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# ON SOME THEOREMS OF LITTLEWOOD AND SELBERG II

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## 1. Introduction

In a previous paper with the same title [1] we proved some theorems about the Riemann zeta-function under the assumption of Riemann hypothesis. In this paper we prove some unconditional results on $\zeta(s)$. Stating somewhat more generally we prove the following.

**Theorem.** *Let* $s = \sigma + it$ *and*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1} \qquad (\sigma > 1), \tag{1}$$

*where* $p$ *runs through all primes and* $\omega(p)$ *are some complex numbers (independent of* $s$*) with absolute value not exceeding* $1$*. Suppose* $\alpha$ *and* $\delta$ *are positive constants satisfying* $\frac{1}{2} \le \alpha \le 1 - \delta$ *and that in* $\{\sigma \ge \alpha - \delta, T - H \le t \le T + H\} F(s)$ *can be continued analytically and there* $|F(s)| < T^A$*. Here* $A$ *is a positive constant,* $T \ge T_0, H = C \log\log\log T$ *where* $T_0$ *and* $C$ *are large positive constants. Let* $F(s) \ne 0$ *in* $\{\sigma > \alpha, T - H \le t \le T + H\}$*. Then for* $\alpha + \delta \le \sigma \le 1 - \delta, t = T$*, we have*

$$\frac{F'(s)}{F(s)} = O\big((\log T)^{\Theta}\big) \tag{2}$$

*and*

$$\log F(s) = O\big((\log T)^{\Theta} (\log\log T)^{-1}\big), \tag{3}$$

*where* $\Theta = (1 - \sigma)/(1 - \alpha)$*.*

**Remark 1.** The application to $\zeta(s)$ is immediate by density results. By standard methods we can also prove density results for $F(s)$ provided in, say $\{\sigma \ge 3/4,\ t \ge T_0\}$ $F(s)$ can be continued analytically and there $|F(s)| < t^A$.

**Remark 2.** The theorem can be generalised further by allowing some growth condition for $\omega(p)$. We can state our theorem in a slightly different way to allow $F(s) = L(s, \chi)$ for characters $\chi (\mathrm{mod}\, q)$, for example for $|t| \le q$.

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**Remark 3.** We can state a result for $\alpha+\delta \leq \sigma \leq 1+\delta$ analogous to the remark made by D.R. Heath-Brown in Section 14.33 of [3].

**Remark 4.** In a later paper with the same title we hope to obtain inequalities dealing with $\left|\arg F(\sigma+it)\right|$ for $\sigma \geq \alpha$ and $\log\left|F(\sigma+it)\right|$ for $\sigma > \alpha$, analogous to what we proved in [1].

**Remark 5.** The $t$-interval condition $T-H \leq t \leq T+H$ is made possible by the kernel function $\exp\left((\sin w)^2\right)$ used extensively by Ramachandra in his papers.

## 2. Notation

In Lemmas 1 and 2 we borrow results from [4] and [3] in the same notation. But in Lemma 2 we have changed the result contained in [3] to suit our needs (see Remark below Lemma 2). We use $z = x+iy$, $w = u+iv$ and $s = \sigma+it$ in various contexts and we hope this does not cause confusion. For any analytic function $F(s)$ we write $(F'/F)(s)$ for $F'(s)/F(s)$. The symbol $\equiv$ denotes a definition.

**Lemma 1.** *Let $f(z)$ be analytic in $|z| < R$. Suppose $f(0)$ is different from zero. For $0 \leq x < R$ let $n(x)$ denote the number of zeros of $f(z)$ in $|z| \leq x$. Then for $0 \leq r < R$ we have*

$$\int_0^r n(x)\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f\left(re^{i\theta}\right)}{f(0)}\right| d\theta. \tag{4}$$

**Remark.** This result is called Jensen's theorem. For its proof see pages 124 to 126 of [4].

**Lemma 2.** *If $f(s)$ is regular and*

$$\left|\frac{f(s)}{f(s_0)}\right| \leq e^M \qquad (M > 1) \tag{5}$$

*in $|s-s_0| \leq r$, then for any constant $\epsilon$ (with $0 < \epsilon < \frac{1}{2}$,) we have*

$$\left|\frac{f'}{f}(s) - \sum_{\rho}\frac{1}{s-\rho}\right| \ll_{\epsilon} \frac{M}{r} \qquad \text{in } |s-s_0| \leq (1-2\epsilon)r, \tag{6}$$

*where $\rho$ runs over all zeros of $f(s)$ such that*

$$|\rho - s_0| \leq (1-\epsilon)r. \tag{7}$$

**Remark.** From Lemma 1 above and the concluding remarks in [2] on (24), (25) and (26) of that paper we obtain Lemma 2 which is nearly contained as Lemma 2 of Section 3.9 of [3].

**Lemma 3.** *Let $z = x + iy$ be a complex variable and*

$$F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z} = \prod_p \left(1 - \frac{\omega(p)}{p^z}\right)^{-1} \qquad (x > 1), \tag{8}$$

*where $p$ runs over all primes and $\omega(p)$ are complex numbers independent of $z$ with $|w(p)| \leq 1$. Let $F(z)$ be regular in $\{x \geq \alpha - \delta, T - H \leq y \leq T + H\}$ and there $|F(z)| < T^A$. Here $T \geq T_0$, $\frac{1}{2} \leq \alpha \leq 1 - \delta$, $H = C \log\log\log T$ where $\delta$ is a small positive constant, $\alpha$ is a positive constant, and $T_0$ and $C$ are large positive constants. Put $z_0 = 2 + iy_0$ where $T - \frac{1}{2}H \leq y_0 \leq T + \frac{1}{2}H$. Then for $z_1 = x_0 + iy_0$ with $\alpha - \delta_1 \leq x_0 \leq 2$, we have,*

$$\left|\frac{F'}{F}(z_1) - \sum_{\rho \in D} \frac{1}{z_1 - \rho}\right| \ll \log T, \tag{9}$$

*where $\rho$ runs over all the zeros of $F(z)$ in the disc $D = D(z_0, 2 - \alpha + 2\delta_1)$ defined by*

$$|z - z_0| \leq 2 - \alpha + 2\delta_1. \tag{10}$$

*Here $\delta_1$ is any positive constant such that $2\delta_1 < \delta$. In particular the lemma holds for $z_1 = \alpha + iy_0$.*

**Remark.** The lemma is trivially true for $z_1 = x_0 + iy_0$ with $x_0 \geq 1 + \delta$.

## 4. Proof of the theorem

**Lemma 4.** *Let $s = \sigma + it$ where $\alpha + \delta_1 \leq \sigma \leq 1 - \delta_1$, $w = u + iv$, $2 \leq X \leq \exp\left(10(\log\log T)/(1 - \alpha)\right)$, $B \geq 10000$. Then*

$$I \equiv \frac{1}{2\pi i} \int_{u=2} \frac{F'}{F}(s + w) X^w \exp\left(\left(\sin \frac{w}{B}\right)^2\right) \frac{dw}{w} \tag{11}$$

$$= O(X^{1-\sigma}). \tag{12}$$

*Proof.* The proof follows from

$$\frac{1}{2\pi i} \int_{u=2} \left(\frac{X}{n}\right)^w \exp\left(\left(\sin \frac{w}{B}\right)^2\right) \frac{dw}{w} = 1 + O\left(\frac{n}{X}\right) \quad \text{or } O\left(\frac{X}{n}\right).$$

according as $n \leq X$ or $n > X$.

**Lemma 5.** *Let $3V \sim H$ and $|v| \leq V'$ ($V'$ will be chosen to be asymptotic to $V$). Then for (fixed $s = \sigma + it$ and all $w = u + iv$), $u + \sigma \geq \alpha - \delta_1$, we have,*

$$\left|\frac{F'}{F}(s+w) - \sum_{\rho} \frac{1}{s+w-\rho}\right| \ll \log T \tag{13}$$

*where $\rho$ runs over all the zeros of $F(z)$ in the disc $D = D(z_0, 2-\alpha+2\delta_1)$ defined by $|z - z_0| \leq 2 - \alpha + 2\delta_1$ where $z_0 = 2 + it + iv$.*

*Proof.* The proof follows from Lemma 3.

**Lemma 6.** *Let*

$$\mu(\rho) = \frac{2^{s+w-\rho} - 1}{(s+w-\rho)^2 \log 2} \tag{14}$$

*and*

$$\mu = \sum_{\rho} \mu(\rho), \tag{15}$$

*where $\rho$ runs over all the zeros of $F(z)$ in the rectangle $R$ defined by*

$$R : \left\{ \operatorname{Re} z \geq \alpha - 2\delta_1, |t - y| \leq 2V \right\}. \tag{16}$$

*Then for $|v| \leq V'$ ($V'$ will be chosen asymptotic to $V$) and $u + \sigma \geq \alpha - \delta_1$ we have,*

$$\left|\frac{F'}{F}(s+w) - \mu\right| \ll \log T. \tag{17}$$

*Proof.* For $D$ as in Lemma 5, we have

$$\sum_{\rho \epsilon D} \frac{1}{s+w-\rho} - \sum_{\rho \epsilon D} \mu(\rho) = O(\log T),$$

since (by Jensen's theorem) there are $O(\log T)$ zeros involved and for any fixed $\rho$

$$\left|\frac{1}{s+w-\rho} - \mu(\rho)\right| \ll 1$$

since it is so on $|s + w - \rho| = 10$. Again

$$\left|\sum_{\rho \notin D, \rho \epsilon R} \mu(\rho)\right| \ll \log T$$

since for $\rho \notin D$, we have

$$\begin{aligned} |s+w-\rho| &\geq |z_0 - \rho| - |s + w - z_0| \\ &\geq 2 - \alpha + 2\delta_1 - (2 - \alpha + \delta_1) = \delta_1 \end{aligned}$$

**Lemma 7.** *It is possible to choose $V' \sim V$ such that on $v = \pm V'$ and $u + \sigma \geq \alpha - 10\delta_1$ we have,*

$$\left|\sum_{\rho \epsilon R} \mu(\rho)\right| \ll (\log T)^2. \tag{18}$$

*Here $10\delta_1 < \delta$.*

*Proof.* By Jensen's theorem the number of zeros of $F(z)$ in $\{x \geq \alpha - 11\delta_1, Y \leq y \leq Y+1\}$ with $11\delta_1 < \delta$ is $O(\log T)$ provided $T-2V \leq Y \leq T+2V$. Hence the number of zeros of $F(s+w)$ in $\{u+\sigma \geq \alpha - 11\delta_1, V-1 \leq v \leq V+1\}$ is $O(\log T)$ and so there exists a line $v = V'$ such that on this line $|s+w-\rho| \gg 1/\log T$. This proves Lemma 7 since the number of zeros in $2 \geq |s+w-\rho| \gg 1/\log T$ is $O(\log T)$ and also the zeros $\rho$ with $|s+w-\rho| \geq 2$ contribute $O(\log T)$. The total contribution to $\mu$ is therefore $O\big((\log T)^2\big)$ and this proves Lemma 7 completely.

**Lemma 8.** *We have (if there are no zeros of $F(z)$ in $x > \alpha$ and $T - H \leq y \leq T + H$)*

$$(19)\quad I = \frac{F'}{F}(s) + \frac{1}{2\pi i}\int_{u=\alpha-\sigma}\Big\{\Big(\frac{F'}{F}(s+w)-\mu\Big)+\mu\Big\}X^w \exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\frac{dw}{w} + o(1)$$

*where the integration is restricted to $|v| \leq V'$ and we take the integral to mean the limit as we move from $u = 2$ to $u = \alpha - \sigma$.*

*Proof.* First, the contribution to $I$ of Lemma 4 from $|v| \geq V'$ is $o(1)$. The lemma now follows on moving the line of integration to $u = \alpha - \sigma$ since by Lemmas 6 and 7 the horizontal bits contribute $o(1)$. Note that $\exp\big((\sin w)^2\big)$ decays like $\big(\exp\exp\big(|v|/10\big)\big)^{-1}$ uniformly in $|u| \leq 1/10$.

**Lemma 9.** *We have,*

$$(20)\quad \int_{u=\alpha-\sigma, |v|\leq V'}\Big(\frac{F'}{F}(s+w)-\mu\Big)X^w\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\frac{dw}{w} = O(X^{\alpha-\sigma}\log T).$$

*Proof.* The proof follows by Lemma 6.

**Lemma 10.** *We have*

$$(21)\qquad J \equiv \frac{1}{2\pi i}\int_{u=\alpha-\sigma, |v|\leq V'}\mu X^w\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\frac{dw}{w} = O(X^{\alpha-\sigma}\log T).$$

*Proof.* Let as before $11\delta_1 < \delta$. We move the line of integration to $u = \alpha - \sigma - 10\delta_1$. We obtain

$$J = \sum_{\rho\epsilon R}\frac{X^{\rho-s}}{\rho-s}\exp\Big(\Big(\sin\frac{\rho-s}{B}\Big)^2\Big) + \frac{1}{2\pi i}\int_{u=\alpha-\sigma-10\delta_1}\mu X^w\exp\Big(\Big(\sin\frac{w}{B}\Big)^2\Big)\frac{dw}{w}$$

the last integration being subject to $|v| \leq V'$. Now since $\operatorname{Re}\rho \leq \alpha$ and $\sigma \geq \alpha + \delta$ and $\exp\big((\sin w)^2\big)$ tapers in $|u| \leq 1/10$ uniformly as fast as $\big(\exp\exp\big(|v|/10\big)\big)^{-1}$, the lemma follows.

**Lemma 11.** *We have, uniformly for* $\{\alpha + \delta \le \sigma \le 1 - \delta, t = T\}$

$$\frac{F'}{F}(s) = O\big((\log T)^{\Theta}\big) \tag{22}$$

*provided* $F(z) \neq 0$ *in* $\{x > \alpha, T - H \le y \le T + H\}$. *Here* $\Theta$ *is as stated in the theorem.*

*Proof.* The proof follows from Lemmas 4, 8, 9 and 10 on choosing $X$ by $X^{1-\alpha} = \log T$.

**Lemma 12.** *Subject to the conditions of Lemma 11,*

$$\log F(s) = O\Big((\log T)^{\Theta}(\log\log T)^{-1}\Big). \tag{23}$$

*Proof.* The proof follows by integrating (22) with respect to $\sigma$ from $\sigma$ to $\sigma' \equiv \frac{1}{2}(1+\sigma)$, since (as will be proved in the next lemma) $\log F(\sigma' + it) = O\big((\log T)^{\Theta-\epsilon}\big)$, for some fixed $\epsilon > 0$.

**Lemma 13.** *We have, with* $\sigma' = \frac{1}{2}(1+\sigma)$,

$$\log F(\sigma' + it) = O\big((\log T)^{\Theta-\epsilon}\big) \tag{24}$$

*for some fixed* $\epsilon = \epsilon(\sigma) > 0$.

*Proof.* By a simple application of the Borel–Carathéodory theorem we have $\log F(\sigma + \epsilon + it) = O(\log T)$ for $\{\sigma \ge \alpha, T - \frac{1}{2}H \le t \le T + \frac{1}{2}H\}$. Put $s' = \sigma' + it$. We proceed by considering (as in Lemma 4) the integral

$$\frac{1}{2\pi i}\int_{u=2} F(s'+w)X^{w}\exp\left(\left(\sin\frac{w}{B}\right)^{2}\right)\frac{dw}{w}$$

and moving the portion $|v| \le H/3$ of the line of integration to $u + \sigma' = \alpha + \epsilon$ i.e. $u = \alpha - \sigma' + \epsilon$. This leads to the Lemma. With Lemmas 11, 12 and 13 the theorem stated in the introduction is completely proved.

**Remark.** If $F(s) \neq 0$ in $\{\sigma > \alpha, T - C \le t \le T + C\}$ where $\frac{1}{2} \le \alpha \le 1 - \delta$ and here $|F(s)| < T^{A}$, $(T \ge 10)$, it follows by the proof of Theorem 14.2 of [3] that if $C = C(\delta, \epsilon, \sigma_0)$ then uniformly in $\alpha < \sigma_0 \le \sigma \le 1$ and $t = T$, we have $F(s) = O\big((\log T)^{\Theta+\epsilon}\big)$, where the $O$-constant depends only on $A$, $\sigma_0$, $\delta$ and $\epsilon$.

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