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ON THE JULIA SET OF THE POLYNOMIAL $f(z) = pz + z^m$ WITH p REAL

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1. Introduction

Let $f(z)$ be a rational function of a complex variable z with $\deg(f) \geq 2$ and

$$f^0(z) = z, \quad f^{n+1}(z) = f\big(f^n(z)\big), \qquad n = 0, 1, 2, \ldots.$$

According to Blanchard [3], the Julia set $J(f)$ for $f(z)$ is the set of those points $z \in \overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ where $\{f^n(z)\}$ is not a normal family in the sense of Montel. The following general properties are classical [5]:

1. $J(f)$ is a nonempty perfect set.
2. $J(f^n) = J(f)$ for any integer $n \geq 1$.
3. $J(f)$ is completely invariant under the mapping $z \mapsto f(z)$, i.e., $f\big(J(f)\big) = f^{-1}\big(J(f)\big) = J(f)$.

Certainly, the structure of $J(f)$ depends on the function $f(z)$. If $f(z)$ is a polynomial, $J(f)$ depends on the coefficients of $f(z)$ in a very complicated manner. Myrberg [6–10], Brolin [4], and Bhattacharyya and Arumaraj [1–2] have considered the cases where $f(z)$ is a polynomial of $\deg(f) = 2, 3$ and 4 with real coefficients.

In this note we investigate the structure of $J(f)$ where $f(z) = pz + z^m$ with p real.

We need the following definitions.

Definition 1. If the equation $f^n(z) - c = 0$ has a multiple root, then c is called a critical point of the inverse function $f^{-n}(z)$.

Definition 2. $\alpha \in \overline{\mathbf{C}}$ is a fixpoint of $f(z)$ if $f(\alpha) = \alpha$, and α is an attractive fixpoint if $|f'(\alpha)| < 1$. The immediate attractive set $A^*(\alpha)$ of an attractive fixpoint α is the maximal domain of normality of $\{f^n(z)\}$ which contains α. The attractive set $A(\alpha)$ of α is defined by

$$A(\alpha) = \big\{z \mid \lim_{n\to\infty} f^n(z) = \alpha\big\}.$$

Definition 3. Two polynomials $f(z)$ and $g(z)$ are conjugate if there exist constants $a, b \in \mathbf{C}$ such that $f(az + b) = ag(z) + b$.

Clearly, $z_0 \in A(\alpha)$ if and only if $\lim_{n\to\infty} g^n(az_0 + b) = a\alpha + b$ (i.e., $az_0 + b \in A'(a\alpha + b)$ if $A'(\beta)$ denotes the attractive set of β for $g(z)$).

We shall refer the reader to [3] and [4] for the results needed for our proofs.

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2. Results and their proofs

Let $f(z) = pz + z^m$, where p is real, $m \geq 2$ being a positive integer. Then $f(z)$ has m finite fixpoints

$$q_1 = 0,\ q_2 = (1-p)^{1/(m-1)},\ q_3 = q_2\omega,\ q_4 = q_2\omega^2,\ \dots, q_m = q_2\omega^{m-2}$$

and $f^{-1}(z)$ has $m-1$ finite critical points

$$c_1 = \frac{(m-1)p}{m}\Big(-\frac{p}{m}\Big)^{1/(m-1)},\ c_2 = c_1\omega,\ c_3 = c_1\omega^2,\ \dots, c_{m-1} = c_1\omega^{m-2},$$

where ω is one of the complex $(m-1)$th roots of unity.

From now on k, p means a positive integer and a real number, respectively.

Lemma 1. *If $|p| < 1$, then $c_i \in A^*(0)$, $i = 1, 2, \dots, m-1$.*

This can be proved in the same way as Lemma 1 of [2].

Lemma 2. *The polynomial equation*

$$g(t) = t^{4k-1}(t^{2k-1} - 2k)^{2k-1} + (2k-1)^{2k-1}(2kt - 2k + 1) = 0 \tag{1}$$

has only one negative real root.

Proof. Since $g(0) = -(2k-1)^{2k} < 0$ and $g(-\infty) = +\infty$, there is at least one negative real root. If there are more than one negative roots, $g''(t) = 0$ must have at least one negative root. But

$$g''(t) = 4k^2(2k-1)t^{4k-3}(t^{2k-1} - 1)(t^{2k-1} - 2k)^{2k-3}((2k+1)t^{2k-1} - 8k + 2) > 0$$

for $t < 0$. Thus the lemma is proved.

Lemma 3. 1) *The polynomial equation*

$$h_1(t) = t^{2k} + 2kt - (2k-1) = 0 \tag{2}$$

has only one negative real root. Furthermore, if we denote the only negative real root of (1), (2) by $-\theta_0$, $-\theta_1$ ($\theta_0 > 0$, $\theta_1 > 0$), respectively, then $\theta_1 > \theta_0$; thus $h_1(t) \leq 0$ for $-\theta_0 \leq t \leq 0$.

2) *The polynomial equation*

$$h_2(t) = t^{2k+1} - (2k+1)t + 2k = 0$$

has only one negative real root. Furthermore, if we denote this root by $-\theta_2$ ($\theta_2 > 0$), then $h_2(t) < 0$ if and only if $t < -\theta_2$.

3) *θ_2 is the only positive real root of the polynomial equation*

$$h_3(t) = t^{2k+1} - (2k+1)t - 2k = 0.$$

Furthermore, $h_3(t) > 0$ if and only if $t > \theta_2$.

Proof. 1) As in the proof of Lemma 2, by $h_1(0) = -(2k-1) < 0$ and $h_1(-\infty) = +\infty$ and $h''(t) = 2k(2k-1)t^{2k-1} > 0$ for $t < 0$, (2) has only one negative real root. Furthermore, note that $\theta_1 > 1$ (since $h_1(-1) = -4k+2 < 0$), and we have

$$\begin{aligned} g(-\theta_1) &= \theta_1^{4k-1}(\theta_1^{2k-1} + 2k)^{2k-1} - (2k-1)^{2k-1}(2k\theta_1 + 2k - 1) \\ &= \theta_1^{2k}\big(h_1(-\theta_1) + 4k\theta_1 + 2k - 1\big)^{2k-1} - (2k-1)^{2k-1}(2k\theta_1 + 2k - 1) \\ &> (4k\theta_1 + 2k - 1)^{2k-1} - (2k-1)^{2k-1}(2k\theta_1 + 2k - 1) \\ &> (2k-1)(2k)^{2k-2}(2k\theta_1 + 2k - 1) - (2k-1)^{2k-1}(2k\theta_1 + 2k - 1) > 0. \end{aligned}$$

Hence $-\theta_1 < -\theta_0$, that is, $\theta_1 > \theta_0$.

2) Since $h_2(-1) = 4k > 0$ and $h_2(-\infty) = -\infty$ and $h'(t) = (2k+1)(t^{2k}-1) > 0$ for $t < -1$, there is only one root $-\theta_2 \in (-\infty, -1)$ $(\theta_2 > 0)$ and $h_2(t) < 0$ for $t < -\theta_2$. On the other hand, the minimum value of $h_2(t)$ on $[-\theta_2, +\infty)$ is $h_2(1) = 0$. So we have $h_2(t) \geq 0$ when $-\theta_2 \leq t \leq +\infty$. Hence the conclusion follows.

3) This proof is similar to the proof of 2).

Lemma 4. 1) *Let* $f(z) = pz + z^{2k+1}$, $-(2k+1)\theta_2/2k \leq p \leq 0$. *Then*

$$\big|f(x)\big| \leq q_2 = (1-p)^{1/2k} \qquad \text{if} \quad |x| \leq q_2.$$

2) *Let* $f(z) = pz - z^{2k+1}$, $0 \leq p \leq (2k+1)\theta_2/2k$. *Then*

$$\big|f(x)\big| \leq (1+p)^{1/2k} \qquad \text{if} \quad |x| \leq (1+p)^{1/2k}.$$

Proof. 1) If $0 \leq x \leq q_2$, then by $x^{2k} \leq 1 - p$,

$$f(x) - q_2 = x(x^{2k} - 1 + p) + x - q_2 \leq 0.$$

So we have $f(x) \leq q_2$. On the other hand, by $-(2k+1)\theta_2/(2k) \leq p \leq 0$ and 2) of Lemma 3, $f\big((-p/(2k+1))^{1/2k}\big) \geq -q_2$. But $f\big((-p/(2k+1))^{1/2k}\big)$ is the minimum value of $f(x)$ on $[0, +\infty)$. Hence $f(x) \geq -q_2$ for $0 \leq x \leq q_2$. So we obtain $\big|f(x)\big| \leq q_2$.

If $-q_2 \leq x \leq 0$, then

$$f(x) + q_2 = x(x^{2k} - 1 + p) + x + q_2 \geq 0.$$

So we have $f(x) \geq -q_2$. On the other hand, analogously, $f\big(-(-p/(2k+1))^{1/2k}\big) \leq q_2$. But $f\big(-(-p/(2k+1))^{1/2k}\big)$ is the maximum value of $f(x)$ on $(-\infty, 0]$. Hence $f(x) \leq q_2$ for $-q_2 \leq x \leq 0$. We also obtain $\big|f(x)\big| \leq q_2$.

2) By 3) of Lemma 3, we can prove this in the same way as 1).

Lemma 5. *Let* $f(z) = pz + z^{2k}$. *If* $-2k\theta_0/(2k-1) \le p \le 0$, *then*

$$|f(c_1)| \le q_2 = (1-p)^{1/(2k-1)} \tag{3}$$

and

$$|f^2(x)| \le q_2 \qquad \text{if} \quad |f(x)| \le q_2. \tag{4}$$

Proof. We first consider (3). Now

$$c_1 = -\frac{(2k-1)p}{2k}\left(\frac{p}{2k}\right)^{1/(2k-1)}.$$

So, by $p \le 0$, we have $f(c_1) \ge 0$. On the other hand, by Lemma 2, $g(t) \le 0$ for $-\theta_0 \le t \le 0$. Hence $g\big((2k-1)p/2k\big) \le 0$ if $-2k\theta_0/(2k-1) \le p \le 0$, i.e. $f(c_1) \le q_2$ if $-2k\theta_0/(2k-1) \le p \le 0$. Thus (3) is proved.

Next we prove (4). If $0 \le f(x) \le q_2$, then

$$f^2(x) - q_2 = f(x)\big(f(x)^{2k-1} - 1 + p\big) + f(x) - q_2 \le 0,$$

and we have $f^2(x) \le q_2$. On the other hand, by $-2k\theta_0/(2k-1) \le p \le 0$ and 1) of Lemma 3,

$$f\left(-\left(\frac{p}{2k}\right)^{1/(2k-1)}\right) \ge q_2.$$

But

$$f\left(-\left(\frac{p}{2k}\right)^{1/(2k-1)}\right)$$

is the minimum value of $f(x)$ on $[0, q_2]$. Hence $f^2(x) \ge -q_2$. So we obtain $|f^2(x)| \le q_2$.

If $-q_2 \le f(x) \le 0$, then

$$f^2(x) + q_2 = pf(x) + f(x)^{2k} + q_2 \ge 0,$$

and we have $f^2(x) \ge -q_2$. On the other hand, c_1 is the minimum value of $f(x)$; thus $f(x) \ge c_1$ for any $x \in (-\infty, +\infty)$. Hence, by $-q_2 < c_1 < 0$ and (3) and $f'(x) < 0$ when $-q_2 \le x \le 0$, we have

$$f^2(x) \le f(c_1) \le q_2.$$

We also obtain $|f^2(x)| \le q_2$. Thus Lemma 5 is proved.

Lemma 6. *Let* $f(z) = pz + z^{2k+1}$. *Then* $c_i \notin A(\infty)$ $(i = 1, 2, \ldots, 2k)$ *if and only if* $|p| \le (2k+1)\theta_2/2k$.

Proof. Now

$$c_1 = \frac{2kp}{2k+1}\left(-\frac{p}{2k+1}\right)^{1/(2k)}.$$

Suppose first that $p \le 0$. Then $|f(x)| > |x|$ for $|x| > q_2 = (1-p)^{1/2k}$. Thus $x \in A(\infty)$ for $|x| > q_2$. By 2) of Lemma 3, $|c_1| > q_2$ when $p < -(2k+1)\theta_2/2k$. So we have $c_1 \in A(\infty)$. Since

$$|f^n(c_1)| = |f^n(c_2)| = \cdots = |f^n(c_{2k})|$$

for any positive integer n, it follows that $c_i \in A(\infty)$ $(i = 1, 2, \ldots, 2k)$ when $p < -(2k+1)\theta_2/2k$. If $-(2k+1)\theta_2/2k \le p \le 0$, then $|c_1| \le q_2$. By 1) of Lemma 4, $|f^n(c_1)| \le q_2$ for any positive integer n. Thus $c_i \notin A(\infty)$ $(i = 1, 2, \ldots, 2k)$ when $-(2k+1)\theta_2/2k \le p \le 0$.

Suppose now that $p > 0$. Since $f(z)$ and $f_*(z) = pz - z^{2k+1}$ are conjugate (since $af(z) = f_*(az)$ where $a = \exp(-\pi i/2k)$), and

$$c_1 = \frac{2kp}{2k+1}\left(\frac{p}{2k+1}\right)^{1/(2k)} \exp\left(\frac{\pi i}{2k}\right),$$

we consider the behavior of $c_i^* = \exp(-\pi i/2k) \cdot c_i$ $(i = 1, 2, \ldots, 2k)$ under the iterates of $f_*(z) = pz - z^{2k+1}$. Since $|f_*(x)| > |x|$ for $|x| > (1+p)^{1/2k}$, $f_*^n(x) \to \infty$ as $n \to +\infty$ for $|x| > (1+p)^{1/2k}$. But, by (3) of Lemma 3,

$$|c_1^*| = \frac{2kp}{2k+1}\left(\frac{p}{2k+1}\right)^{1/(2k)} > (1+p)^{1/2k}$$

when $p > (2k+1)\theta_2/2k$. So we have $f_*^n(c_1^*) \to \infty$ $(n \to +\infty)$ and $f_*^n(c_i^*) \to \infty$ $(n \to +\infty)$ $(i = 1, 2, \ldots, 2k)$. Hence $c_i \in A(\infty)$ $(i = 1, 2, \ldots, 2k)$ when $p > (2k+1)\theta_2/2k$. If $0 \le p \le (2k+1)\theta_2/2k$, then $|c_1^*| \le (1+p)^{1/2k}$. By (2) of Lemma 4, $|f_*^n(c_1^*)| \le (1+p)^{1/2k}$ for any positive integer n. Thus $f_*^n(c_i^*) \not\to \infty$ $(n \to +\infty)$ $(i = 1, 2, \ldots, 2k)$. Hence $c_i \notin A(\infty)$ when $0 \le p \le (2k+1)\theta_2/2k$. Thus the proof of Lemma 6 is completed.

Lemma 7. *Let $f(z) = pz + z^{2k}$. Then $c_i \notin A(\infty)$ $(i = 1, 2, \ldots, 2k-1)$ if and only if $-2k\theta_0/(2k-1) \le p \le (2k/(2k-1))(2k)^{1/(2k-1)}$.*

Proof. By Lemma 2 and Lemma 5, we can prove this lemma in the same way as Lemma 3 of [2].

Theorem 1. *Let $f(z) = pz + z^{2k+1}$, where p is real and k is a positive integer. If $|p| \le (2k+1)\theta_2/2k$, then $J(f)$ is connected. Otherwise $J(f)$ is totally disconnected, $m_2J = 0$ (where m_2J denotes the planar measure of $J(f)$) and $j|_J$ is isomorphic to the one-sided shift on $2k+1$ symbols* (cf. [3, pp. 124]).

Theorem 2. *Let $f(z) = pz + z^{2k}$, where p is real and k is a positive integer. If $-2k\theta_0/(2k-1) \leq p \leq \big(2k/(2k-1)\big)(2k)^{1/(2k-1)}$, then $J(f)$ is connected. Otherwise $J(f)$ is totally disconnected, $m_2 J = 0$ and $f|_J$ is isomorphic to the one-sided shift on $2k$ symbols.*

By Lemmas 6 and 7, Theorems 1 and 2 immediately follow from Theorems 11.2 and 11.4 of [4] and Theorem 9.9 of [3].

Theorem 3. *Let $f(z) = pz + z^m$, where p is real and $m \geq 3$ is a positive integer. Then*

1) *$J(f)$ is a Jordan curve if and only if $|p| < 1$.*

2) *$J(f) \subset \{z \mid |z| \leq (1+|p|)^{1/(m-1)}\}$.*

Proof. 1) As in [2], sufficiency immediately follows from Lemma 1 and Theorem 11.3 of [4].

Considering the rays $z = r \cdot \exp \alpha_s$, where $\alpha_s = (2s\pi i)/(m-1)$, $0 < r < +\infty$ $(s = 0, 1, 2, \ldots, m-2)$ when $p \geq 1$, and the rays $z = r \cdot \exp \beta_s$, where $\beta_s = \big((2s+1)\pi i\big)/(m-1)$, $0 < r < +\infty$ $(s = 0, 1, 2, \ldots, m-2)$ when $p \leq -1$, we can prove the necessity in the same way as ii) of Theorem 1 of [2].

2) This proof is similar to the proof of iii) of Theorem 1 of [2].

Since $f(z)$ and $P(z) = z^2 - \frac{1}{4}p(p-2)$ are conjugate when $k = 1$ in Theorem 2, we have the following known result (cf. [4, Theorem 12.1]).

Corollary. *Let $P(z) = z^2 - r$, where r is real. If $-\frac{1}{4} \leq r \leq 2$, then $J(P)$ is connected. Otherwise $J(P)$ is totally disconnected and $m_2 J = 0$.*

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