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A NOTE ABOUT AN AHLFORS INEQUALITY AND INNER RADIUS OF UNIVALENCE

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1. Introduction and main results

Let $f(z)$ be a holomorphic function defined on unit disc $U = \{z : |z| < 1\}$ and $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$ be its Schwarzian derivative. In 1973, Ahlfors [1] showed that the inequality

$$(1) \qquad \left|\tfrac{1}{2}S_f + v^2 - v_z\right| \leq k|v_{\bar z}| \qquad (0 < k < 1)$$

together with $v \to \infty$ for $|z| \to 1$ and $v_{\bar z/v^2} \neq 0$ is sufficient to imply the existence of a quasiconformal extension of $f(z)$.

Writing $\frac{1}{2}v$ instead of v, (1) becomes

$$(2) \qquad \left|S_f - (v_z - \tfrac{1}{2}v^2)\right| \leq k|v_{\bar z}| \qquad (0 < k < 1).$$

If $f(z)$ is defined on the upper half plane $H = \{z : \operatorname{Im}(z) > 0\}$, it is easy to see that (2) together with $v \to \infty$ for $\operatorname{Im}(z) \to 0$ and $v_{\bar z}/v^2 \neq 0$ also is a sufficient condition for quasiconformal extension of $f(z)$.

Let A be any simply connected domain of hyperbolic type in $\overline{\mathbf{C}}$. We define the *Poincaré density* ρ_A of A by

$$\rho_A = \frac{|h'(z)|}{1 - |h(z)|^2},$$

where $h(z)$ is any conformal mapping of A onto the unit disc U. For complex-valued functions ϕ on A we set the norm

$$\|\phi\|_A = \sup_{z \in A} \frac{|\phi(z)|}{\rho_A(z)^2}.$$

Let $F(z)$ be any meromorphic function on A. Lehto [2] has defined the *inner radius of univalence* $\sigma_I(A)$ as the supremum of the numbers $a \geq 0$ with the property that $F(z)$ is injective whenever $\|S_F\|_A \leq a$.

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Let $g(z)$ be defined in U (or in H) and $A = g(U)$ (or $g(H)$) be a quasidisc. Let $\sigma_I(A)$ be the inner radius of univalence for A. Assume that $f(z)$ is any meromorphic function on U (or in H). It is clear that the inner radius of univalence $\sigma_I(A)$ is also the supremum of the numbers $a \geq 0$ with the property that $f(z)$ is injective whenever $\|S_f - S_g\|_U \leq a$ (or $\|S_f - S_g\|_H \leq a$).

In this note we want to show that the Ahlfors inequality is a very powerful tool for investigating $\sigma_I(A)$. Some special choices of v can yield valuable lower bounds for $\sigma_I(A)$ including some well-known results. In fact we obtain the following results:

Theorem 1. *Let $g(z)$ be holomorphic in U and $A = g(U)$. Then*

$$\sigma_I(A) \geq 2 - 2 \sup_{|z|<1} \left| z(1-|z|^2)\left(\frac{g''}{g'} - \frac{2g'}{g+c}\right)\right| \tag{3}$$

and

$$\sigma_I(A) \geq 2 \inf_{|z|<1} \left|\frac{zg'}{g}\right| \left(1 - \left|\frac{zg'}{g} - 1\right|\right), \tag{4}$$

where c is any complex number.

Theorem 2. *Let $g(z)$ be holomorphic in H and $A = g(H)$. Then*

$$\sigma_I(A) \geq 2 - 4 \sup_{\mathrm{Im}(z)>0} \left| y\left(\frac{g''}{g'} - \frac{2g'}{g+c}\right)\right| \tag{5}$$

and

$$\sigma_I(A) \geq 2 \inf_{\mathrm{Im}(z)>0} \left|\frac{zg'}{g}\right| \left(1 - \left|\frac{zg'}{g} - 1\right|\right), \tag{6}$$

where c is any complex number.

2. Proofs of theorems

Because the proofs are routine, we omit the details.

(i) In the case of $A = g(U)$, for any complex number c, choose

$$v = \frac{g''}{g'} - \frac{2g'}{g+c} + \frac{2\bar{z}}{1-|z|^2}.$$

Then (2) becomes

$$\left| S_f - S_g + \frac{2\bar{z}}{1-|z|^2}\left(\frac{g''}{g'} - \frac{2g'}{g+c}\right)\right| \leq \frac{2k}{(1-|z|^2)^2}.$$

So

$$\sigma_I(A) \geq 2 - 2 \sup_{|z|<1} \left| z\left(1-|z|^2\right) \left(\frac{g''}{g'} - \frac{2g'}{g+c} \right) \right| . \tag{7}$$

Let $c = \infty$. We have

$$\left| S_f - S_g + \frac{2\bar{z}}{1-|z|^2} \left(\frac{g''}{g'} \right) \right| \leq \frac{2k}{\left(1-|z|^2\right)^2}, \tag{8}$$

and

$$\sigma_I(A) \geq 2 - 2 \sup_{|z|<1} \left| z\left(1-|z|^2\right) \frac{g''}{g'} \right| . \tag{9}$$

The inequality (8) was first obtained by Epstein under some additional assumptions and was proved by Pommerenke later [4].

Now choose

$$v = \frac{g''}{g'} - \frac{2g'}{g\left(1-|z|^{-2}\right)}.$$

Then (2) becomes

$$\left| S_f - S_g - \frac{2\bar{z}g'(zg'-g)}{g^2\left(1-|z|^2\right)^2} \right| \leq \left| \frac{2kzg'}{g\left(1-|z|^2\right)^2} \right| .$$

Thus

$$\sigma_I(A) \geq 2 \inf_{|z|<1} \left| \frac{zg'}{g} \right| \left(1 - \left| \frac{zg'}{g} - 1 \right| \right) . \tag{10}$$

(ii) In the case of $A = g(H)$, choose

$$v = \frac{g''}{g'} - \frac{2g'}{g+c} - \frac{2}{z-\bar{z}}.$$

Then (2) becomes

$$\left| S_f - S_g + \frac{2}{z-\bar{z}} \left(\frac{g''}{g'} - \frac{2g'}{g+c} \right) \right| \leq \frac{2k}{|z-\bar{z}|^2}.$$

We get

$$\sigma_I(A) \geq 2 - 4 \sup_{\mathrm{Im}(z)>0} \left| y \left(\frac{g''}{g'} - \frac{2g'}{g+c} \right) \right| \tag{11}$$

and in the special case for $c = \infty$

$$\sigma_I(A) \geq 2 - 4 \sup_{\mathrm{Im}(z)>0} \left| y \left(\frac{g''}{g'} \right) \right|. \tag{12}$$

If we choose

$$v = \frac{g''}{g'} - \frac{2g'}{g(1 - \bar{z}/z)}.$$

Then (2) becomes

$$\left| S_f - S_g - \frac{2\bar{z}g'(zg' - g)}{g^2(z - \bar{z})^2} \right| \leq \left| \frac{2kzg'}{g(z - \bar{z})^2} \right|.$$

Thus

$$\sigma_I(A) \geq 2 \inf_{\mathrm{Im}(z)>0} \left| \frac{zg'}{g} \right| \left(1 - \left| \frac{zg'}{g} - 1 \right| \right). \tag{13}$$

This inequality makes sense only for $\left|(zg'/g) - 1\right| < 1$. If we take

$$g(z) = z^k = \exp(k \log z) \qquad (z \in H,\ |k - 1| < 1,\ \log i = \tfrac{1}{2}\pi i),$$

$A = g(H)$ is a spiral-like domain for non-real k. Because $zg'/g = k$, we have

$$\sigma_I(A) \geq 2|k|\bigl(1 - |k - 1|\bigr). \tag{14}$$

When k is real, Lehtinen and Lehto obtained [3]

$$\sigma_I(A) = 2k\bigl(1 - |k - 1|\bigr). \tag{15}$$

We do not know whether (14) is sharp for non-real k.

3. A general formula

Generally, let $h(z)$ be any quasiconformal self-mapping of the whole plane. Define $\tau(z) = h(\bar{z})/h(z)$ for $A = g(H)$ and $\tau(z) = h(1/\bar{z})/h(z)$ for $A = g(U)$. We choose

$$v = \frac{g''}{g'} - \frac{2g'}{g(1 - \tau)}.$$

Then

$$\sigma_I(A) \geq 2 \inf \frac{|g'|\bigl(|\bar{\partial}(g\tau)| - |\partial(g\tau)|\bigr)}{|g - g\tau|^2 \eta^2}, \tag{16}$$

where $\eta = 1/2y$ or $1/(1 - |z|^2)$.

Let $h(z)$ be any quasiconformal extension of $g(z)$, denote $g^* = g(\bar{z})$ or $g(1/\bar{z})$, then

$$\sigma_I(A) \geq 2 \inf \frac{|g'|\bigl(|\bar{\partial} g^*| - |\partial g^*|\bigr)}{|g - g^*|^2 \eta^2}. \tag{17}$$

This is just another form of Lehto's result [3, p. 121].

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