$ whose Hausdorff dimension is arbitrary near $n-1$ and $\omega(E) = 0$ for all A. By the above example this condition cannot be replaced by the condition that the $(n-1)$-dimensional Hausdorff measure of E is $= 0$.

For $p = n$ these problems were first studied in [GLM 2] and [HM]. Conditions for $\omega(E) > 0$ were given in [GLM 2, 4.10] and [M]. If ∂G is "thick", then these results can be used to prove the counterpart of B. Øksendal's theorem for the A-harmonic measures ω, see [HM, Theorem 4.1] and [H, Theorem A]. Our main theorem, Theorem 3.1, can also be used to study sets E in ∂G which cannot be seen easily from G. We say that such sets E are buried in ∂G and prove that $\omega(E) = 0$ for all A; this result slightly generalizes [H, Theorem A]. Using stochastic methods Øksendal [Ø] has also studied the corresponding problems for $p = 2$ and for linear operators A.

Suppose that G is a bounded domain in R^n and that $1 < p \le n$. We shall study partial differential operators $A\colon G \times R^n \to R^n$ which satisfy the following assumptions:

doi:10.5186/aasfm.1989.1418

a) For each $\varepsilon > 0$ there exists a compact subset F of G such that $A|F \times R^n$ is continuous and $m(G \setminus F) < \varepsilon$.

b) There exist positive constants γ_1 and γ_2 such that for a.e. $x \in G$

$$|A(x,h)| \le \gamma_1 |h|^{p-1}, \tag{1.1}$$

$$A(x,h) \cdot h \ge \gamma_2 |h|^p \tag{1.2}$$

for all $h \in R^n$.

c) For a.e. $x \in G$

$$\big(A(x,h_1) - A(x,h_2)\big) \cdot (h_1 - h_2) > 0, \qquad h_1 \ne h_2.$$

d) For a.e. $x \in G$

$$A(x, \lambda h) = |\lambda|^{p-2} \lambda A(x,h)$$

for $\lambda \in R \setminus \{0\}$ and $h \in R^n$.

A continuous function $u\colon G \to R$ is a solution of the equation

$$\nabla \cdot A\big(x, \nabla u(x)\big) = 0 \tag{1.3}$$

if u belongs to the Sobolev space $\operatorname{loc} W_p^1(G)$, i.e., u is ACL^p, and if

$$\int_G A\big(x, \nabla u(x)\big) \cdot \nabla \phi(x)\, dm(x) = 0 \tag{1.4}$$

for all $\phi \in C_0^\infty(G)$. We call solutions of (1.3) A-harmonic. A lower semicontinuous function $u\colon G \to R \cup \{\infty\}$ is A-superharmonic if it satisfies the A-comparison principle, i.e., if for every domain $D \subset\subset G$ and every A-harmonic function $h \in C(\overline{D})$ in D, $h \le u$ in ∂D implies $h \le u$ in D. These functions form a similar, but in general non-linear, potential theory as ordinary harmonic and superharmonic functions do, see [GLM 1] and [HK].

Finally, let E be a subset of ∂G. The upper class $\mathcal{U}$ consists of all A-superharmonic functions $u\colon G \to R \cup \{\infty\}$ such that

$$\liminf_{x \to y} u(x) \ge \chi_E(y)$$

for each $y \in \partial G$. Here χ_E is the characteristic function of E. It can be shown that

$$\omega(E, G; A)(x) = \inf_{u \in \mathcal{U}} u(x), \qquad x \in G,$$

defines an A-harmonic function $\omega = \omega(E, G; A)$, called the A-harmonic measure of E with respect to G. For this construction see [HK] and [GLM 2]. The set E has zero A-harmonic measure, if $\omega(x) = 0$ for some $x \in G$, or equivalently $\omega(x) = 0$ for all $x \in G$. The last assertion follows from Harnack's inequality, see Lemma 3.3 below. In this case we simply write $\omega = 0$.

2. Sets of A-harmonic measure zero

Let G be a bounded domain in R^n. We assume that G is A-Dirichlet regular, i.e., for each $\psi \in C(\partial G)$ there is a (unique) function $u \in C(\overline{G})$ such that u is A-harmonic in G and that $u|\partial G = \psi$. The function u is called the A-harmonic function with boundary values ψ. The following lemma is a generalization of [GLM 2, 4.9].

2.1. Lemma. *Suppose that E is a compact subset of ∂G. Let $\omega = \omega(E, G; A)$. Then $\omega = 0$ if and only if there is $c \in [0,1)$ and a sequence of neighborhoods $\mathcal{U}_i$, $i = 1, 2, \ldots$, of E such that*

$$(a) \qquad \bigcap \mathcal{U}_i \cap G = \emptyset$$

and

$$(b) \qquad \omega(x) \le c \qquad \text{for each } x \in G \cap \partial\mathcal{U}_i, \quad i = 1, 2, \ldots.$$

Proof. For the only if part choose $c = 0$ and $\mathcal{U}_i = E + B(1/i)$, $i = 1, 2, \ldots$. Here $B(r)$ denotes the open ball of radius $r > 0$ centered at 0.

For the converse part we first show that

$$(2.2) \qquad u(x) \le c$$

for each $x \in G$. Fix $x \in G$. By (a) there is $\mathcal{U}_i$ such that $x \notin \mathcal{U}_i$. If $x \in \partial\mathcal{U}_i$, then (2.2) follows from (b). Assume that $x \in G \setminus \overline{\mathcal{U}}_i$. Let V be the x-component of $G \setminus \overline{\mathcal{U}}_i$. Let $y \in \partial V$. If $y \in G$, then $y \in \partial\mathcal{U}_i$ and hence $\omega(y) \le c$ by (b). If $y \notin G$, then let $\psi \in C(\partial G)$ be such that $\psi(y) = 0$, $\psi|E = 1$ and $0 \le \psi \le 1$. Let u be the A-harmonic function with boundary values ψ. Then $u(y) = 0$ and since u belongs to the upper class $\mathcal{U}$, $\omega \le u$ in G. Hence we obtain

$$(2.3) \qquad \lim_{z \to y} \omega(z) = 0.$$

Thus in both cases

$$\limsup_{z \to y} \omega(z) \le c$$

and this holds for every $y \in \partial V$. Now constants are A-harmonic functions, hence the A-comparison principle yields $\omega \le c$ in V and we have shown $\omega(x) \le c$ as required.

Next we complete the proof for the converse part. If $c = 0$, then $\omega = 0$ as required. If $c > 0$ and $\omega \ne 0$, then $\omega > 0$ and hence

$$(2.4) \qquad \omega < \omega/c \qquad \text{in } G.$$

On the other hand, $\omega/c \le 1$ in G by (2.2) and if u belongs to the upper class $\mathcal{U}$ for ω, then

$$(2.5) \qquad \omega/c \le u$$

by the A-comparison principle. Note that $\lim_{z \to y} \omega(z) = 0$ for every $y \in \partial G \setminus E$; this can be proved as (2.3). By (2.5), $\omega/c \le \omega$ and hence we obtain a contradiction from (2.4). This completes the proof.

3. Quasihyperbolic distance and A-harmonic measure

Let E be a closed set in R^n and $D = R^n \setminus E$. If x_1, $x_2 \in D$, then the quasihyperbolic distance $k_D(x_1, x_2)$ of x_1 and x_2 is

$$k_D(x_1, x_2) = \inf_\gamma \int_\gamma d(x, E)^{-1} ds$$

where the infimum is taken over all rectifiable curves γ joining x_1 and x_2 in D. Here $d(x, E)$ denotes the distance from x to E. If no such curves exist, i.e., if x_1 and x_2 belong to different components of D, then we set $k_D(x_1, x_2) = \infty$.

Let G be a domain in R^n. We say that G satisfies a p-capacity density condition if for some $c_0 > 0$ and $r_0 > 0$

$$\operatorname{cap}_p\big(\bar{B}(x, r) \cap \mathcal{C}G, B(x, 2r)\big) \geq c_0 r^{n-p}$$

for all $x \in \partial G$ and $0 < r \leq r_0$. Here cap_p refers to the variational p-capacity of the condenser $E = \big(\bar{B}(x, r) \cap \mathcal{C}G, B(x, 2r)\big)$, i.e.,

$$\operatorname{cap}_p E = \inf \int_{B(x,2r)} |\nabla u|^p dm$$

where the infimum is taken over all functions $u \in C_0^\infty\big(B(x, 2r)\big)$ such that $u \geq 1$ in $\bar{B}(x, r) \cap \mathcal{C}G$.

3.1. Theorem. *Let G be a bounded domain satisfying a p-capacity density condition. Suppose that E is a compact subset of ∂G such that there exist a sequence of neighborhoods $\mathcal{U}_i$, $i = 1, 2, \ldots$, of E and $M < \infty$ with*
(a) *$\bigcap \mathcal{U}_i \cap G = \emptyset$ and*
(b) *for each $i = 1, 2, \ldots$ and $x \in \partial \mathcal{U}_i \cap G$ there is $y \in \partial G$ with $k_D(x, y) \leq M$, $D = R^n \setminus E$.*
Then $\omega(E, G; A) = 0$.

The proof is based on two lemmas. The first is essentially due to V.G. Maz'ya [Maz]. We shall employ the short argument due to Heinonen [H, Lemma 5.2].

3.2. Lemma. *Let F be a closed set in a ball $B(x_0, 2r)$. If u is a continuous function in $B(x_0, 2r)$ such that $u \,|\, F = 1$, $0 \leq u \leq 1$ and u is a solution of (1.3) in $B(x_0, 2r) \setminus F$, then*

$$u(x) \geq c_1 r^{(p-n)/(p-1)} \operatorname{cap}_p\big(F \cap \bar{B}(x_0, r), B(x_0, 2r)\big)^{1/(p-1)}$$

for each $x \in B(x_0, r)$. Here the constant c_1 depends only on γ_1, γ_2, p and n.

Proof. Let $\omega = \omega\big(F \cap \bar{B}(x_0,r), B(x_0,2r) \setminus (F \cap \bar{B}^n(x_0,r)); A\big)$. By [H, Lemma 5.2]

$$\omega(x) \geq c_1 r^{(p-n)/(p-1)} \operatorname{cap}_p\big(F \cap \bar{B}^n(x_0,r), B(x_0,2r)\big)^{1/(p-1)}$$

for each $x \in B(x_0,r) \setminus F$ and $c_1 > 0$ depends only on γ_1, γ_2, p and n. Next fix $x \in B(x_0,r) \setminus F$ and let V be the x-component of $B(x_0,2r) \setminus F$. Now $\liminf u(z) \geq \limsup \omega(z)$ as z approaches $y \in \partial V$ in V; note that $0 \leq \omega \leq 1$ and that $\lim_{z \to y} \omega(z) = 0$ for all $y \in \partial B(x_0,2r)$ because balls are always A-Dirichlet regular. Hence by the A-comparison principle $u \geq \omega$ in V and thus the required inequality follows from the corresponding inequality for ω.

The next lemma is the well known Harnack inequality, see e.g. [S, pp. 264–269].

3.3. Lemma. *Let u be a non-negative solution of (1.3) in $B(x_0,2r)$. Then*

$$\sup_{x \in B(x_0,r)} u(x) \leq c_2 \inf_{x \in B(x_0,r)} u(x)$$

where the constant c_2 depends only on γ_1, γ_2, p and n.

Proof for Theorem 3.1. Since G is bounded, we may assume that the inequality in the p-capacity density condition holds for all $r \in (0, \operatorname{diam} G)$. Write $\omega = \omega(E,G;A)$. We shall show that there is $c \in [0,1)$ such that

$$\omega(x) \leq c$$

for all $x \in \partial \mathcal{U}_i \cap G$, $i = 1,2,\ldots$. Lemma 2.1 then completes the proof. Observe that since G satisfies a p-capacity density condition, G is A-Dirichlet regular, see [Maz]. This implies that $\lim_{x \to y} \omega(x) = 0$ for all $y \in \partial G \setminus E$.

Fix $i = 1,2,\ldots$ and let $x \in \partial \mathcal{U}_i \cap G$. Choose $y \in \partial G$ with $k_D(x,y) \leq M$. Let γ be a rectifiable curve in D joining x to y with

$$\int_\gamma d(z,E)^{-1} ds \leq M + 1. \tag{3.4}$$

Next choose points $z_1, \ldots, z_j$ and radii $r_1, \ldots, r_j$ inductively as follows. Set $z_1 = x$ and $r_1 = d(z_1,E)/4$. Assume that $z_1, \ldots, z_i$ have been chosen and let γ_i denote the part of γ from z_i to y. If $\partial G \cap \bar{B}(z_i, 2r_i) \neq \emptyset$, then we set $j = i$ and end the process. If $\partial G \cap \bar{B}(z_i, 2r_i) = \emptyset$, then choose z_{i+1} to be the last point where γ_i meets $\partial B(z_i, r_i)$ and put $r_{i+1} = d(z_{i+1},E)/4$. Since $y \in \partial G \setminus E$, this process ends after a finite number of steps.

Next we obtain an upper bound for j in terms of M. Fix $i = 1, \ldots, j-1$ and let γ_i be the part of γ from z_i to z_{i+1}. Pick $z' \in E$ such that

$$4r_i = d(z_i,E) = |z_i - z'|.$$

Then for $z \in \gamma_i \cap B(z_i, r_i)$,

$$d(z, E) \le |z - z'| \le |z - z_i| + |z_i - z'| \le r_i + 4r_i = 5r_i$$

and thus

$$\int_{\gamma_i} d(z, E)^{-1} ds \ge \int_{\gamma_i \cap B(z_i, r_i)} d(z, E)^{-1} ds \ge r_i / 5r_i = 1/5.$$

Hence

$$\int_{\gamma} d(z, E)^{-1} ds \ge \sum_{i=1}^{j-1} \int_{\gamma_i} d(z, E)^{-1} ds \ge (j-1)/5$$

and we obtain from (3.4)

$$j \le 5M + 6. \tag{3.5}$$

By the above construction $\partial G \cap \bar{B}(z_j, 2r_j) \ne \emptyset$, hence there is $x_0 \in \partial G \cap \bar{B}^n(z_j, 2r_j)$. Set $u = 1 - \omega$. Then u is a solution of (1.3) in G, $0 \le u \le 1$ and if we set $u(x) = 1$ for $x \in \mathcal{C}G \cap B(z_j, 4r_j)$, then u is continuous in $B(z_j, 4r_j)$. Consequently, u is a continuous function in $B(x_0, 2r_j)$ and a solution of (1.3) in $B(x_0, 2r_j) \setminus \mathcal{C}G$. Let $F = \mathcal{C}G \cap \bar{B}(x_0, r_j)$. Thus Lemma 3.2 and the p-capacity density condition yield for $z \in B(x_0, r_j)$

$$\begin{aligned} u(z) &\ge c_1 r_j^{(p-n)/(p-1)} \operatorname{cap}_p\big(F, B(x_0, 2r_j)\big)^{1/(p-1)} \\ &\ge c_1 r_j^{(p-n)/(p-1)} c_0 r_j^{(n-p)/(p-1)} = c_1 c_0 > 0. \end{aligned}$$

Hence for $z \in B(z_j, r_j)$ we have

$$u(z) \ge c_1 c_0. \tag{3.6}$$

Set $B_i = B(z_i, r_i)$, $i = 1, \dots, j$, and $u = 1 - \omega$. Then (3.6) and Lemma 3.3 yield

$$c_1 c_0 \le \inf_{B_j} u \le \sup_{B_{j-1}} u \le c_2 \inf_{B_{j-1}} u \le \cdots \le c_2^{j-1} \inf_{B_1} u.$$

Hence we obtain

$$\omega(x) = 1 - u(x) \le 1 - \inf_{B_1} u \le 1 - c_1 c_0 c_2^{1-j}$$

and (3.5) implies $\omega(x) \le c < 1$ where

$$c = 1 - c_1 c_0 c_2^{-5M-5}.$$

This shows that $\omega(x) \le c$ and the proof is complete.

3.7. Remark. In the case $p = n$ it was shown in [GLM 2, 4.18 and 4.19] that if E is a compcat set in the boundary of the unit ball B and if the domain $R^n \setminus E$ is a uniform domain in the sense of [MS], then $\omega = \omega(E, B; A) = 0$. Note that B satisfies a p-capacity density condition for all p, $1 < p \leq n$. Now Theorem 3.1 *implies this result for all* A. Hence it is easy to construct compact sets $E \subset \partial B$ whose Hausdorff-dimension is arbitrary close to $n-1$ and yet $\omega(E, B; A) = 0$ for all A.

On the other hand, since the neighborhoods $\mathcal{U}_i$ of Theorem 3.1 are at our disposal, it is easy to construct a compact set E in ∂B which satisfies (a) and (b) of 3.1 and yet $R^n \setminus E$ is not a uniform domain.

4. Buried sets

Let G be a bounded domain in R^n. Write $C = \partial G$. For $r > 0$ set

$$C_G(r) = \big(C + B(r)\big) \cap G$$

and for $c > 0$ put

$$C_c(r) = \big\{x \in C \,:\, d(x, \partial C_G(r) \cap G\big) \geq (1 + c)r\big\}.$$

Then $C_c(r)$ is a compact subset of ∂G.

A subset E of ∂G is said to be *buried* in ∂G if there is a number $c > 0$ and a sequence of positive numbers $r_i \to 0$ such that

$$E \subset \cap_i C_c(r_i). \tag{4.1}$$

It is easy to see that if ∂G is a C^1-manifold, then no subset E of ∂G is buried in ∂G. Roughly speaking, a set E is buried in ∂G if there are numbers $r_i \searrow 0$ with the following property: If one stands at the distance r_i from ∂G in G, then the set E is slightly further away than ∂G.

The following theorem generalizes [H, Theorem A].

4.2. Theorem. *Suppose that* G *is a bounded domain which satisfies a* p*-capacity density condition. If a set* E *is buried in* ∂G*, then* $\omega(E, G; A) = 0$.

Proof. We may assume that E is compact. Let $c > 0$ and (r_i) be such that (4.1) holds. For each $i = 1, 2, \ldots$ write $\mathcal{U}_i = \partial G + B(r_i)$. Then $\mathcal{U}_i$ is a neighborhood of ∂G and hence of E. Moreover, $\bigcap \mathcal{U}_i \cap G = \emptyset$. It remains to show that the condition (b) of Theorem 3.1 is satisfied.

To this end let $x \in \partial \mathcal{U}_i \cap G$. Then there exists $y \in \partial G$ such that

$$|x - y| = d(x, \partial G) = r_i.$$

Now

$$d(x,E) \geq (1+c)r_i \tag{4.3}$$

because in the opposite case

$$(1+c)r_i > d(x,E) \geq d\big(x, C_c(r_i)\big) \geq (1+c)r_i,$$

a contradiction. Let $\gamma(t) = \big(ty+(r_i-t)x\big)$, $t \in [0, r_i]$, be the straight line segment from x to y. If we let $D = R^n \setminus E$, then

$$\begin{aligned} k_D(x,y) &\leq \int_\gamma d(z,E)^{-1} ds \leq \int_0^{r_i} \big[(1+c)r_i - t\big]^{-1} dt \\ &= \log \frac{1+c}{c} = M < \infty \end{aligned}$$

because by (4.3) for each $t \in [0, r_i]$

$$d\big(\gamma(t), E\big) \geq (1+c)r_i - t.$$

Hence the condition (b) of Theorem 3.1 is satisfied and $\omega(E,G;A) = 0$ follows from Theorem 3.1.

4.4. Remark. Simple examples show that there are bounded domains G and sets E buried in ∂G such that $\partial G \setminus E$ is countable. Hence the p-capacity density condition in Theorem 4.2 cannot be completely removed. Slight modifications of the above example show that this condition cannot by replaced by the condition that G is A-Dirichlet regular.

References

[CFK] CAFFARELLI, L., E. FABES, and C. KENIG: Completely singular elliptic harmonic measures. - Indiana Univ. Math. J. 30, 1981, 917–924.

[GP] GEHRING, F.W., and B. PALKA: Quasiconformally homogeneous domains. - J. Analyse Math. 30, 1976, 172–199.

[GLM1] GRANLUND, S., P. LINDQVIST, and O. MARTIO: Conformally invariant variational integrals. - Trans. Amer. Math. Soc. 277, 1983, 43–73.

[GLM2] GRANLUND, S., P. LINDQVIST, and O. MARTIO: F-harmonic measure in space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 7, 1982, 233–247.

[H] HEINONEN, J.: Boundary accessibility and elliptic harmonic measures. - Complex Variables 10, 1988, 273–282.

[HK] HEINONEN, J., and T. KILPELÄINEN: A-superharmonic functions and supersolutions of degenerate elliptic equations. - Ark. Mat. 26, 1988, 87–105.

[HM] HEINONEN, J., and O. MARTIO: Estimates for F-harmonic measures and Øksendal's theorem for quasiconformal mappings. - Indiana Univ. Math. J. 36, 1987, 659-683.

[M] MARTIO, O.: F-harmonic measures, quasihyperbolic distance and Milloux's problem. - Ann. Acad. Sci. Fenn. Ser. A I Math. 12, 1987, 151–162.

[MS] MARTIO, O., and J. SARVAS: Injectivity theorems in plane and space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 4, 1978/1979, 383–401.

[Maz] MAZ'YA, V.G.: On the continuity on a boundary point of solutions of quasi-linear elliptic equations. - Vestnik Leningrad Univ. Mat. 13, 1970, 42–55 (Russian).

[Ø] ØKSENDAL, B.: Dirichlet forms, quasiregular functions and Brownian motion. - Invent. Math. 91, 1988, 273–297.

[S] SERRIN, J.: Local behavior of solutions of quasilinear equations. - Acta Math. 111, 1964, 247–302.

University of Jyväskylä
Department of Mathematics
Seminaarinkatu 15
SF–40100 Jyväskylä
Finland

Received 10 October 1987