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GENERALIZED CONFORMAL WELDING

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Abstract. Let Φ be a homeomorphism of the unit circle $\mathbf{T}$ onto itself. Suppose that for any Borel set $G \subset \mathbf{T}$ with $\dim G = 0$ we have measure $m_1(\Phi(G)) = m_1(\Phi^{-1}(G)) = 0$. We prove that the unit disk $\mathbf{D}$ and its exterior $\mathbf{D}^*$ are mapped by conformal mappings f and f^* onto disjoint domains Ω and Ω^*, respectively, so that the radial boundary values satisfy

$$(f \circ \Phi)(z) = f^*(z)$$

for $z \in T - E$, where $m_1(E) = 0$.

1. Introduction

A homeomorphism Φ of the unit circle $\mathbf{T} = \{|z| = 1\}$ onto itself is conformally welded in the the classical sense if the unit circle $\mathbf{D}$ and its exterior $\mathbf{D}^*$ may be mapped by some conformal mappings f and f^* into disjoint Jordan domains Ω and Ω^*, such that

$$f^* = f \circ \Phi$$

holds on $\mathbf{T}$. The present paper generalizes the concept of conformal welding to the case where Ω and Ω^* are not necessarily Jordan domains. It follows from a result of Beurling and Ahlfors, see [4], that any quasi-symmetric Φ is conformally welded, see Lehto and Virtanen [10], Pfluger [14]. Lehto [9] and David [5] prove conformal welding for other classes of homeomorphisms. One problem is that there exists Φ for which there is no Jordan curve α, e.g.

$$\Phi(e^{i\theta}) = \begin{cases} e^{i\pi(\theta/\pi} & 0 < \theta < \pi, \\ e^{-i\pi(-\theta/\pi)^b} & -\pi < \theta < 0, \end{cases}$$

and $0 < a < b < 1$. For counterexamples see Oikawa [13], Huber [8], Semmes [16] and Bishop [2], [3].

Nevertheless Bers has asked when is some form of conformal welding possible. This is important for the uniformization of Riemann surfaces. In a companion paper [6] we introduced a generalized conformal welding. The use of Fuchsian groups made it more specialized than the general results we now develop.

We measure sets E by using p-Hausdorff measures $m_p(E)$, $0 \le p \le 1$. A set E has dimension 0 if $m_p(E) = 0$ for all $p > 0$.

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Definition 1. A homeomorphism $\Phi\colon \mathbf{T} \to \mathbf{T}$ is regular if for every $E \subset \mathbf{T}$ with $\dim E = 0$ we have $m_1(\Phi(E)) = m_1(\Phi^{-1}(E)) = 0$.

Examples. It is easy to see that any homeomorphism Φ of the Lusin class, i.e., for each

$$G \subset \mathbf{T}, \quad m_1(G) = 0 \qquad \text{implies} \qquad m_1(\Phi(G)) = m_1(\Phi^{-1}(G)) = 0,$$

is regular. Another example is "Bihölder" homeomorphisms, i.e., there exist positive constants k, α:

$$k^{-1}|z_1 - z_2|^{-\alpha} < |\Phi(z_1) - \Phi(z_2)| \le k|z_1 - z_2|^{\alpha},$$

for all z_1, $z_2 \in \mathbf{T}$. This includes all quasisymmetric maps.

Beurling, see [15], proved that any conformal map f on $\mathbf{D}$ has radial boundary values

$$f(e^{i\theta}) = \lim_{r\to 1} f(e^{i\theta})$$

except for a set of dimension zero. Thus for any regular homeomorphism Φ, $f \circ \Phi$ is defined on $\mathbf{T}$ (a.e). Our conformal maps will always be bounded on $\mathbf{T}$. Thus one can regard $f \circ \Phi = f^*$ as an identity in $L^\infty(\mathbf{T})$. We prove

Theorem 1. *Let $\Phi\colon \mathbf{T} \to \mathbf{T}$ be a regular homeomorphism. Then there exists conformal maps f, f^* on $\mathbf{D}$, $\mathbf{D}^*$ respectively so that*
(i) $f(\mathbf{D}) \cap f^*(\mathbf{D}^*) = \emptyset$,
(ii) $(f \circ \Phi)(z) = f^*(z)$,
for all $z \in \mathbf{T} - E$, where $m_1(E) = 0$.

2. Background results

Our results depend on the theory of the "fractional-derivative capacity" as well as conformal and quasiconformal mapping.

2.1. Capacity. Maz'ya (Chapter 7 [12]) is the reference for this section. For $u \in C_0^\infty(\mathbf{R}^2)$ and vanishing in $\{|z| > 2\}$, let $\hat{u}$ denote the Fourier transform. For $0 \le p \le 1$ define a norm

$$\|u\|_p = \Big[\iint |z|^{2p}|\hat{u}|^2\, dm_2\Big],$$

i.e., the L^2 norm of the p^{th} order fractional derivative. The space $\mathcal{D}_p$ is the closure of $C_0^\infty(\mathbf{R}^2)$ (vanishing in $\{|z| \ge 2\}$) in the $\|\cdot\|_p$ norm. The capacity C_p is defined for compact sets $E \subset \{|z| \le 1\}$ by

$$C_p(E) = \inf\{\, \|u\|_p : u \in \mathcal{D}_p, u \ge 1 \text{ on } E\}.$$

This is indeed a capacity in the sense of Choquet, essentially coinciding with the logarithmic capacity for $p = 1$.

The first property we need is:

(i) For $p < q$, $\mathcal{D}_q$ is compactly embedded in $\mathcal{D}_p$, i.e., suppose we have a sequence $u_n \in \mathcal{D}_q$ with $\|u_n\|_q \leq 1$. Then there is a subsequence u_{n_k} and $u \in \mathcal{D}_q$ so that

$$\|u_{n_k} - u\|_p \to 0.$$

It follows from the definition and property (i) that: For any sequence $u_n \in \mathcal{D}_q$ with $\|u_n\|_q \leq 1$, any $p < q$, there is a set E, $C_p(E) < \varepsilon$,

$$u_{n_k} \to u(z)$$

uniformly on $\mathbf{R}^2 - E$.

We also make use of the relation between p-capacity and Hausdorff dimension.

(ii) For any $0 < p \leq 1$, $E \subset \mathbf{R}^2$

$$\dim E > 2 - 2p \qquad \text{implies} \qquad C_p(E) > 0.$$

We apply these results to functions $h(z) = \sum_{k=1}^{\infty} b_k z^k$ analytic on $\mathbf{D}$ with

$$\iint_D |h'|^2 dx\, dy = \pi \sum_{k=1}^{\infty} k|b_k|^2 < \infty.$$

By Beurling, see [15],

$$\lim_{r \to 1} h(re^{i\theta})$$

exists and is finite except for $e^{i\theta} \in E$, with $C_1(E) = 0$. We refer to this limit as $h(e^{i\theta})$ (when it exists). (In general, $h(z)$ may be extended to $u \in \mathcal{D}_1$.)

Applying (i), (ii) immediately yields:

Theorem 2. *Let $h(z) = \sum_{k=1}^{\infty} a_{k,n} z^k$ be analytic on $\mathbf{D}$ with $\sum_{k=1}^{\infty} k|a_{k,n}|^2 \leq 1$. Then there exists a subsequence h_{n_k} and a limit $h(z)$ so that for every $p < 1$ and $\varepsilon > 0$ there is a set $E \subset \mathbf{T}$ with $C_p(E) \leq \varepsilon$ and*

$$h_{n_k}(z) \to h(z)$$

uniformly on $\mathbf{T} - E$.

2.2. Conformal mapping. We shall be considering pairs $\{f, f^*\}$ where f is conformal on $\mathbf{D}$ and f^* is conformal on $\mathbf{D}^*$. It is important that $f(\mathbf{D}) \cap$

$f^*(\mathbf{D}^*) = \emptyset$. However, as we can just as well consider $\{j \circ f, j \circ f^*\}$ for any Möbius transformation j, we shall assume the following normalization:

$$f^*(z) = z + \sum_{k=1}^{\infty} a_k z^{-k}, \qquad |z| > 1,$$

$$f(z) = \sum_{k=1}^{\infty} b_k z^k, \qquad |z| < 1.$$

This implicitly assumes $b_0 \in f(\mathbf{D})$, $b_0 \notin f^*(\mathbf{D}^*)$. The classical theory shows that

$$\left.\begin{matrix} \partial f^*(\mathbf{D}^*) \\ f(\mathbf{D}) \end{matrix}\right\} \subset \{|w| \leq 2\},$$

see Pommerenke [15]. We shall call this the S-normalization. Now as f and $f^* - z$ may be extended to $\mathcal{D}_1$, from Section 2.1 we immediately deduce

Lemma 1. *Let $\{f_n, f_n^*\}$ be a sequence of S-normalized pairs. Then there is an analytic function f on $\mathbf{D}$, f^* conformal on $\mathbf{D}^*$, and a subsequence n_k so that for any $p < 1$, $\varepsilon > 0$*

$$f_{n_k}^*(e^{i\theta}) \to f^*(e^{i\theta}), \qquad f_{n_k}(e^{i\theta}) \to f(e^{i\theta})$$

uniformly for $e^{i\theta} \in \mathbf{T} - E$, $C_p(E) \leq \varepsilon$.

Remarks. 1. Thus $f^*(e^{i\theta}) \to f^*(e^{i\theta})$ pointwise except for a set of dimension zero.

2. The exceptional set cannot be replaced by one with logarithmic capacity zero.

3. The function f may be identically constant. We make use of a lemma of Beurling (see [15]), to ensure that f_{n_k} does not collapse.

Lemma 2. *Let f^* be conformal on $\mathbf{D}^*$. Then for any constant b*

$$C_1\{e^{i\theta} : f^*(e^{i\theta}) = b\} = 0.$$

Remarks. This fails in $\mathcal{D}_1$.

2.3. Quasiconformal mappings. References may be found in [11]. Let a homeomorphism $\Phi \colon \mathbf{T} \to \mathbf{T}$ be quasisymmetric on $\mathbf{T}$. Beurling and Ahlfors, see [11], proved that Φ extends to a quasiconformal mapping of $\mathbf{C}$, i.e., $\partial\Phi$ and $\bar{\partial}\Phi \in L^2(\mathbf{C})$ and $\|\bar{\partial}\Phi/\partial\Phi\|_\infty < 1$. Let the quasiconformal map Φ^{-1} have complex dilatation

$$\mu(z) = \frac{\bar{\partial}\Phi^{-1}}{\partial\Phi^{-1}}, \qquad \text{for } z \in \mathbf{C}.$$

Bojarski's theorem says that for any measureable λ with $\|\lambda\|_\infty < 1$ there is a quasiconformal homeomorphism ψ: $\mathbf{C} \to \mathbf{C}$ so that $\bar\partial\psi = \lambda\partial\psi$ (a.e. with respect to area). Applying this to

$$\lambda(z) = \begin{cases} \mu(z) & z \in \mathbf{D}^* \\ 0 & z \in \mathbf{D} \end{cases}$$

yields a quasiconformal map f with dilatation 0 on $\mathbf{D}$. The composition formulae shows that $f^* \equiv f \circ \Phi$ has dilatation 0 on $\mathbf{D}^*$. Thus f is conformal on $\mathbf{D}$ and f^* is conformal on $\mathbf{D}^*$. This is the argument of Lehto and Virtanen [10] or Pfluger [14] for

Lemma 3. *For quasisymmetric* Φ: $\mathbf{T} \to \mathbf{T}$ *there are complementary Jordan domains* Ω, Ω^* *and conformal maps*

$$f\colon \mathbf{D} \to \Omega, \qquad f^*\colon \mathbf{D}^* \to \Omega^*$$

so that the boundary values (on $\mathbf{T}$*) satisfy*

$$f^* = f \circ \Phi.$$

2.4. Regular homeomorphisms. We now give some basic properties of regular homeomorphisms. First we observe that a regular homeomorphism is absolutely continuous in the sense of:

Definition 2. A homeomorphism Φ: $\mathbf{T} \to \mathbf{T}$ is RC ("regularly continuous") if for every $p < 1$ and $\varepsilon > 0$ there is a $\delta > 0$ so that for any $E \subset \mathbf{T}$ with $C_p(E) < \delta$, we have

$$m_1(\Phi(E)), m_1(\Phi^{-1}(E)) < \varepsilon.$$

The proof is given in

Lemma 4. *A homeomorphism* Φ: $\mathbf{T} \to \mathbf{T}$ *is* RC *if and only if it is regular.*

We only show that every regular Φ: $\mathbf{T} \to \mathbf{T}$ is RC. Otherwise there is a sequence of $E_n \subset \mathbf{T}$, $C_p(E_n) \leq 1/2^n$, so that $m_1\Phi E_n \geq \varepsilon > 0$, say. Setting $F_n = \sum_{k=n}^{\infty} E_k$, and as $\dim F_n \leq 1/2^{n-1}$ we obtain $F_n \supset F_{n+1} \supset \cdots$ with $m_1\Phi F_n \geq \varepsilon$. Thus $F_n \downarrow \lim F_n \equiv F$ with $\dim F = 0$ but $m_1\Phi(F) \geq \varepsilon$ by monotone convergence.

We cannot approximate a regular homeomorphism Φ: $\mathbf{T} \to \mathbf{T}$ by smooth Φ_n which are "uniformly RC". However a one sided approximation is possible.

Lemma 5. *Let* Φ_n: $\mathbf{T} \to \mathbf{T}$ *be a regular homeomorphism with*

$$\omega(\delta, p) = \sup\{m_1\Phi^{-1}(E) : C_p(E) \leq \delta\}.$$

Then there exists a sequence of smooth homeomorphisms Φ_n: $\mathbf{T} \to \mathbf{T}$*:*

(i) $\sup\{m\Phi_n^{-1}(E) : C_p(E) \leq \delta\} \leq \omega(\delta, p)$,

(ii) $k_n^{-1}|z - w| \leq |\Phi_n(z) - \Phi_n(w)| \leq k_n|z - w|$ *for all* $z, w \in \mathbf{T}$, *with constants* k_n,

(iii) $\Phi_n^{-1}(z) \to \Phi^{-1}(z)$, $\Phi_n(z) \to \Phi(z)$ *uniformly on* T.

This result is only about Φ^{-1} which we write as

$$\Phi^{-1}(e^{i\theta}) = e^{i\varphi(\theta)}$$

where $\varphi\colon \mathbf{R} \to \mathbf{R}$ is a homeomorphism so that for all integers k

$$\varphi(\theta + 2\pi k) = \varphi(\theta) + 2\pi k.$$

Up to a constant the p-capacity on $\mathbf{T}$ is equivalent to the obvious capacity for $\mathbf{R}$. Clearly φ is "RC", i.e. for any $\varepsilon > 0$ there is a $\delta > 0$ so that for any $E \subset [0, 2\pi]$, $C_p(E) < \delta$,

$$m_1\varphi(E) < \varepsilon.$$

Let $\tau_n \in C^\infty$ be an approximation to the identity
(i) $\tau_n(x) > 0$ on $[-1/n, 1/n]$,
(ii) $\tau_n(x) = 0$, $|x| \geq 1/n$,
(iii) $\int_{-\infty}^{\infty} \tau_n(x)\, dx = 1$.

We set

$$\varphi_n(x) = \int_{x-1/n}^{x+1/n} \tau_n(y)\varphi(x+y)\, dy.$$

Clearly φ_n is a smooth homeomorphism with $\varphi_n(x + 2\pi k) = \varphi_n(x) + 2\pi k$. Also for any $E \subset [0, 2\pi)$, $C_p(E) \leq \delta$,

$$\begin{aligned} m_1\big(\varphi_n(E)\big) &= \int_E d\varphi_n = \iint_{\mathbf{R}} \tau_n(y)\, dy\, dx \\ &= \int_{\mathbf{R}} \tau_n(y) \int_E d\varphi(x+y)\, dx\, dy \\ &= \int_{\mathbf{R}} \tau_n(y) m_1\big(\varphi(E+y)\big)\, dy \leq \omega_p(\delta). \end{aligned}$$

Thus setting $\Phi_n^{-1}(e^{i\theta}) = e^{i\varphi_n(\theta)}$ we obtain a homeomorphism with the required properties.

3. Proof of Theorem 1

Let $\Phi\colon \mathbf{T} \to \mathbf{T}$ be a regular homeomorphism. We use the approximating homeomorphisms Φ_n constructed in Lemma 5. These quasisymmetric maps are extended to $\mathbf{C}$ and as in 2.2 we obtain maps f_n, f_n^* conformal on $\mathbf{D}$, $\mathbf{D}^*$ respectively so that

$$f_n^*(z) = f_n \circ \Phi_n(z)$$

for $z \in \mathbf{T}$. We assume that $\{f_n, f_n^*\}$ are S-normalized and thus that f_n^*, f_n converge normally on $\mathbf{D}^* \cup \mathbf{D}$ to f^*, f (which may be constant). Observe that

$f^*, f \circ \Phi$ are well defined functions of $L^\infty(\mathbf{T})$. Let p be any polynomial in z, $\bar{z}$. Now

$$\int_{\mathbf{T}} p(z) f_n^*(z)\,|dz| \to \int_{\mathbf{T}} p(z) f^*(z)\,|dz|$$

by normal convergence. The left hand side is equal to

$$\int p(z) f_n \circ \Phi_n(z)\,|dz| = \int_{\mathbf{T}} p(\Phi_n^{-1}(w)) f_n(w)\,|d\Phi_n^{-1}|$$

where $w = \Phi_n(z)$. Now for any $\delta > 0$ and $p < 1$ there is a set E with $C_p(E) < \delta$ and $f_n(w) \to f(w)$ uniformly on $\mathbf{T} - E$. For any $\varepsilon > 0$, for small enough δ,

$$\int_E |d\Phi_n^{-1}| < \varepsilon,$$

by the results of Sections 2.1–2.4. Thus as $p \circ \Phi_n^{-1} \to p \circ \Phi^{-1}$ uniformly on $\mathbf{T} - E$:

$$\int_{\mathbf{T}-E} p(\Phi_n^{-1}(w)) f_n(w)\,|d\Phi_n^{-1}| \to \int_{\mathbf{T}-E} (p \circ \Phi^{-1}) f(w)\,|d\Phi^{-1}|.$$

Now as p, f_n are bounded on E

$$\left| \int_E p(\Phi_n^{-1}(w)) f_n(w)\,|d\Phi_n^{-1}| \right| < C\varepsilon.$$

Combining these results implies

$$\int_{\mathbf{T}} p(z) f_n \circ \Phi_n(z)\,|dz| \to \int_{\mathbf{T}} (p \circ \Phi^{-1}) f(w)\,|d\Phi^{-1}|.$$

Now as Φ is regular and f may be approximated uniformly (except for a set of small capacity) by continuous functions, we change variable of integration to obtain

$$\int_{\mathbf{T}} p(z)(f \circ \Phi)(z)\,|dz| = \int_{\mathbf{T}} p(z) f^*(z)\,|dz|.$$

Therefore $f \circ \Phi = f^*$ in $L^\infty(\mathbf{T})$.

Finally observe that f cannot be constant, as otherwise so is the conformal mapping f^*, by Lemma 2.

4. Uniqueness of representation

Let q: $\mathbf{C} \to \mathbf{C}$ be any homeomorphism which is conformal on $\Omega \cup \Omega^*$, so that $q(z) = z + \sum_{k=1}^{\infty} c_k z^{-k}$ near ∞. Then if $\{f, f^*\}$ is a (S-normalized) conformal welding for a homeomorphism Φ: $\mathbf{T} \to \mathbf{T}$ we obtain a second (S-normalized) conformal welding $\{q \circ f, q \circ f^*\}$.

Thus even in the classical case that Ω, Ω^* are the inner and outer domain of a closed Jordan curve α there may exist such a q. This is obvious if Area $\alpha > 0$ for then we define dilatation μ on α and let q be the normalized quasiconformal solution of the Beltrami equation

$$\bar{\partial} q = \mu \partial q.$$

However there exists examples of nonuniqueness even when Area $\alpha = 0$ (but $\dim \alpha > 1$), see Bishop [2]. Our example is constructed by classification results of Ahlfors and Beurling.

Let Ω be any domain containing ∞ with complement E. The Dirichlet class $\mathcal{D}(\Omega)$ is the set of functions

$$h(z) = \sum_{k=1}^{\infty} a_k z^{-k}, \qquad (|z| > R)$$

analytic on Ω with

$$\|h\| = \sqrt{\int_\Omega |h'|^2 dx\, dy} < \infty.$$

The capacity with respect to $\mathcal{D}(\Omega)$ is

$$c(E) = \sup\{|a_1|\colon h \in \mathcal{D}(\Omega), \|h\| \le 1\}.$$

Then there are nonmöbius conformal maps defined on Ω if and only if $c(E) > 0$.

Now let E be a totally disconnected compact set. Let Ω, Ω^* be disjoint simply connected domains with $\infty \in \Omega^*$ and constructed so that

$$\partial \Omega = \partial \Omega^* \supset E.$$

By scaling and translation we may assume that the conformal maps

$$f\colon \mathbf{D} \to \Omega, \qquad f^*\colon \mathbf{D}^* \to \Omega^*$$

are S normalized.

We can construct Ω and Ω^* by a family of Jordan arcs connecting components of E. Thus we may define a map

$$\Phi = f^{-1} \circ f^*$$

defined at first only on $\mathbf{T}-F$, where $f^*(F)=E$. By suitable construction of Ω, Ω^* one can ensure that $\dim F>0$ and Φ extends to a regular homeomorphism of $\mathbf{T}$. Thus $\{f,f^*\}$ is a generalized conformal welding of Φ.

Now provided $c(E)>0$ there is a nonmöbius conformal map q of $\mathbf{C}-E$ (not necessarily a homeomorphism of $\mathbf{C}$). We may assume that q is normalized. Thus we obtain a pair $\{q\circ f, q\circ f^*\}$ which forms a conformal welding in our generalized sense.

5. Non Jordan case

Now we discuss the case that $\partial\Omega$ (or $\partial\Omega^*$) is not a Jordan curve. The example in the introduction cannot be conformally welded in the classical sense. Our proof of Theorem 1 shows that there is a dense subset $F\subset\partial\Omega=\partial\Omega^*$ so that every $z\in F$ is the endpoint of open Jordan arcs $\beta\subset\Omega$, $\beta^*\subset\Omega^*$. Furthermore $f^{-1}(F)$ is a dense subset of $\mathbf{T}$ (of positive dimension). Thus if $\lim_{r\to1}f(re^{i\theta})$ does not exist there is then a nontrivial cluster set χ of f at $e^{i\theta}$. In prime-end theory (see [15]) χ is the impression associated with $e^{i\theta}$. The arcs β, β^* separate these continua so that each $e^{i\theta}$ maps to a *unique* impression $I(e^{i\theta})$. Thus for a regular homeomorphism Φ

$$\Phi=f^*\circ f^{-1}$$

in the sense of prime-ends.

As soon as there is a single nontrivial continua $\chi=I(e^{i\theta})$ then the conformal welding of Φ by $\{f,f^*\}$ is not unique. Any normalized conformal map q of $\mathbf{C}-\chi$ gives a conformal welding of Φ by $\{q\circ f, q\circ f^*\}$.

Note that we can always find a conformal mapping q so that $q(\chi)$ is a horizontal line segment. In the next section we observe that there is always a conformal welding so that *all* the impressions χ are horizontal line segments.

6. Class of representations

For each regular homeomorphism $\Phi\colon\mathbf{T}\to\mathbf{T}$ let $\mathcal{F}_\Phi$ be the class of pairs $\{f,f^*\}$ of (S-normalized) conformal maps

$$f\colon\mathbf{D}\to\Omega,\qquad f^*\colon\mathbf{D}^*\to\Omega^*,$$

$\Omega\cap\Omega^*=\emptyset$, $f^*(e^{i\theta})=(f\circ\Phi)(e^{i\theta})$ (a.e. on $\mathbf{T}$). The following is an immediate deduction from Section 2:

Lemma 6. *$\mathcal{F}_\Phi$ is compact in the topology of uniform convergence on compact subsets of $\mathbf{D}\cup\mathbf{D}^*$.*

Theorem 3. *For every regular homeomorphism $\Phi\colon\mathbf{T}\to\mathbf{T}$ there is a conformal welding $\{f,f^*\}$ so that*

(i) Area$\left(\mathbf{C}-\{f(\mathbf{D})\cup f^*(\mathbf{D}^*)\}\right)=0$,

(ii) *Each impression of $\partial f(\mathbf{D})$ (and $\partial f^*(\mathbf{D}^*)$) is a horizontal line segment.*

On $\mathcal{F}_\Phi$ consider the problem of maximizing $\operatorname{Re} a_1$ for $f^*(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \in \mathcal{F}_\Phi$.

Now for each pair

$$f^* = z + \sum_{k=1}^{\infty} a_k z^{-k}, \qquad f = \sum_{k=0}^{\infty} b_k z^k$$

of $\mathcal{F}_\Phi$ we have the area formulae

$$\operatorname{Area}\bigl(\mathbf{C} - \{f(\mathbf{D}) \cup f^*(\mathbf{D}^*)\}\bigr) = \pi - \pi \sum_{k=1}^{\infty} k\bigl(|a_k|^2 + |b_k|^2\bigr), \qquad a_0 = 0.$$

Thus

$$1 \geq \sum_{k=0}^{\infty} k\bigl(|a_k|^2 + |b_k|^2\bigr).$$

Regarding $\mathcal{F}_\Phi$ as a bounded subset of the obvious Hilbert space we see that $\{f^*, f\} \in \mathcal{F}_\Phi$ is an extreme point if and only if

$$\operatorname{Area}\bigl(\mathbf{C} - \{f(\mathbf{D}) \cup f^*(\mathbf{D}^*)\}\bigr) = 0.$$

Now by Lemma 6 there exists an extremal $f \in \varepsilon \times \tau(\mathcal{F}_\Phi)$ maximizing $\operatorname{Re} a_1$. Thus we may assume the existence of an extremal satisfying (i).

Now let χ be any impression of $\partial f(\mathbf{D})$ which is not a horizontal line segment. By the variational theory of Schiffer, see [15], there is a function

$$g(z) = z + \sum_{k=1}^{\infty} d_k z^{-k}$$

conformal on $\mathbf{C} - \chi$, so that $\operatorname{Re} d_1 > 0$. Consider the pair $\{g \circ f^*, g \circ f\} \in \mathcal{F}_\Phi$. Then if $g \circ f^* = z + \sum_{k=1}^{\infty} e_k z^{-k}$ as $e_1 = d_1 + a_1$, we get $\operatorname{Re} e_1 > \operatorname{Re} a_1$ which is a contradiction.

Remarks. Using quasiconformal variations one can prove every extremal satisfies (i).

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