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COMPACT AND WEAKLY COMPACT MULTIPLICATIONS ON C*-ALGEBRAS

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Dedicated to Professor T. T. West on the occasion of his 50 th birthday.

Abstract. Let ρ_l be a left and ρ_r a right centralizer of a C*-algebra. We characterize when $\rho_l\rho_r$ is a compact or a weakly compact operator.

Following Vala [8], an element a of a C*-algebra A is called *compact* if the mapping $x \mapsto axa$ is a compact operator on A. For our purposes, the following equivalent definition due to Ylinen [10; Theorem 3.1] is more adequate: $a \in A$ is compact if and only if the left multiplication $L_a : x \mapsto ax$, or equivalently the right multiplication $R_a : x \mapsto xa$, is a weakly compact operator on A. Suppose for a moment that A is the algebra $L(H)$ of all bounded operators on some Hilbert space H and take $a, b \in A$, both non-zero. Vala had proved in [7] that L_aR_b is compact if and only if both a and b are compact, whilst Akemann and Wright showed in [1; Proposition 2.3] that L_aR_b is weakly compact if and only if either a or b is compact. This was extended to arbitrary prime C*-algebras and a, b in $M(A)$, the multiplier algebra of A, in [4], and similar results were obtained for linear combinations of left and right multiplications.

In the case of a general C*-algebra, Ylinen [9; Theorem 3.1] and also Akemann and Wright [1; p.146] proved that L_aR_a is weakly compact if and only if a is a compact element. In this short note we will characterize both L_aR_b compact and L_aR_b weakly compact for arbitrary C*-algebras. However, we will undertake this in the slightly more general framework of left and right centralizers. To this end, we first recall some facts from [6; 3.12].

Let A be a C*-algebra. A linear map ρ_l: $A \to A$ having the property $\rho_l(xy) = \rho_l(x)y$ for all $x, y \in A$ is called a *left centralizer of* A. Each left centralizer is bounded, and if we consider A canonically embedded in its enveloping W*-algebra A^{**}, then to each left centralizer ρ_l of A corresponds uniquely a *left multiplier* $a \in A^{**}$, i.e., $\rho_l = L_a$ and $aA \subseteq A$. The norm closed subspace of all left multipliers of A will be denoted by $LM(A)$. The analogous concepts of *right centralizer* and *right multiplier* can be defined by $\rho_r = J\rho_lJ$, where J is the involution of A, and

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$RM(A) = J(LM(A))$. Therefore, all compactness properties stated below for left centralizers apply equally well to right centralizers.

Let $K(A)$ denote the norm closed two-sided ideal of compact elements of A. We refer to [2] for some basic properties of $K(A)$. We first determine the weakly compact left centralizers of a C*-algebra.

Lemma 1. *A left centralizer ρ_l of A is weakly compact if and only if $\rho_l = L_a$ for some $a \in K(A)$.*

Proof. First observe that the second adjoint L_a^{**} of L_a, $a \in LM(A)$, is nothing but the mapping $x \mapsto ax$, $x \in A^{**}$. If L_a is weakly compact, $L_a^{**}A^{**} \subseteq A$ by [3; VI.4.2], which implies $a \in A$ and thus $a \in K(A)$ by the aforementioned characterization of compact elements [10; Theorem 3.1]. This proves the "only if"-part, and the "if"-part is clear. □

Definition. Let $\rho_l = L_a$ and $\rho_r = R_b$, $a \in LM(A)$, $b \in RM(A)$, be a left, respectively right, centralizer of a C*-algebra A. The mapping $\rho_l\rho_r = L_aR_b$ is called a *(two-sided) multiplication on* A and is denoted by $M_{a,b}$.

Suppose that either $a \in K(A)$ or $b \in K(A)$. Since the weakly compact operators form an ideal, $M_{a,b}$ is weakly compact. Suppose that both $a \in K(A)$ and $b \in K(A)$. Then $M_{a,b}$ is compact by virtue of the polarization identity

$$M_{a,b} = \frac{1}{4}\sum_{k=0}^{3} i^k M_{(b+i^k a^*)^*, b+i^k a^*}.$$

The converse implications do not hold for arbitrary C*-algebras. However, they are true for prime algebras. We prepare this by the following simple result.

Lemma 2. *Let A be a prime C*-algebra and $a \in LM(A)$, $b \in RM(A)$. Then $M_{a,b} = 0$ if and only if $a = 0$ or $b = 0$.*

Proof. If $M_{a,b} = 0$, then $AaAAbA = 0$. Thus AaA and AbA are orthogonal ideals of A and the primeness of A forces either $AaA = 0$, i.e., $a = 0$, or $AbA = 0$, i.e., $b = 0$. □

Lemma 3. *Let A be a prime C*-algebra and $a \in LM(A)$, $b \in RM(A)$.*

(i) *If $M_{a,b}$ is weakly compact, $a \in K(A)$ or $b \in K(A)$.*

(ii) *If $M_{a,b} \neq 0$ is compact, $a \in K(A)$ and $b \in K(A)$.*

The proof is very similar to that of the corresponding result in [4]. Therefore, we merely outline the argument for assertion (i). The weak compactness of $M_{a,b}$ implies $M_{a,b}A \subseteq K(A)$ since $M_{axb,axb} = M_{a,b}M_{xb,ax}$ is weakly compact for every $x \in A$. If $K(A) = 0$, then $M_{a,b} = 0$, and the assertion follows by Lemma 2. If $K(A) \neq 0$, we may assume that A acts irreducibly on some Hilbert space H and $K(A) = K(H)$, the compact operators on H [2; C*.4 and F.4.3]. Thinking of a

and b as left, respectively right, multipliers of A in $A'' = L(H)$, as we may do by [6; 3.12.3 and 3.12.5], it follows as in Lemma 1 that $M_{a,b}\, L(H) \subseteq K(H)$. Hence the induced multiplication on the Calkin algebra $C(H) = L(H)/K(H)$ vanishes, and since $C(H)$ is prime, either $a \in K(H)$ or $b \in K(H)$.

It is interesting to compare Lemma 3 with similar results in [5], where it is proved that the product $\delta_1\delta_2$ of two derivations of a prime C*-algebra is weakly compact if and only if either δ_1 or δ_2 is weakly compact [5; Lemma 4], and is compact if and only if both δ_1 and δ_2 are weakly compact and some additional condition holds [5; Lemma 7].

Before we can extend Lemma 3 to arbitrary C*-algebras we need another essentially well known result; for the sake of completeness we give a proof.

Lemma 4. *Let A be a C*-algebra and $a \in LM(A)$, $b \in RM(A)$. Then $M_{a,b} = 0$ if and only if a and b are centrally orthogonal.*

Proof. Denote by z_x the central support projection of $x \in A^{**}$. Since the set $\{x \in A^{**} \mid xA^{**}b = 0\}$ is an ultraweakly closed ideal of A^{**} and contains a, it contains z_a. Similarly, the ideal $\{y \in A^{**} \mid z_a A^{**} y = 0\}$ contains z_b. Therefore, $z_a z_b = 0$, i.e., a and b are centrally orthogonal. This proves the "only if"-part, and the "if"-part is obvious. □

We call $\rho_l = L_a$ and $\rho_r = R_b$ *orthogonal* if a and b are centrally orthogonal.

In what follows we assume that the C*-algebra A acts in its reduced atomic representation on $H = \oplus_{t\in\hat{A}} H_t$, where $\hat{A}$ is the spectrum of A (cf. [6; 4.3.7]). Let $p_t \in A'$ be the projection onto $H_t \hookrightarrow H$, $t \in \hat{A}$, and p the central projection in A^{**} with $A^{**}p = A''$. The *multiplier algebra* $M(A)$ of A is the intersection $LM(A) \cap RM(A)$. If T is a bounded linear map on a C*-subalgebra B of A^{**} and $c \in M(B)$ is central, then cT shall denote the map $x \mapsto cT(x)$ on B. Put $I_l = \overline{A\varrho_l A}$ and $I_r = \overline{A\varrho_r A}$. We can now characterize the (weakly) compact multiplications on A.

Theorem 1. *Let ρ_l, respectively ρ_r, be a left, respectively right, centralizer of a C*-algebra A. Then $\rho_l\rho_r$ is weakly compact if and only if there exist orthogonal central projections e_1, e_2, e_3 in A^{**} with $e_1 + e_2 + e_3 = 1$ and $a, b \in A^{**}$, $c \in Z(A^{**})$ such that $ce_i \in M(I_i)$ where $I_1 = I_l e_1$, $I_2 = I_r e_2$, $i = 1, 2$, both $c\rho_{l|Ae_1} = L_{ae_1}$ and $c\rho_{r|Ae_2} = R_{be_2}$ are weakly compact, $\rho_{r|Ae_1} = R_{cbe_1}$ and $\rho_{l|Ae_2} = L_{cae_2}$, and $\rho_{l|Ae_3}$ and $\rho_{r|Ae_3}$ are orthogonal.*

Proof. Let $\rho_l = L_{a_0}$ with $a_0 \in LM(A)$ and $\rho_r = R_{b_0}$ with $b_0 \in RM(A)$. Under the hypotheses on the projections e_j and the elements a, b and c, the identity

$$\begin{aligned} M_{a_0,b_0} &= M_{a_0,b_0e_1} + M_{a_0e_2,b_0} = M_{a_0,cbe_1} + M_{cae_2,b_0} \\ &= M_{ca_0e_1,b} + M_{a,cb_0e_2} = M_{ae_1,b} + M_{a,be_2} \end{aligned}$$

immediately proves the "if"-part.

Suppose now that M_{a_0,b_0} is weakly compact. Let T_0 be the set of those t for which $M_{a_0,b_0|Ap_t} \neq 0$. As in [1; Lemma 2.4] (see also [4; Lemma 3.5]), it follows that $T_n := \{t \mid \|M_{a_0,b_0|Ap_t}\| > 1/n\}$ is finite for each $n \in \mathbf{N}$ and thus $T_0 = \bigcup_{n\in\mathbf{N}} T_n$ is countable. Put $e_3 = \sum_{t\notin T_0} p_t + 1 - p$. Then a_0e_3 and b_0e_3 are centrally orthogonal (Lemma 4). Let $T_{0,1}$ be the set of those $t \in T_0$ such that $\rho_{l|Ap_t}$ is weakly compact and $T_{0,2} = T_0 \setminus T_{0,1}$. By Lemma 3, $\rho_{r|Ap_t}$ is weakly compact for all $t \in T_{0,2}$. Put $e_1 = \sum_{t\in T_{0,1}} p_t$, $e_2 = \sum_{t\in T_{0,2}} p_t$, $c = \mathrm{c}(b_0b_0^*)^{\frac{1}{4}}e_1 + \mathrm{c}(a_0^*a_0)^{\frac{1}{4}}e_2$, where $\mathrm{c}(y)$ denotes the central cover of a self-adjoint element $y \in A^{**}$ [6; 2.6.2], and $a_1 = ca_0e_1$, $b_2 = cb_0e_2$.

Take $t \in T_{0,1}$. Then

$$\begin{aligned} a_1p_t &= \mathrm{c}(b_0b_0^*)^{\frac{1}{4}}p_ta_0 = \|\mathrm{c}(b_0b_0^*)p_t\|^{\frac{1}{4}}p_ta_0 \\ &= \|\mathrm{c}(b_0b_0^*p_t)\|^{\frac{1}{4}}p_ta_0 = \|b_0b_0^*p_t\|^{\frac{1}{4}}p_ta_0 = \|b_0p_t\|^{\frac{1}{2}}p_ta_0, \end{aligned}$$

where we used [6; 2.6.2 and 2.6.4] and the fact that p_t is a minimal projection in $Z(A^{**})$. Thus

$$\|a_1p_t\| = \|b_0p_t\|^{\frac{1}{2}}\|a_0p_t\| = \|M_{a_0p_t,b_0p_t}\|^{\frac{1}{2}}\|a_0p_t\|^{\frac{1}{2}}$$

tends to zero when t runs through $T_{0,1}$ (recall that $T_{0,1} \cap T_n$ is finite for each $n \in \mathbf{N}$). We conclude that $c\rho_{l|Ae_1} = L_{a_1}$ is the norm limit of a sequence of weakly compact operators and hence weakly compact itself [3; VI.4.4]. Similarly, $c\rho_{r|Ae_2} = R_{b_2}$ is weakly compact.

Since $Ap_t \cap K(H_t) \neq 0$, if $t \in T_0$ (it contains $M_{a_0,b_0}Ap_t$), $K(H_t) = K(Ap_t) \subseteq Ap_t$ by [6; 6.1.4]. Therefore, $(a_1p_t)_{t\in T_{0,1}}$ corresponds to a sequence $(a^{(n)})_{n\in\mathbf{N}}$ in A where each $a^{(n)}$ belongs to a closed ideal $I^{(n)}$ of A, $I^{(n)} \cong K(H_t)$ for a unique $t \in T_{0,1}$, $I^{(n)} \cap I^{(m)} = 0$ if $n \neq m$, and $\lim_{n\to\infty} \|a^{(n)}\| = 0$. By [4; Proposition 2.1], there is thus $a_1' \in K(A)$ such that $a_1'p_t = a_1p_t$ for every $t \in T_{0,1}$; so, in order to simplify the notation, we may assume that $a_1 = a_1' \in K(A)$ and similarly $b_2 \in K(A)$. It follows that $cxa_0ye_1 = xa_1ye_1 \in Ae_1$ for all $x, y \in A$; hence $ce_1 \in M(I_1)$ and similarly $ce_2 \in M(I_2)$.

Finally, observe that since $\|b_0p_t\| > 0$ and $\left\|\|b_0p_t\|^{-\frac{1}{2}}b_0p_t\right\| \leq \|b_0\|^{\frac{1}{2}}$ for each $t \in T_{0,1}$, we may define

$$b_1 = b_1e_1 = \sum_{t\in T_{0,1}}{}^{\oplus}\|b_0p_t\|^{-\frac{1}{2}}b_0p_t \in A^{**}e_1.$$

Clearly, $cb_1 = \sum_{t\in T_{0,1}}^{\oplus} \mathrm{c}(b_0b_0^*)^{\frac{1}{4}}\|b_0p_t\|^{-\frac{1}{2}}b_0p_t = b_0e_1$ and therefore $\rho_{r|Ae_1} = R_{cb_1}$. Similarly, we put

$$a_2 = a_2e_2 = \sum_{t\in T_{0,2}}{}^{\oplus}\|a_0p_t\|^{-\frac{1}{2}}a_0p_t \in A^{**}e_2$$

and obtain $\rho_{l|Ae_2} = L_{ca_2}$.

Hence we end up with $a = a_1 + a_2 + a_0 e_3$ and $b = b_1 + b_2 + b_0 e_3$, which completes the proof. □

Theorem 2. *Let ρ_l, respectively ρ_r, be a left, respectively right, centralizer of a C^*-algebra A. Then $\rho_l \rho_r$ is compact if and only if there exist a, b in $K(A)$ such that $\rho_l \rho_r = M_{a,b}$. In addition, a and b can be chosen such that for some central projection e in A^{**} and for some positive central multipliers c and d of $I_l(1-e)$ and $I_r(1-e)$, respectively, both $c\rho_{l|A(1-e)} = L_{da(1-e)}$ and $d\rho_{r|A(1-e)} = R_{cb(1-e)}$ are weakly compact.*

Proof. "If"-part: This follows from the remarks preceding Lemma 2.

"Only if"-part: Write $\rho_l = L_{a_0}$, $\rho_r = R_{b_0}$ with $a_0 \in LM(A)$, $b_0 \in RM(A)$ as before, let T_0 be as in the proof of Theorem 1 and put $e = e_3$. Take $t \in T_0$. Since $\rho_{l|Ap_t} \neq 0$ and $\rho_{r|Ap_t} \neq 0$, Lemma 3 shows that both $a_0 p_t$ and $b_0 p_t$ have to be compact elements in Ap_t. Putting

$$a = \sum_{t \in T_0} {}^{\oplus} \|a_0 p_t\|^{-\frac{1}{2}} \|b_0 p_t\|^{\frac{1}{2}} a_0 p_t \qquad \in A(1-e)$$

and

$$b = \sum_{t \in T_0} {}^{\oplus} \|a_0 p_t\|^{\frac{1}{2}} \|b_0 p_t\|^{-\frac{1}{2}} b_0 p_t \qquad \in A(1-e)$$

yields $M_{a_0, b_0} = M_{a,b}$. Since $\|ap_t\| = \|b_0 p_t\|^{\frac{1}{2}} \|a_0 p_t\|^{\frac{1}{2}} = \|bp_t\|$ tends to zero when t runs through T_0, both a and b are compact elements in A by [4; Proposition 2.1] as in the proof of Theorem 1. Put $c = \mathrm{c}(b_0 b_0^*)^{\frac{1}{4}}$ and $d := \mathrm{c}(a_0^* a_0)^{\frac{1}{4}}$. The relations $ca_0(1-e) = da(1-e)$ and $db_0(1-e) = cb(1-e)$ are obviously valid; hence $c\rho_{l|A(1-e)}$ and $d\rho_{r|A(1-e)}$ are both weakly compact (Lemma 1). As in the proof of Theorem 1, we conclude that $c \in M\big(I_l(1-e)\big)$ and $d \in M\big(I_r(1-e)\big)$. □

Corollary. *Every compact multiplication on a C^*-algebra is the norm limit of multiplications of finite rank.*

These results show that, apart from a direct summand where $\rho_l \rho_r$ can be zero, the (weak) compactness of $\rho_l \rho_r$ is completely determined by the weak compactness of ρ_l and ρ_r up to central 'scaling' factors. An extension of Theorems 1 and 2 to linear combinations of left and right centralizers (elementary operators) would first of all need a corresponding result as in Lemma 4, which is not available yet.

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