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EXAMPLES OF NON-DISTINGUISHED FRÉCHET SPACES

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1. Introduction

A Fréchet space is non-distinguished, if its strong dual is not a barreled or bornological locally convex space. The existence of such Fréchet spaces has been well-known for a long time; in this paper we shall use the non-distinguished space constructed by Köthe and Grothendieck, see [K1], 31.7.

Our first main result is the following: Given a Köthe echelon space λ of order 1 we can find a Fréchet–Montel (in short (FM)-) space F such that λ is isomorphic to a complemented subspace of the projective tensor product $F\hat{\otimes}_\pi F$. Using, for example, the Köthe–Grothendieck example mentioned above we see that $F\hat{\otimes}_\pi F$ can be chosen non-distinguished. This result answers a question of Bierstedt and Bonet in [BB], Section 1.5. It also gives a new counterexample to Grothendieck's question, whether the projective tensor product of two distinguished Fréchet spaces is again distinguished. However, this question was already solved by S. Dierolf, see [D].

The rest of this paper is devoted to the study of the Fréchet space

$$C(\mathbf{R}) \cap L_1(\mathbf{R})$$

of continuous, L_1-integrable real functions on the real line. It seems that this simply definable and concrete space is quite complicated and pathological from the TVS-theoretical point of view. In Section 3 we shall show that $C(\mathbf{R}) \cap L_1(\mathbf{R})$ is not distinguished and in Section 4 we shall study the projective tensor product of $C(\mathbf{R}) \cap L_1(\mathbf{R})$ and $C(0,1)$. We shall present a condition concerning finite dimensional Banach spaces, the validity of which would imply that this tensor product does not have property (BB) in the sense of [T1]. However, the question about property (BB) remains unsolved.

For locally convex spaces and their topological tensor products we shall use the notation and definitions of [K1,2]. Let us only remark that the topological dual of a space E is denoted by E' and the space of continuous linear mappings from E in E (respectively F) by $L(E)$ (respectively $L(E,F)$). If A is a set in

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some locally convex space, we mean by $\Gamma(A)$ the absolutely convex hull of A and by A° the polar of A.

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2. On the projective tensor product of (FM)-spaces

We begin with the following generalization of Theorem 3.3 in [T2].

Theorem 2.1. *If λ is a Köthe echelon space of order 1, then there is an (FM)-space $\tilde{F}$ such that λ is isomorphic to a complemented subspace of $\tilde{F}\hat{\otimes}_\pi\tilde{F}$.*

Proof. In the following we shall use constantly the construction of Section 3 of [T2].

Let $(\alpha_{km})_{k,m=1}^\infty$ be the Köthe matrix of λ, i.e., we assume that the topology of λ is determined by the seminorms

$$\|(x_m)\|_k := \sum_{m=1}^\infty \alpha_{km}|x_m|.$$

We assume that $0 \le \alpha_{km} \le \alpha_{k+1,m}$ for all k and m and that for each m there is an $\alpha_{km} \neq 0$.

We define the finite dimensional vector spaces M_n, E_n, N_n, N_{tn} and A_n and the norm p as in [T2], Section 3. We replace the norms $(p_k \,|\, A_n)$ of [T2] by

$$(2.1)\qquad (\tilde{p}_k \,|\, A_n) := \alpha_{kn}^{1/2}\Big(n^{(k-1)/2k}p(x + \sum_{t=1}^n y_t) + \sum_{t=1}^n t^k p(y_t) + \sum_{t=1}^{n\wedge k} n^k p(y_t)\Big),$$

where $x + \sum_t y_t \in M_n \oplus \bigoplus_{t=1}^n N_{tn} = A_n$. Finally, we set as in [T2]

$$\tilde{F} := \Big\{z = (z_n)_{n=1}^\infty | z_n \in A_n, \tilde{p}_k(z) := \sum_{n=1}^\infty (\tilde{p}_k|A_n)(z_n) < \infty\Big\}.$$

1°. To show that $\tilde{F}$ is a Montel space we use the proof of Proposition 3.2 in [T2]; we only replace (3.2) by

$$\alpha_{kn}^{1/2} n^{(k-1)/2k} < \frac{\alpha_{k+1,n^\varepsilon}^{1/2}}{4r_{k+1}}\, n^{k/2(k+1)},$$

(3.3) by

$$\alpha_{kn}^{1/2} t^k < \frac{\alpha_{k+1,n^\varepsilon}^{1/2}}{4r_{k+1}}\, t^{k+1}$$

and (3.4) by

$$\alpha_{kn}^{1/2} n^k < \frac{\alpha_{sn}^{1/2}\varepsilon}{8r_s}\, n^s.$$

The other changes are straightforward.

2°. Let us denote by P_n the natural projection from $\tilde{F}$ onto A_n. If $z \in \tilde{F}\otimes\tilde{F}$ and $z = \sum_i a_i \otimes b_i$ is an arbitrary finite representation, then for all k

$$\begin{aligned}\sum_i \tilde{p}_k(a_i)\tilde{p}_k(b_i)\delta &= \sum_i\Big(\sum_n (\tilde{p}_k|A_n)(P_n a_i)\Big)\Big(\sum_n (\tilde{p}_k|A_n)(P_n b_i)\Big)\\ &= \sum_{n,m}\sum_i\Big((\tilde{p}_k|A_n)(P_n a_i)\Big)\Big((\tilde{p}_k|A_m)(P_m b_i)\Big)\\ &\geq \sum_{n,m}\big((\tilde{p}_k|A_n)\otimes(\tilde{p}_k|A_m)\big)\big((P_n\otimes P_m)(z)\big)\\ &= \sum_{n,m}(\tilde{p}_k\otimes\tilde{p}_k)\big((P_n\otimes P_m)(z)\big).\end{aligned} \tag{2.2}$$

This shows that the sum

$$\sum_{n,m=1}^{\infty}(P_n\otimes P_m)(z) \tag{2.3}$$

converges absolutely in $\tilde{F}\otimes_\pi\tilde{F}$ to z for all $z\in\tilde{F}\otimes\tilde{F}$. Moreover, by the definition of the projective tensor norm and (2.2),

$$(\tilde{p}_k\otimes\tilde{p}_k)(z) \geq \sum_{n,m}(\tilde{p}_k\otimes\tilde{p}_k)(z_{nm}), \tag{2.4}$$

where $z_{nm} := (P_n\otimes P_m)(z)$. The converse of (2.4) is true by the triangle inequality so that

$$(\tilde{p}_k\otimes\tilde{p}_k)(z) = \sum_{n,m}(\tilde{p}_k\otimes\tilde{p}_k)(z_{nm}). \tag{2.5}$$

Since $\tilde{F}\otimes\tilde{F}$ is dense in $\tilde{F}\hat{\otimes}_\pi\tilde{F}$, the equality (2.5) holds also for all $z\in\tilde{F}\hat{\otimes}_\pi\tilde{F}$. This result will be needed later.

3°. We show that λ is isomorphic to a subspace of $\tilde{F}\hat{\otimes}_\pi\tilde{F}$.

For all $n\in\mathbf{N}$ let $z_n\in A_n\otimes A_n\subset F\hat{\otimes}_\pi F$ be as in the proof of Theorem 3.5 of [T2]. By [T2], there are constants $C_k>0$ such that

$$\tfrac{1}{4}\leq(p_2\otimes p_2)(z_n)\leq(p_k\otimes p_k)(z_n)\leq C_k \tag{2.6}$$

for all k and n. We can consider the tensors z_n as elements of $\tilde{F}\hat{\otimes}_\pi\tilde{F}$ in the natural way. By comparing the definitions of p_k and $\tilde{p}_k$ we get $\alpha_{kn}(p_k\otimes p_k)(z_n) = (\tilde{p}_k \otimes \tilde{p}_k)(z_n)$. Hence, (2.6) implies

$$\tfrac{1}{4}\alpha_{kn} \leq (\tilde{p}_k \otimes \tilde{p}_k)(z_n) \leq C_k\alpha_{kn} \tag{2.7}$$

for all n and k.

The equality (2.5) shows that

$$(\tilde{p}_k \otimes \tilde{p}_k)\Big(\sum_{n\in\mathbf{N}} a_n z_n\Big) = \sum_{n\in\mathbf{N}} |a_n|(\tilde{p}_k \otimes \tilde{p}_k)(z_n) \tag{2.8}$$

for all scalar sequences (a_n) with only a finite number of non-zero elements.

Let $(e_n)_{n=1}^{\infty}$ be the natural basis of λ. We define the linear mapping ψ from λ into $\tilde{F}\hat{\otimes}_\pi\tilde{F}$ by $\psi(e_n) = z_n$. The formulas (2.7) and (2.8) imply that ψ is a topological isomorphism from λ onto the subspace $E := \psi(\lambda)$.

$4°$. We show that E is a complemented subspace.

It follows directly from (2.5) that the natural projection $\tilde{P}_0$ from $\tilde{F}\hat{\otimes}_\pi\tilde{F}$ onto

$$A := \overline{\bigoplus_{n=1}^{\infty} A_n \otimes A_n}$$

is continuous. Hence, it is enough to show that E is complemented in A.

For all $n \in \mathbf{N}$ the linear span sp (z_n) is a 1-dimensional subspace of $A_n\otimes A_n$. Hence, there exists a projection $\tilde{P}_n$ from $A_n \otimes A_n$ onto sp (z_n) such that $\|\tilde{P}_n\| = 1$, where the operator norm is taken with respect to $(A_n \otimes A_n, \tilde{p}_{k(n)} \otimes \tilde{p}_{k(n)})$. Here $k(n)$ is the smallest integer for which $\alpha_{k(n)n} \neq 0$. We claim that

$$\tilde{P} = \bigoplus_{n=1}^{\infty} \tilde{P}_n$$

is a continuous projection from A onto E.

Let $k \geq 2$ and let $z \in A$ be such that

$$(\tilde{p}_k \otimes \tilde{p}_k)(z) \leq 4^{-1}C_k^{-1},$$

where C_k is as in (2.6). We denote by $\mathbf{N}'$ the subset of $\mathbf{N}$ for which $\alpha_{kn} \neq 0$ if and only if $n \in \mathbf{N}'$. Then, by (2.1), (2.5), (2.7) and the definitions above (note that $k(n) \le k$ for $n \in \mathbf{N}'$)

$$\begin{aligned}(\tilde{p}_k \otimes \tilde{p}_k)(\tilde{P}z) &= \sum_{n\in\mathbf{N}'} (\tilde{p}_k \otimes \tilde{p}_k)\big(\tilde{P}_n(P_n \otimes P_n)z\big)\\ &\le \sum_n 4C_k \frac{\alpha_{kn}}{\alpha_{k(n)n}} \big(\tilde{p}_{k(n)} \otimes \tilde{p}_{k(n)}\big)\big(\tilde{P}_n(P_n \otimes P_n)z\big)\\ &\le 4C_k \sum_n \frac{\alpha_{kn}}{\alpha_{k(n)n}} \big(\tilde{p}_{k(n)} \otimes \tilde{p}_{k(n)}\big)\big((P_n \otimes P_n)z\big)\\ &\le 4C_k \sum_n (\tilde{p}_k \otimes \tilde{p}_k)\big((P_n \otimes P_n)z\big)\\ &\le 4C_k(\tilde{p}_k \otimes \tilde{p}_k)(z) \le 1.\end{aligned}$$

Hence, $\tilde{P}$ is a continuous projection from A onto E. Q.E.D.

As an immediate application of this theorem we get

Corollary 2.2. *There exists an (FM)-space $\tilde{F}$ such that the space $\tilde{F}\hat{\otimes}_\pi\tilde{F}$ is not distinguished.*

Proof. We take the Köthe–Grothendieck non-distinguished space for λ in the previous theorem.

Bierstedt and Bonet ask in [BB], Section 1.5., if the projective tensor product of two Fréchet spaces E and F with Heinrich's density condition (for definition, see [BB]) also satisfies the density condition. By Corollary 1.3.(2) of [BB], every Fréchet space with the density condition is distinguished. On the other hand, all (FM)-spaces satisfy the density condition, see [BB], Section 1.1. Using our Corollary 2.2. we thus get

Corollary 2.3. *There exists a Fréchet space F with Heinrich's density condition such that $F\hat{\otimes}_\pi F$ is not distinguished and does not satisfy the density condition.*

This corollary is also a partial answer to Question 13.11.3 in [PB].

It should be noted that, as a consequence of the preceding constructions, the projective tensor product of (FM)-spaces need not always be reflexive.

3. On the space $C(\mathbf{R}) \cap L_1(\mathbf{R})$

Counterexamples to problems concerning topological vector spaces are often obtained by very abstract constructions. However, there exist also more concrete spaces (occurring in analysis), the structure of which is non-trivial from the TVS-theoretical point of view: for example, in this section we study the function space $C(\mathbf{R}) \cap L_1(\mathbf{R})$. We shall show that this space is not distinguished; hence, finding a satisfactory topology in the dual of the space is difficult.

By $C(\mathbf{R}) \cap L_1(\mathbf{R})$ we mean the space of continuous, L_1-integrable functions $f : \mathbf{R} \to \mathbf{R}$ endowed with the seminorms

$$p_k(f) := \max\left\{ \sup_{x\in[-k,k]} |f(x)|, \int_{-\infty}^{\infty} |f(x)|\, dx \right\}. \tag{3.1}$$

The completeness of this space can be verified by considering it as a subspace of $L_1^{\mathrm{loc}}(\mathbf{R})$ of locally integrable functions on the real line.

We shall now define the subspace λ of $C(\mathbf{R}) \cap L_1(\mathbf{R})$, the structure of which resembles that of the Köthe–Grothendieck non-distinguished space mentioned in Section 2.

For all $m \in \mathbf{N}$ let us denote by χ_m the characteristic function of the interval $[2^{-m}, 2^{-m} + 2^{-2m+1}]$ and let us define the continuous mapping $f_m : \mathbf{R} \to \mathbf{R}$ by

$$f_m(x) = \chi_m(x) \sin(2^{2m}\pi x).$$

These functions have the following properties:

1°. The supports are distinct for different f_m.

2°. For all $m \in \mathbf{N}$

$$\sup_{x\in\mathbf{R}} |f_m(x)| = 1.$$

3°. For all $m \in \mathbf{N}$ we have $\int_{-\infty}^{\infty} f_m(x)dx = 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} |f_m(x)|\, dx &= \int_{2^{-m}}^{2^{-m}+2^{-2m+1}} |\sin(2^{2m}\pi x)|\, dx \\ &= \int_{2^{-m}}^{2^{-m}+2^{-2m}} \sin(2^{2m}\pi x)\, dx = 2^{-2m+2}\pi^{-1}\end{aligned}$$

and

$$\int_{-\infty}^{\infty} f_m(x)^2\, dx = \int_{2^{-m}}^{2^{-m}+2^{-2m+1}} \sin^2(2^{2m}\pi x)\, dx = 2^{-2m}.$$

4°. If $f \in C(\mathbf{R})$, then

$$\lim_{m\to\infty} 2^{2m} \int_{-\infty}^{\infty} f(x) f_m(x)\, dx = 0.$$

For if $\varepsilon > 0$, we choose m_0 such that $|f(x) - f(0)| < \varepsilon$ for $0 \le x \le 2^{-m_0} + 2^{-2m_0+1}$. Then, if $m > m_0$,

$$\Big|\int_{-\infty}^{\infty} f(x)f_m(x)\,dx\Big| = \Big|\int_{-\infty}^{\infty} f(x)f_m(x)\,dx - \int_{-\infty}^{\infty} f(0)f_m(x)\,dx\Big|$$

$$= \Big|\int_{2^{-m}}^{2^{-m}+2^{-2m+1}} \big(f(x) - f(0)\big)\sin(2^{2m}\pi x)\,dx\Big|$$

$$\le \sup_{0\le x\le(2^{-m}+2^{-2m+1})} \big|f(x) - f(0)\big| \cdot \int_{2^{-m}}^{2^{-m}+2^{-2m+1}} \big|\sin(2^{2m}\pi x)\big|\,dx$$

$$< \varepsilon 2^{-2m+2}\pi^{-1}.$$

Finally, for all $n, m \in \mathbf{N}$ we denote $e_{nm}(x) = f_m(x - n + 1)$ and we define λ as the closed linear span of $(e_{nm})_{n,m\in\beta N}$. Note that the supports of the functions e_{nm} are distinct for different n or m. For simplicity, we also denote $C(\mathbf{R}) \cap L_1(\mathbf{R}) =: E$.

Lemma 3.1. *The subspace λ is complemented in E.*

Proof. We define the mapping P from E into the space of all real functions on the real line by

$$Pf := \sum_{n,m=1}^{\infty} e_{nm} \frac{\int_{-\infty}^{\infty} f e_{nm} dx}{\int_{-\infty}^{\infty} e_{nm}^2 dx}.$$

We shall show that P is in fact a continuous projection from E into λ.

1°. We check first that Pf is a continuous function $\mathbf{R} \to \mathbf{R}$ for all $f \in E$. The continuity is clear everywhere else except in the points $x = 0, 1, 2, 3, \ldots$. But in these points the continuity follows from the facts that $|e_{nm}(x)| \le 1$ for all x, n and m and that

$$\lim_{m\to\infty}\Big(\int_{-\infty}^{\infty} f e_{nm}\,dx\Big)\Big(\int_{-\infty}^{\infty} e_{nm}^2\,dx\Big)^{-1} = 0$$

for all n and continuous f; see the properties 3° and 4° of the functions f_m.

2°. We show that P has the other required properties. Let $k \ge 1$ and let $f \in E$ with $p_k(f) \le 1$. Then

$$\sup_{x\in[-k,k]} \big|f(x)\big| \le 1, \qquad \int_{-\infty}^{\infty} |f|\,dx \le 1$$

and

$$p_k(Pf) \le p_k \left(\sum_{n \le k, m \in \mathbf{N}} e_{nm} \frac{\int\limits_{-\infty}^{\infty} f e_{nm} dx}{\int\limits_{-\infty}^{\infty} e_{nm}^2 dx} \right)$$

$$+ \sum_{n > k, m \in \mathbf{N}} p_k(e_{nm}) \frac{\left| \int\limits_{n+2^{-m}}^{n+2^{-m}+2^{-2m+1}} f e_{nm}\, dx \right|}{\int\limits_{-\infty}^{\infty} e_{nm}^2\, dx}$$

$$\le 2k \sup_{x \in [-k,k]} \left(\sum_{n \le k, m \in \mathbf{N}} e_{nm}(x) 2^{2m} \int\limits_{-\infty}^{\infty} |e_{nm}|\, dx \right)$$

$$+ \sum_{n > k, m \in \mathbf{N}} 2^{2m} \int\limits_{-\infty}^{\infty} |e_{nm}|\, dx \int\limits_{n+2^{-m}}^{n+2^{-m}+2^{-2m+1}} |f|\, dx \sup_{x \in [n+2^{-m}, n+2^{-m}+2^{-2m+1}]} |e_{nm}(x)|$$

$$\le 2k \sup_{x \in [-k,k]} \left(\sum_{n \le k, m \in \mathbf{N}} 4\pi^{-1} e_{nm}(x) \right) + \sum_{n > k, m \in \mathbf{N}} 4\pi^{-1} \int\limits_{n+2^{-m}}^{n+2^{-n}+2^{-2m+1}} |f|\, dx$$

$$\le 8k\pi^{-1} + 4\pi^{-1} \int\limits_{-\infty}^{\infty} |f|\, dx \le 8k\pi^{-1} + 4\pi^{-1}.$$

This shows that P maps E continuously into E. It is easy to see that $P(E) \subset \lambda$. Moreover, P is idempotent on the dense subspace $\mathrm{sp}(e_{nm})_{n,m \in \mathbf{N}}$ of λ. Hence, P is also a projection onto λ. Q.E.D.

It is convenient to introduce the following

Definition 3.2. By Λ we mean the Fréchet space of c_0^w-valued sequences $(X_n)_{n=1}^{\infty}$, for which the norms

$$q_k\big((X_n)\big) := \sum_{n=1}^{\infty} \|X_n\|_{nk}$$

are finite for all k. Here $\|\cdot\|_{nk}$ is the c_0^w-norm

$$\|X_n\|_{nk} = \sup_{m \in \mathbf{N}} \{2^{2m} |x_{nm}|\}$$

($X_n = (x_{nm})_{m=1}^{\infty}$, $x_{nm} \in \mathbf{R}$; c_0^w is the space of sequences $(a_m)_{m=1}^{\infty}$ for which lim $2^{2m}|a_m| = 0$), if $n \le k$, and the ℓ_1-norm

$$\|X_n\|_{nk} = \sum_{m=1}^{\infty} |x_{nm}|,$$

if $n > k$.

By $\hat{e}_{ij}$ we denote the sequence $(X_n)_{n=1}^{\infty}$ for which $X_n = 0$, if $n \neq i$, and for which the elements of X_i are 0 except the j-th element, which is equal to 1.

Lemma 3.3. *The space Λ is topologically isomorphic to λ.*

Proof. The isomorphism is determined by mapping the element $e_{nm} \in \lambda$ to $2^{-2m}\hat{e}_{nm}$. We leave the details to the reader.

By [K1] 22.6.(3) and 19.9.(1), the dual of Λ is the space of such ℓ_1^w-valued sequences $(Y_n)_{n=1}^{\infty}$ for which one of the norms

$$q'_k((Y_n)) := \sup_n \{\|Y_n\|_{nk}\}$$

is finite; here $\|\cdot\|_{nk}$ is the ℓ_1^w-norm

$$\|Y_n\|_{nk} = \|(y_{nm})_{m=1}^{\infty}\|_{nk} = \sum_{m=1}^{\infty} 2^{-2m}|y_{nm}|,$$

if $n \leq k$ (l_1^w is the space of sequences (a_m) for which $\sum 2^{-2m}|a_m| < \infty$), and the norm of ℓ_∞, if $n > k$.

By τ we denote the topology of Λ' determined by the neighbourhoods

$$W = \Gamma\left(\bigcup_{k=1}^{\infty} c_k B_k\right),$$

where Γ means the absolutely convex hull, (c_k) runs over all positive sequences and B_k is the set

$$\{Y = (Y_n)_{n=1}^{\infty} \in \Lambda' \mid q'_k((Y_n)) \leq 1\}.$$

This is the associated bornological space of the strong dual Λ'_b (see [K1], Sections 28, 29).

A bounded set in Λ is contained in a set $\cap_{k=1}^{\infty}(c_k B_k)^{\circ}$ for some positive sequence (c_k). Hence, a basis of Λ'_b-neighbourhoods is defined by the sets

$$\left(\bigcap_{k=1}^{\infty}(c_k B_k)^{\circ}\right)^{\circ} = \left(\Gamma(\bigcup_{k=1}^{\infty} c_k B_k)\right)^{\circ\circ}.$$

Theorem 3.4. *The space $C(\mathbf{R}) \cap L_1(\mathbf{R})$ is non-distinguished.*

Proof. In view of Lemmas 3.1 and 3.3 it is enough to show that Λ'_b is not bornological. But the structure of Λ is very similar to that of the Köthe–Grothendieck non-distinguished space, [K1], 31.7, and we can use the same proof: We take a τ-neighbourhood $V_0 = \Gamma(\bigcup_{k=1}^{\infty} B_k/k)$ and show that if

$$U = (U_n)_{n=1}^{\infty} = (u_{nm})_{n,m=1}^{\infty} \in V_0,$$

then there is an $N \in \mathbf{N}$ such that $|u_{nm}| \leq 1$ for all $n > N$ and $m \in \mathbf{N}$. On the other hand as in [K1] we see that if a Λ'_b-neighbourhood V is given, then an element $2\sum_{n=1}^{\infty} \hat{e}_{nk(n)}$ belongs to V, provided the numbers $k(n)$ are large enough. This consideration shows that $V \not\subset V_0$. Hence, the topologies of (Λ', τ) and Λ'_b do not coincide, and the space Λ'_b is not bornological. Q.E.D.

4. On the projective tensor product of $C(\mathbf{R}) \cap L_1(\mathbf{R})$ and $C(0,1)$

In this section we shall study the question whether the space

$$(C(\mathbf{R}) \cap L_1(\mathbf{R}))\hat{\otimes}_\pi C(0,1) \tag{4.1}$$

has property (BB) in the sense of [T1]. Unfortunately, we can not solve the problem here, but we shall describe a condition concerning finite dimensional Banach spaces, the presence of which would imply that (4.1) does not have property (BB).

We shall again begin the study by choosing some elements and subspaces of $E = C(\mathbf{R}) \cap L_1(\mathbf{R})$. This time we shall not, however, specify the functions so carefully.

Using, for example, suitable convolutions of the characteristic functions of some subintervals of $[0, m^{-1}]$, where $m \in \mathbf{N}$, it is possible to find positive functions $f_m \in E$ with the following properties:

1°. $f_m(x) = 0$, if $x \notin [0, m^{-1}]$,

2°. $\sup_{x\in\mathbf{R}} |f_m(x)| = 1$,

3°. $\int\limits_{-\infty}^{\infty} f_m(x)\,dx = (2m)^{-1}$,

4°. $\int\limits_{-\infty}^{\infty} f_m(x)^2\,dx \geq \frac{1}{2} \int\limits_{-\infty}^{\infty} f_m(x)\,dx$.

For all $n, m \in \mathbf{N}$ and $i = 1, \ldots, m$ we denote $e_{nmi}(x) := f_m(x - (i-1)m^{-1} - n)$. Finally, we define for all $n, m \in \mathbf{N}$ the subspace $M_{nm} = \mathrm{sp}\{e_{nmi} | i = 1, \ldots, m\} \subset E$. We note the following facts:

5°. If $k \leq n$, then (M_{nm}, p_k) is isometric to ℓ_1^m for all m. The isometry is defined by identifying $(e_{nmi})_{i=1}^m$ with the usual basis of ℓ_1^m. Note that $p_k(e_{nmi}) = (2m)^{-1}$.

6°. Similarly, if $k > n$, it is easy to see that for all $f \in M_{nm}$

$$p_k(f) = \sup_{x\in\mathbf{R}} |f(x)|,$$

and (M_{nm}, p_k) becomes isometric to ℓ_∞^m. Again, $(e_{nmi})_{i=1}^m$ corresponds to the natural basis of ℓ_∞^m, and, by definition, $p_k(e_{nmi}) = 1$ for all i.

Lemma 4.1. *There is for every $n, m \in \mathbf{N}$ a projection Q_{nm} from E onto M_{nm} such that for all $k \in \mathbf{N}$ the operator norm of Q_{nm} with respect to the space (E, p_k) is not greater than 2.*

Proof. We define $Q_{nm} := \sum_{i=1}^{m} Q_{nmi}$, where

$$Q_{nmi}(f) := \frac{\int\limits_{-\infty}^{\infty} f(x)e_{nmi}\,dx}{\int\limits_{-\infty}^{\infty} \left(e_{nmi}(x)\right)^2 dx}\, e_{nmi}$$

for $f \in E$. It is clear that Q_{nm} and Q_{nmi} are idempotent and, hence, projections.

Assume that $k \leq n$. Let $f \in E$ be such that $p_k(f) \leq 1$. Then

$$\sum_{i=1}^{m} \int\limits_{n+m^{-1}(i-1)}^{n+m^{-1}i} |f(x)|\,dx \leq \int\limits_{-\infty}^{\infty} |f(x)|\,dx \leq p_k(f) \leq 1$$

and, by 4° and the Hölder inequality,

$$\begin{aligned} p_k(Q_{nm}f) &\leq 4m \sum_{i=1}^{m} \left| \int\limits_{-\infty}^{\infty} f(x)e_{nmi}(x)\,dx \right| p_k(e_{nmi}) \\ &= 2\sum_{i=1}^{m} \left| \int\limits_{n+m^{-1}(i-1)}^{n+m^{-1}i} f(x)e_{nmi}(x)\,dx \right| \\ &\leq 2\sum_{i=1}^{m} \int\limits_{n+m^{-1}(i-1)}^{n+m^{-1}i} |f(x)|\,dx \cdot \sup_{x\in\mathbf{R}} \{e_{nmi}(x)\} \leq 2. \end{aligned}$$

Hence, $\|Q_{nm}\| \leq 2$ in the space $L\big((E,p_k)\big)$ of continuous linear mappings on (E,p_k), if $k \leq n$.

Assume now that $k > n$. It is clear that if $f \in E$, $p_k(f) \leq 1$, then

$$p_k(Q_{nmi}f) \leq \frac{\int\limits_{-\infty}^{\infty} e_{nmi}(x)\,dx}{\int\limits_{-\infty}^{\infty} \left(e_{nmi}(x)\right)^2 dx}\, p_k(e_{nmi}) \leq 2.$$

Hence,

$$p_k(Q_{nm}f) \leq \max_{i=1,\ldots,m} \{p_k(Q_{nmi}f)\} \leq 2. \qquad \text{Q.E.D.}$$

We can now make the following

Remark 4.2. Let U_{1m} and $U_{\infty m}$ be the ℓ_1^m- and ℓ_∞^m-unit balls of $\mathbf{R}^m$ and let V be the unit ball of $C(0,1)$. If there exist increasing, unbounded positive sequences $(r_n)_{n=1}^\infty$, $(s_m)_{m=1}^\infty$ and $(m_n)_{n=1}^\infty$ such that

$$\Gamma(mU_{1m} \otimes V) \cap \Gamma(nU_{\infty m} \otimes V) \not\subset \Gamma\big((r_n m U_{1m} \cap s_m U_{\infty m}) \otimes V\big) \tag{4.1}$$

for all (m,n), where m and n run through some infinite sets A_m and A_n of positive integers and $m > m_n$, then the space $\big(C(\mathbf{R}) \cap L_1(\mathbf{R})\big)\hat{\otimes}_\pi C(0,1)$ does not have property (BB).

Proof. Let $W_k \subset E$ be the closed unit ball of p_k and let

$$B := \bigcap_{k=1}^{\infty} \Gamma(2^{k-1} W_k \otimes V)$$

be a bounded set in $E\hat{\otimes}_\pi F$. Suppose by antithesis that there is an increasing positive sequence (r'_k) such that

$$B \subset \overline{\Gamma\big((\cap_k r'_k W_k) \otimes V\big)}. \tag{4.2}$$

We choose an $n_0 \in A_n$ such that

$$r_{n_0} > 2r'_1 \tag{4.3}$$

and then an $m_0 \in A_m$, $m_0 > m_{n_0}$, such that

$$s_{m_0} > 2r'_{n_0+1}. \tag{4.4}$$

Using Lemma 4.1., the hypothesis, properties 5° and 6°, a consideration similar to that in the proof of Theorem 4.5, [T1], and (4.3) and (4.4) one can see that

$$\Big(\bigcap_k \Gamma(2^{k-1} W_k \otimes V)\Big) \cap (M_{n_0 m_0} \otimes V) \not\subset \overline{\Gamma\big((\cap_k r'_k W_k) \otimes V\big)} \cap (M_{n_0 m_0} \otimes V)$$

which contradicts (4.2). Q.E.D.

We shall still present an approach to the hypothesis of Remark 4.2 by specifying certain tensors in the spaces involved.

We denote the canonical basis of $\mathbf{R}^m$ by $(e_i)_{i=1}^m$ and the ℓ_p^m-norm of $\mathbf{R}^m$ by $\|\cdot\|_p$. We choose the numbers $q_{nm} < 2$, where $n, m \in \mathbf{N}$ and $m > n$ such that

$$m^{(2-q_{mn})/2q_{mn}} = n.$$

Then we imbed the spaces $\ell^m_{q_{nm}}$ isometrically in $C(0,1)$. In this way we can consider the tensors

$$z_{nm} := \sum_{i=1}^{m} e_i \otimes e_i \in \mathbf{R}^m \otimes \ell^m_{q_{nm}}$$

as elements of $\mathbf{R}^m \otimes C(0,1)$ for all $n, m \in \mathbf{N}$, $m > n$.

By [P], 22.4.6., we get:

1°. $(\|\cdot\|_1 \otimes \|\cdot\|_{q_{nm}})(z_{nm}) = m$.

2°. $(\|\cdot\|_\infty \otimes \|\cdot\|_{q_{nm}})(z_{nm}) = m^{1/2}n$.

3°. The tensor z_{nm} defines an element of the space $L(\ell^m_1, C(0,1))$. We factorize z_{nm} as follows:

$$\ell^m_1 \xrightarrow{\tilde{z}_{nm}} \ell^m_{q_{nm}} \xrightarrow{\varphi_1} \ell^m_2 \xrightarrow{\varphi_2} \ell^m_{q_{nm}} \xrightarrow{I} C(0,1),$$

where $\tilde{z}_{nm}$ is defined by z_{nm}, φ_1 and φ_2 are the natural, algebraic identity mappings and I is the embedding. It is immediate that $\|\varphi_1\tilde{z}_{nm}\| = 1$ and $\|I\varphi_2\| = m^{(q_{nm}-2)/2q_{nm}}$. Grothendieck's theorem (see [Pi], Theorem 5.19) now implies that

$$(\|\cdot\|_\infty \otimes \|\cdot\|_C)(z_{nm}) \leq Gn$$

where G is a universal constant and $\|\cdot\|_C$ is the norm of $C(0,1)$.

We see from 1° and 3° that the tensors $G^{-1}z_{nm}$ belong to the left-hand side of (4.1), but the problem is to get a suitable lower bound to the norm connected with the right-hand side of (4.1). However, the following intuitive remark makes the previous approach interesting: A typical "good" representation of z_{nm} with respect to $\|\cdot\|_1 \otimes \|\cdot\|_C$ (i.e., a representation $z_{nm} = \sum a_i \otimes b_i$ for which

$$\sum_i \|a_i\|_1\|b_i\|_C \cong (\|\cdot\|_1 \otimes \|\cdot\|_C)(z_{nm})$$

is $z_{nm} = \sum_{i=1}^m e_i \otimes e_i$, since

$$\sum_{i=1}^{m} \|e_i\|_1\|e_i\|_C = \sum_{i=1}^{m} \|e_i\|_1\|e_i\|_{q_{nm}} = m.$$

On the other hand, z_{nm} cannot have a "good" representation for $\|\cdot\|_\infty \otimes \|\cdot\|_C$ in $\mathbf{R}^m \otimes \ell^m_{q_{nm}}$ because of 2° and 3°. In order to show that z_{nm} belongs to the right-hand side of (4.1) one should find a representation which is "good" enough for $\|\cdot\|_1 \otimes \|\cdot\|_C$ and $\|\cdot\|_\infty \otimes \|\cdot\|_C$ simultaneously. The representations mentioned above are very different, but we do not know the answer for all representations.

It is clear that this approach to "probléme des topologies" in the case of $(C(\mathbf{R}) \cap L_1(\mathbf{R}))\hat{\otimes}_\pi C(0,1)$ is not the only possible. However, it really seems to the author that the methods of [T3] cannot be applied here to get a positive answer.

Other interesting spaces from this point of view would be $(C(\mathbf{R}) \cap L_1(\mathbf{R}))$ $\hat{\otimes}_\pi X$, where X is an arbitrary $\mathcal{L}_\infty$-space, the tensor product of $C(\mathbf{R}) \cap L_1(\mathbf{R})$ with itself, $(C(\mathbf{R}) \cap L_p(\mathbf{R}))\hat{\otimes}_\pi L_q$, where $1 \leq p < \infty$ and $1/p + 1/q = 1$ etc.

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