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SENSE-REVERSING GENERATORS OF DISCRETE CONVERGENCE GROUPS IN THE PLANE

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Abstract. Suppose that g is a sense-reversing homeomorphism of the 2-sphere S^2 onto itself that generates a discrete convergence group of infinite order. We show that then g is topologically conjugate to $\bar{z}+1$ if g has one fixed point in S^2. Otherwise, g has two fixed points and is known to be conjugate to $2\bar{z}$.

1. Introduction

Following Gehring and Martin ([2, p. 335]), we say that a group G of homeomorphisms of the two-dimensional sphere S^2 onto itself is a *convergence group* if every sequence of elements of G contains a subsequence, say g_n, such that

(i) $g_n \to g$ and $g_n^{-1} \to g^{-1}$ uniformly on S^2, where g is a homeomorphism; or
(ii) there are $x_0, y_0 \in S^2$ (possibly $x_0 = y_0$) such that $g_n \to x_0$ and $g_n^{-1} \to y_0$ uniformly on compact subsets of $S^2 \setminus \{y_0\}$ and $S^2 \setminus \{x_0\}$, respectively.

For example, a group of K-quasiconformal mappings, for a fixed $K \geq 1$, is a convergence group. The group G is discrete if it does not contain a sequence of distinct elements tending to the identity mapping Id, and then only (ii) can occur. We allow the elements of G as well as any Möbius transformations that we consider to be sense-reversing. Any group of Möbius transformations is called a Möbius group. When g is a homeomorphism, we write $g^0 = \mathrm{Id}$, and for $n \geq 1$ we set $g^n = g \circ g^{n-1}$ and $g^{-n} = (g^{-1})^n$.

Let the convergence group G be cyclic, generated by g. We assume that $g \neq \mathrm{Id}$. We ask if g is *topologically conjugate* to a Möbius transformation, that is, if there is a homeomorphism f of S^2 onto itself such that $f \circ g \circ f^{-1}$ is a Möbius transformation. That this is indeed the case when G is nondiscrete, has recently been proved by Martin and the author [3].

Suppose that G is discrete. Gehring and Martin ([2, p. 340]) showed that g must be of one of the following three types:

(i) g is called *elliptic* if g has finite order;
(ii) g is *parabolic* if g has a unique fixed point x_0 and then $g^n(x) \to x_0$ as $n \to \infty$ or $n \to -\infty$;

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(iii) g is *loxodromic* if g has exactly two fixed points x_1 and x_2 and then, say, $g^n(x) \to x_1$ and $g^{-n}(x) \to x_2$ as $n \to \infty$, uniformly on compact subsets of $S^2 \setminus \{x_2\}$ and $S^2 \setminus \{x_1\}$, respectively.

As Gehring and Martin observed in [2, p. 354–356], this conjugacy problem has been solved in many cases. A theorem due in part to Brouwer, Kerékjártó, and Eilenberg [1] shows that an elliptic generator g is topologically conjugate to an orthogonal transformation of S^2 and thus to $h(z) = cz$ or $h(z) = c/\bar{z}$ where c is a root of unity. Here and later, we identify S^2 with the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, whenever convenient.

Kerékjártó [7] proved that a sense-preserving loxodromic g is conjugate to $h(z) = 2z$, and hence to $h(z) = cz$ for any complex nonzero c with $|c| \neq 1$ (cf. [7, p. 235]). In fact, his proof applies to sense-reversing loxodromic functions also and shows that such a function is conjugate to $h(z) = 2\bar{z}$ (cf. [4, p. 366]). If we start with the assumption that g is loxodromic, we only need to use Sections 7 and 8 of [7, p. 261–262].

Kerékjártó [6] proved that a sense-preserving parabolic function g is topologically conjugate to $h(z) = z+1$ and thus to $h(z) = z+b$ for any complex nonzero b. Sperner [9] obtained an analogous characterization of functions that are topologically conjugate to a translation. Kerékjártó's arguments in [6, Section 6 onwards], seem to apply to sense-preserving functions only. Thus it may be of some interest to give an explicit proof of the corresponding result for sense-reversing parabolic functions.

Theorem 1. *Let the sense-reversing function g generate a discrete convergence group of infinite order in S^2. If g has exactly one fixed point in S^2, then g is topologically conjugate to $h(z) = \bar{z} + 1$.*

This together with the preceding remarks yields the following consequence.

Corollary 1. *Any cyclic convergence group in S^2 is topologically conjugate to a group of Möbius transformations.*

We remark that Gehring and Martin ([2, Theorem 7.31, p. 356]) have given an example of a noncyclic discrete convergence group on S^2 that is not topologically conjugate to a Möbius group. A nondiscrete convergence group with the same property can be obtained by modifying their example so that the Fuchsian group acting on the unit disk is replaced by a suitable nondiscrete group, such as the group of all Möbius transformations of the unit disk onto itself.

2. Proof of Theorem 1

2.1. Let g be as in Theorem 1. In view of Kerékjártó's result [6], we may perform a preliminary conjugation and assume that $g(\infty) = \infty$ and that $g^2(z) = z + 2$. (Here $g^2 = g \circ g$.) We need to find a simply connected domain U with $\infty \in \partial U$ such that $\overline{U} \setminus \{\infty\} \subset g(U)$. Then there is a Jordan curve γ going through infinity such that one of the domains determined by γ contains U while the other one contains $\mathbf{C}\setminus g(U)$, and such that $\gamma\setminus\{\infty\}\subset g(U)\setminus\overline{U}$, so that $(g(\gamma)\setminus\{\infty\})\cap\overline{g(U)}=\emptyset$. There is a sense-preserving homeomorphism f of the strip $S=\{z: 0\le \operatorname{Re} z\le 1\}$ onto the closure of that domain bounded by γ and $g(\gamma)$ that intersects $\partial g(U)$. Since g is sense-reversing, we can choose f so that

$$f(\bar z+1)=g\big(f(z)\big) \tag{2.1}$$

when $\operatorname{Re} z=0$. Now (2.1) and the definition $f(\infty)=\infty$ extend f to a homeomorphism of S^2 onto itself such that $(f^{-1}\circ g\circ f)(z)=\bar z+1$, as required. It will be clear from the construction of U that (2.1) indeed extends f to all of $\mathbf{C}$.

To find a suitable domain U, we define $H_q=\{z\in\mathbf{C}:\operatorname{Re} z<q\}$ for real q and set $V_q=H_q\cap g(H_q)$. Then $g(V_q)=g(H_q)\cap H_{q+2}\supset V_q$. We claim that for each $M>0$ there is $x_M<0$ such that

$$F_M=\big\{x+iy: x<x_M\ ,\ |y|\le M\big\}\subset V_0. \tag{2.2}$$

Given $M>0$, define $\Omega=\{x+iy: -2\le x<0,\ |y|\le M\}$. Then $g^{-1}(\Omega)$ is a bounded subset of $\mathbf{C}$. We write $E+b=\{z+b: z\in E\}$ for $E\subset\mathbf{C}$ and $b\in\mathbf{C}$. Since g commutes with $g^2=z+2$, we have $g^{-1}(\Omega-2n)=g^{-1}(\Omega)-2n\subset H_0$ for $n\ge n_0$, say. Since $\Omega-2n\subset H_0$, we have $\Omega-2n\subset V_0$. Thus (2.2) holds with $x_M=-2n_0$. It follows that V_0 is not empty and that V_0 has a unique component D such that $F_M\subset D$ for all $M>0$.

We claim that $g(D)\supset D$. In any case D is contained in some component of $g(V_0)$ since $D\subset V_0\subset g(V_0)$. There is x_0 such that $x_0-2n\in D$ for all $n\ge 0$. Thus $g(x_0-2n)=g(x_0)-2n\in D$ for all large n. Hence $D\cap g(D)\ne\emptyset$ and so $D\subset g(D)$ since $g(D)$ is one of the components of $g(V_0)$.

We note that D, being one of the components of $\mathbf{C}\setminus(\partial H_0\cup\partial g(H_0))$, is a Jordan domain. This is clear if ∂H_0 and $\partial g(H_0)$ have at most one point in common, and follows from a theorem of Kerékjártó otherwise ([5, p. 87], see also [8, p. 168]).

If $\overline{D}\setminus\{\infty\}\subset g(D)$, we take $U=D$. Otherwise, we obtain U by modifying D in essentially the same way as in the proof of the loxodromic case given by Kerékjártó in [7, p. 261–262]. However, we have to be slightly more careful since g has no finite attractive fixed point that we could make use of.

We write H for H_0. We have $D\subset H$, $\partial D\subset\partial H\cup\partial g(H)$ and $\partial g(D)\subset\partial g(H)\cup\partial H_2$. So if $z\in\partial D\setminus\partial g(H)$, then $z\in\partial H$ and so $z\notin\partial g(D)$. Since $\partial D\subset\overline{D}\subset\overline{g(D)}$, we thus have $z\in g(D)$. Therefore $\partial D\setminus g(D)\subset\overline{H}\cap\partial g(H)$. Note that $g(H)\setminus\overline{H}\ne\emptyset$ and thus both $\partial H\cap\partial D$ and $\partial H\setminus\partial D$ are nonempty.

The subset $H\cap\partial D$ of ∂D is open and consists of at most countably many disjoint open arcs. The same applies to the interior of $\partial D\cap\partial H\cap\partial g(H)$. Let all these arcs, if there are any, be numbered as I_i for $i\ge 1$. Let F be a homeomorphism of $\overline{H}$ onto $\overline{D}$ with $F(\infty)=\infty$, for example a conformal mapping. Then the arcs $F^{-1}(I_i) = I_i' \subset \partial H \setminus \{\infty\}$ are open and disjoint. Let L_i be the open semicircle in H joining the endpoints of I_i'.

Note that $I_i \subset \partial g(H)$ and so $g(I_i) \subset \partial H_2$ for all i. Thus $g\big(F(L_i)\big)$ is an arc in $g(D) \cap H_2$ joining two points a_i and b_i that lie on ∂H_2. Now the segment $a_i b_i$ together with $g\big(F(L_i)\big)$ bounds a Jordan domain Ω_i contained in $g(D)$. Let γ_i be a Jordan arc in $\Omega_i \setminus \overline{H_1}$ joining a_i to b_i, and set $\gamma_i' = g^{-1}(\gamma_i)$. The unbounded component W of $D \setminus \bigcup_i \gamma_i'$ is a Jordan domain. We have

$$g(W) \supset g(D) \setminus \bigcup_i (\Omega_i \setminus \overline{H_1}) \supset D \supset W.$$

Furthermore, the set $E_1 = \partial W \setminus g(W)$ is a compact nowhere dense subset of ∂H in the topology of $\overline{\mathbf{C}}$. Let F_1 be a homeomorphism of $\overline{H}$ onto $\overline{W}$ with $F_1(\infty) = \infty$.

For each integer $n \geq 1$, we cover $E_1 \cap \{iy : n-1 \leq |y| \leq n\}$ by finitely many vertical open segments J_{ni} of length less than ε, where $\varepsilon \in (0,1)$ is chosen so that $\big|g\big(F_1(z)\big) - g\big(F_1(w)\big)\big| < \frac{1}{2}$ if $z, w \in \overline{H_0} \setminus H_{-1}$ with $|z|, |w| \leq n+1$ and $|z-w| \leq \varepsilon$. Let L_{ni} be the semicircle contained in H_0 and joining the endpoints of J_{ni}. The component U_1 of $H_0 \setminus \bigcup_n \bigcup_i L_{ni}$ containing H_{-1} is a Jordan domain, and $U = F_1(U_1)$ is a domain with the required properties. This completes the proof of Theorem 1.

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