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THE COMPONENTS OF A JULIA SET

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Let R be a rational map of degree d of the complex sphere $\mathbf{C}_\infty$ onto itself, and let J and F be the Julia and Fatou sets of R respectively. We assume throughout that $d \geq 2$; then J is the smallest compact set E which contains at least three points, and which satisfies

$$R(E) = E = R^{-1}(E).$$

We call this property the *complete invariance* of E, and the fact that J is the smallest such set is referred to as the *minimality* of J. For details of the general theory, we refer the reader to [1], [2] and [3]. It is known that J is a perfect set (so J is uncountable, and no point of J is isolated), and also that if J is disconnected, then it has infinitely many components. The following result, which seems not to have been noticed before, contains both of these results (when J is disconnected) and more.

Theorem. *If J is disconnected, then it has uncountably many many components, and each point of J is an accumulation point of distinct components of J.*

In [4], McMullen gives an example in which J has a *buried* component (that is, a component of J which is not on the boundary of any component of F). If each component of F has finite connectivity, and if J is disconnected, then there are only countably many components of J which lie on the boundary of some component of F, and our Theorem immediately yields the following general result.

Corollary. *Suppose that J is disconnected, and that every component of F has finite connectivity. Then J has a buried component.*

The major part of the proof of the Theorem is contained in the following

Proposition. *Let K be a compact connected subset of $\mathbf{C}_\infty$. Then $R^{-1}(K)$ has at most d components, and each is mapped by R onto K.*

The proof of the Proposition is easier if we first discuss some preliminary results. The complement of a set A with respect to the plane $\mathbf{C}$ and the sphere $\mathbf{C}_\infty$ are denoted by $\mathbf{C} - A$ and $\mathbf{C}_\infty - A$ respectively. First, we quote

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Lemma 1 ([5], p. 144). *A compact subset W of $\mathbf{C}_\infty$ is connected if and only if every component of $\mathbf{C}_\infty - W$ is simply connected.*

Next, consider a bounded domain D in $\mathbf{C}$ which is bounded by a finite number of Jordan curves γ_j. The winding number of γ_j about any z not on γ_j is denoted by $n(\gamma_j, z)$, and if $z \notin \partial D$ we write

$$n(\partial D, z) = \sum_j n(\gamma_j, z).$$

Obviously,

$$D = \{z : z \notin \partial D, n(\partial D, z) \neq 0\}. \tag{1}$$

Finally, let A and B be disjoint, non-empty, compact subsets of $\mathbf{C}$. We put a rectangular grid on $\mathbf{C}$ which is fine enough so that no square in the grid meets both A and B, and we let $\{Q_j\}$ be the set of those (closed) squares that meet A. Now let Ω be the interior of $\cup Q_j$: then Ω is a bounded open set with a finite number of components Ω_j, each being bounded by a finite number of Jordan curves, and (1) holds with D replaced by Ω_j. Further,

$$A \subset \Omega, \qquad B \cap \Omega = \emptyset, \qquad \partial\Omega \cap (A \cup B) = \emptyset. \tag{2}$$

We now give the

Proof of the Proposition. Let $D = \mathbf{C}_\infty - K$, and let D_j be the components of D; Lemma 1 shows that each D_j is simply connected. Next, it is easy to see that each component of $R^{-1}(D_j)$ is mapped by R onto D_j, and because each D_j is simply connected, we see that any component of $R^{-1}(D_j)$ is either a simply connected domain, or it is a domain of finite connectivity which contains a critical point of R (for if such a component, say Δ, does not contain a critical point then, by the monodromy theorem, the map R of Δ onto D_j is a homeomorphism). As $R^{-1}(D)$ is the union of the $R^{-1}(D_j)$, it follows that $R^{-1}(D)$ is the union of a *finite* number of multiply (but finitely) connected domains, say $M_1, \ldots, M_t$, and a number (possibly infinite) of simply connected domains S_j.

When there are no multiply connected domains M_j present, all of the components of $R^{-1}(D)$ are simply connected and then Lemma 1 implies that the complement of $R^{-1}(D)$, namely $R^{-1}(K)$, is connected: thus the conclusion of the Proposition holds in this case.

We now assume that at least one domain M_j exists, and we consider the minimal, and necessarily finite, set of components $E_1, \ldots, E_q$ of $R^{-1}(K)$ such that

$$\bigcup \partial M_j \subset E_1 \cup \cdots \cup E_q. \tag{3}$$

Next, we show that $E_1, \ldots, E_q$ are all of the components of $R^{-1}(K)$. Suppose, then, that Q is another component of $R^{-1}(K)$ and write $E = E_1 \cup \cdots \cup E_q$: then E and Q are disjoint compact subsets of $R^{-1}(K)$, so from [5] (Theorem 5.6, p. 82), there are compact subsets A and B of $R^{-1}(K)$ such that

$$(4) \qquad A \cup B = R^{-1}(K), \qquad A \cap B = \emptyset, \qquad Q \subset A, \qquad E \subset B.$$

We may assume that $\infty \in R^{-1}(D)$; then A and B are disjoint, compact subsets of $\mathbf{C}$, so we can find an open set Ω (as described above) satisfying (2), and hence from (4), also

$$(5) \qquad \partial\Omega \subset R^{-1}(D).$$

Now let Ω_Q be the component of Ω that contains the connected set Q. Using (3) and (4), we find that for each r,

$$\partial M_r \subset E \subset B,$$

and so we see from (2) that Ω_Q and ∂M_r are disjoint. Now Ω_Q is arcwise connected, and this means that either $\Omega_Q \subset M_r$ or $\Omega_Q \cap M_r = \emptyset$. Now the first possibility cannot occur because if it does, then

$$Q \subset \Omega_Q \subset M_r \subset R^{-1}(D)$$

which violates the fact that $Q \subset R^{-1}(K)$; thus Ω_Q is disjoint from each M_r. As each M_r is open, we deduce that the closure of Ω_Q is disjoint from $\cup M_r$.

As a consequence of this, each boundary component γ_j (a Jordan curve) of Ω_Q lies in some simply connected domain S_m for, by (5), it lies in $R^{-1}(D)$; thus one side of γ_j lies in S_m, while the other side contains $R^{-1}(K)$ and each M_r. It follows that for any z_1 in M_r, and any z_2 in Q,

$$n(\gamma_j, z_1) = n(\gamma_j, z_2),$$

and hence that

$$n(\partial\Omega_Q, z_1) = n(\partial\Omega_Q, z_2) \neq 0.$$

This shows that z_1 is in Ω_Q, contrary to the fact that Ω_Q and M_r are disjoint. It follows that no such component Q exists, and so we have proved that

$$R^{-1}(K) = E_1 \cup \cdots \cup E_q.$$

As $R^{-1}(K)$ is compact, so is each E_j, and hence $R(E_j)$ also: thus $R(E_j)$ is a closed subset of K. We shall now show that each $R(E_j)$ is relatively open in K: then, as K is connected, we find that $R(E_j) = K$. Clearly, this implies that $q \leq d$ and the proof of the Proposition will then be complete.

To show that $R(E_j)$ is relatively open in K, we take any ζ in $R(E_j)$, say $\zeta = R(w)$, where $w \in E_j$. We find a neighbourhood N of w not meeting any other E_i (this is possible because $R^{-1}(K)$ has only finitely many components) and observe that

$$K \cap R(N) = R(E_j \cap N) \subset R(E_j).$$

This shows that $R(E_j)$ is relatively open in K, and the proof of the Proposition is complete.

We end with the

Proof of the Theorem. Let K be the set of points in J at which infinitely many components of J accumulate. Our first objective is to show that $J = K$ and to do this, we prove

(a) K is closed;

(b) K is completely invariant, and

(c) K has at least three points.

With these, the minimality of J shows that $J \subset K$, and hence that $K = J$.

Obviously, K is closed, so (a) holds. By assumption, J has infinitely many components so K is not empty, and with this, (b) implies (c) (for, from the general theory of iteration, any non-empty finite completely invariant set lies in F). We shall now show that (b) holds.

First, take ζ in K, so there is a sequence J_1, J_2, ... of distinct components of J which accumulate at ζ. Obviously, the components $R(J_n)$ accumulate at $R(\zeta)$, and from the Proposition we see that at most d of the J_n can map to any given component of J. We deduce that infinitely many components of J accumulate at $R(\zeta)$, and hence that $R(K) \subset K$.

Next, take any ζ in K and w such that $R(w) = \zeta$: then find neighbourhoods U of w and V of ζ such that for an appropriate k, R is a k-fold map of U onto V. Again, there is a sequence J_1, J_2, ... of distinct components of J which accumulate at ζ, and we may assume that all of these meet V. It follows that some component of each $R^{-1}(J_n)$ meets U, and these components must be distinct as the J_n are. As U and V can be chosen arbitrarily small, this shows that $R^{-1}(K) \subset K$, and hence that (b) holds. We have now shown that $J = K$ and so, in particular, no component of J is isolated.

It only remains to prove that J has uncountably many components. We argue by contradiction, so suppose that the components of J are J_1, J_2, ... Now J is a compact metric space, and J is the countable union of the J_n so, by Baire's category theorem not every J_n is nowhere dense in J. We may suppose that J_1 is not, so the closure of J_1 has a non-empty interior (all in the relative topology on J). But J_1 is a component of J, so it is closed in J. We deduce that J_1

has a non-empty interior in J, and as this violates the statement at the end of the previous paragraph, we can conclude that J must have uncountably many components.

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