Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 16, 1991, 391–397

ON THE ZEROS OF A CLASS OF GENERALISED DIRICHLET SERIES—VII

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Abstract. Let $a_1 = 1 = \lambda_1$ and $\{\lambda_n\}$ $(n = 1, 2, 3, \dots)$ a sequence of real numbers with $1/c \le \lambda_{n+1} - \lambda_n \le c$ ($c \ge 1$ is a constant). Let $\{a_n\}$ $(n = 1, 2, 3, \dots)$ be any sequence of complex numbers satisfying

$$x^A \ge \frac{1}{x} \sum_{n \le x} |a_n|^2 \ge d_1 \exp\left(-\frac{d_2 \log x}{\log\log x}\right)$$

for all $x \ge x_0(A)$ where A, d_1, d_2 are positive constants. Suppose that $F(s) = \sum_{n=1}^{\infty}(a_n \lambda_n^{-s})$ $(\sigma > A+10)$ can be continued analytically in $(\sigma \ge \frac{1}{2} - \delta,\ T \le t \le 2T)$ and there $\max|F(s)| \le T^A$, where A and δ are positive constants with $0 < \delta < \frac{1}{2}$ (A may be the same as before). Let $0 < \varepsilon < 1$ and $D = D(\varepsilon) = D(\varepsilon, c, d_1, d_2, A, \delta)$ a suitable constant depending on the parameters indicated. Then it is proved that $F(s)$ has at least $T^{1-\varepsilon}$ zeros in the rectangle

$$\left\{\sigma \ge \tfrac{1}{2} - \frac{D}{\log\log T}, T \le t \le 2T\right\}$$

provided T exceeds a large constant.

1. Introduction

Let $s = \sigma + it$, $\sigma > 0$ and

$$\zeta(s) = \sum_{n=1}^{\infty}\left(n^{-s} - \int_n^{n+1} u^{-s}\,du\right) + \frac{1}{s-1}. \tag{1}$$

It is well-known that $\zeta(s)$ has infinity of zeros with $\sigma = \frac{1}{2}$. (Here ample use is made of the functional equation for $\zeta(s)$). But a proof that $\zeta(s)$ has infinity of zeros in $\sigma \ge \frac{1}{2}$, a proof which does not use the functional equation is not known. As such a good generalisation of this fact to Dirichlet series $F(s)$ more general than $\zeta(s)$ is not known. For example consider $F_0(s) = \sum_{n=1}^{\infty}\left((-1)^n \lambda_n^{-s}\right)$ where $\lambda_1 = 1$ and $1/c \le \lambda_{n+1} - \lambda_n \le c$ where $c \ge 1$ is a constant. This is clearly analytic in $\sigma > 0$ and we do not know how to prove that $F_0(s)$ has infinity of zeros in $\sigma \ge \frac{1}{2}$. With these problems in view R. Balasubramanian and K. Ramachandra undertook some investigations in the earlier papers of this series. An example of some elegance discovered by them is the following theorem (for more general results see the earlier papers of the series viz. $[\mathrm{R}]_1$, $[\mathrm{R}]_2$, $[\mathrm{BR}]_2$, $[\mathrm{BR}]_3$, $[\mathrm{R}]_3$ and $[\mathrm{BR}]_4$).

AMS 1991 Subject Classification: 11XX, 11MXX, 11M41.

doi:10.5186/aasfm.1991.1620

Theorem 1. *Let $\chi(n)$ $(n = 1, 2, 3, \dots)$ be any sequence of complex numbers with $\sum_{n\le x} \chi(n) = O(1)$. Then the number of zeros of $\zeta(s) + \sum_{n=1}^{\infty} \chi(n) n^{-s}$ in the rectangle $(|\sigma - \frac{1}{2}| \le \delta,\ T \le t \le 2T)$ (δ is an arbitrary constant subject to $0 < \delta < \frac{1}{2}$ is*

$$\gg T \log T. \tag{2}$$

In this paper I consider the series

$$F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \tag{3}$$

where $a_1 = 1$, $\lambda_1 = 1$, $1/c \le \lambda_{n+1} - \lambda_n \le c$ ($c \ge 1$ is a constant) and for all $x \ge e^e$,

$$\frac{1}{x} \sum_{n \le x} |a_n|^2 \ge d_1 \operatorname{Exp}\Big(- \frac{d_2 \log x}{\log \log x} \Big) \tag{4}$$

$d_1 > 0$, $d_2 > 0$ being positive constants. Further I assume that $|a_n| \le n^A$ and that $F(s)$ has analytic continuation in $(\sigma \ge \frac{1}{2} - \delta,\ T \le t \le 2T)$ and here $\big|F(s)\big| \le T^A$ where A and δ are positive constants (with $0 < \delta < \frac{1}{2}$) and $T \ge T_0(A, \delta)$. I prove the following theorem.

Theorem 2. *Let $0 < \varepsilon < 1$ and $D = D(\varepsilon) = D(\varepsilon, c, d_1, d_2, A, \delta)$ a suitable constant depending on the parameters indicated. Let $T \ge T_0 = T_0(\varepsilon, c, d_1, d_2, A, \delta)$. Then $F(s)$ has at least $T^{1-\varepsilon}$ zeros in the rectangle*

$$\Big\{ \sigma \ge \tfrac{1}{2} - \frac{D}{\log \log T}, T \le t \le 2T \Big\}. \tag{5}$$

In particular this theorem is true for the function $F_0(s)$ stated in the introduction. It is also true of the function of Theorem 1.

2. Proof of Theorem 2.

The proof can be broken up conveniently into a few lemmas. I assume that $T \ge T_0$ (a constant which may depend on other constants). I need the condition (4) only from Lemma 10 onwards. We write T_1 for any number between $T + H$ and $2T - H$.

Lemma 1. *Let $\alpha > \delta$ and $R = R(H, \alpha)$ denote the rectangle $(\sigma \ge \alpha, T_1 - H \le t \le T_1 + H)$. Let $F(s)$ be as above and continuable analytically in $R(H, \alpha - \delta)$*

and here $|F(s)| \le T^A$. *Further let* $F(s) \neq 0$ *in* $R(H,\alpha)$. *Then for* $t = T_1$ *in* $R(H,\alpha)$ *we have*

$$(6)\quad -A_1 \frac{\log T}{\log\log T} \max\left[1, \log\left\{\frac{A_2}{(\sigma-\alpha)\log\log T}\right\}\right] \le \log|F(s)| \le A_3 \frac{\log T}{\log\log T},$$

and

$$(7)\qquad |\arg F(s)| \le A_4 \frac{\log T}{\log\log T},$$

provided $\frac{1}{2}T \ge H \ge A_5 \log\log\log T$. *(Here* A_1, A_2, A_3, A_4, *and* A_5 *are suitable constants depending on* A, δ *and* c. *Also* α *can depend on* T *and* H*).*

Proof. See the appendix in [RS].

Lemma 2. *In* $(\sigma \ge \alpha + (\log\log T)^{-1},\ T_1 - H \le t \le T_1 + H)$ *we have*

$$(8)\qquad |\log F(s)| \le A_6 \frac{\log T}{\log\log T},$$

provided $t = T_1$ *and* $H \ge A_5 \log\log\log T$.

Proof. Follows from Lemma 1.

Lemma 3. *Let* $z = x + iy$ *be a complex variable subject to* $|x| \le \frac{1}{4}$. *Then*

$$(9)\qquad \left|\operatorname{Exp}\left((\sin z)^2\right)\right| \le \begin{cases} 2 & \text{for all } y \\ 2\left(\operatorname{Exp}\operatorname{Exp}|y|\right)^{-1} & \text{for } |y| \ge 2. \end{cases}$$

Remark. This important function viz. $\operatorname{Exp}\left((\sin z)^2\right)$ was introduced and used extensively by me in my earlier papers.

Proof. The proof is really simple. See for example Lemma 2.1 of $[\mathrm{BR}]_1$ for a proof.

Lemma 4. *Let* $0 < \delta < \frac{1}{4}$, $H \ge A_7 \log\log T$, $\alpha_1 = \alpha - \delta$, $\alpha_2 = \alpha - (10D)/(\log\log T)$, $\alpha_3 = \alpha$, *where* A_7 *is a large positive constant. Then for* $T_1 - \frac{1}{2}H \le t \le T_1 + \frac{1}{2}H$ *we have uniformly,*

$$(10)\qquad |F(\alpha_2 + it)| \le 1000(T^A)^{\mu} \left(\operatorname{Exp}\left(A_3 \frac{\log T}{\log\log T}\right)\right)^{1-\mu},$$

(where $\mu = 10D\delta^{-1}(\log\log T)^{-1}$*) and so, by maximum modulus principle, the same holds uniformly for* α_2 *replaced by any* $\sigma \ge \alpha - (10D)/(\log\log T)$ *and* $T_1 - \frac{1}{3}H \le t \le T_1 + \frac{1}{3}H$.

Proof. Put $s_0 = \alpha_2 + it$ and (for a complex variable w),

$$\varphi(w) = X^{w-s_0} F(w) \operatorname{Exp}\left(\sin^2(w - s_0)\right), \tag{11}$$

X being a positive parameter. Apply maximum modulus principle to the rectangle

$$T_1 - \frac{3}{4}H \le \operatorname{Im} w \le T_1 + \frac{3}{4}H, \alpha_1 \le \operatorname{Re} w \le \alpha_3. \tag{12}$$

We have

$$\begin{aligned} |\varphi(s_0)| &= |F(s_0)| \\ &\le 2\left\{X^{10D/\log\log T}\left(X^{-\delta}T^A + \operatorname{Exp}\left(\frac{A_3 \log T}{\log\log T}\right)\right)\right\} \\ &\quad + 2T^A(X^\delta + X^{-\delta}) \cdot 2\left(\operatorname{Exp}\operatorname{Exp}\frac{H}{4}\right)^{-1}. \end{aligned} \tag{13}$$

Choosing X by $X^\delta = T^A \operatorname{Exp}\left((-A_3 \log T)/(\log\log T)\right)$ we obtain the first result. Since $F(s) = O(1)$ when the real part of s is large and also $|F(s)| \le T^A$ (in the relevant rectangle) the second result follows from the first by a similar application of the maximum modulus principle. We have to use the function $\operatorname{Exp}\left(\left(\sin(z/B)\right)^2\right)$ (where B is a large constant) in the proof.

Lemma 5. (Borel–Caratheodory Theorem). *Suppose $f(z)$ is analytic in $|z - z_0| \le R$ and on $|z - z_0| = R$ we have $\operatorname{Re} f(z) \le U$. Then in $|z - z_0| \le r < R$, we have,*

$$|f(z)| \le \frac{2rU}{R-r} + \frac{R+r}{R-r}|f(z_0)|. \tag{14}$$

Proof. See page 174 of [T].

Lemma 6. *Let $\alpha_4 \ge \alpha + (\log\log T)^{-1}$, $s_0 = \alpha_4 + it$, $T_1 - \frac{1}{4}H \le t \le T_1 + \frac{1}{4}H$. Consider circles of radius $R = 11D(\log\log T)^{-1}$ and $r = 10D(\log\log T)^{-1}$ both with centre s_0. Then in ($\sigma \ge \alpha - 9D(\log\log T)^{-1}$, $T_1 - \frac{1}{4}H \le t \le T_1 + \frac{1}{4}H$) we have*

$$|\log F(s)| = O\left(D \log T (\log\log T)^{-1}\right), \tag{15}$$

provided that in ($\sigma \ge \alpha - 10D(\log\log T)^{-1}, T_1 - H \le t \le T_1 + H$) we have $F(s) \ne 0$.

Proof. Follows from the fact that both $r(R-r)^{-1}$ and $(R+r)(R-r)^{-1}$ are $O(1)$. We take $f(z)$ to be $\log F(z)$. This proves the lemma. (Of course we have written z for s for the purposes of application of Borel–Caratheodory's theorem in the notation of Lemma 5. But this should not cause confusion.)

Lemma 7. *Let* $A_8 > 0$ *be an arbitrary constant which is fixed independent of* D. *Put* $\sigma_2 = \alpha - A_8(\log\log T)^{-1}$. *Then in* $\big(\sigma = \sigma_2, T_1 - (1/5)H \le t \le T_1 + (1/5)H\big)$ *and so in* $\big(\sigma \ge \sigma_2, T_1 - (1/6)H \le t \le T_1 + (1/6)H\big)$, *we have,*

$$\log F(\sigma + it) = O\big(De^{-D\lambda}\log T(\log\log T)^{-1}\big), \tag{16}$$

where $\lambda > 0$ *is a constant independent of* D, *provided* $F(s) \ne 0$ *in* $\big(\sigma \ge \alpha - (10D)/(\log\log T), T_1 - H \le t \le T_1 + H\big)$.

Proof. We apply convexity argument to the three lines $\sigma = \sigma_1 = \alpha - 9D(\log\log T)^{-1}$, $\sigma = \sigma_2 = \alpha - A_8(\log\log T)^{-1}$ and $\sigma = \sigma_3 = \beta$ (where $\beta > 0$ is such that for $\sigma \ge \beta$ we have $|\log F(s)| \le 2$). Let $s_0 = \sigma_2 + it$ (with $T_1 - (1/5)H \le t \le T_1 + (1/5)H$) and

$$\psi(w) = X^{w-s_0}\big(\log F(w)\big)\,\mathrm{Exp}\left(\sin^2\left(\frac{w-s_0}{B}\right)\right) \tag{17}$$

where $B > 0$ is a large constant. Choose the rectangle $(\sigma_1 \le \sigma \le \sigma_3, T_1 - \frac{1}{4}H \le t \le T_1 + \frac{1}{4}H)$. We get as in the proof of Lemma 4

$$\big|\log F(s_0)\big| \le C_1\left(\frac{C_2 D\log T}{\log\log T}\right)^{\theta_1} 2^{\theta_2} \tag{18}$$

where C_1 and C_2 are positive constants independent of D and

$$\theta_1 = \frac{\sigma_3 - \sigma_2}{\sigma_3 - \sigma_1} \quad \text{and} \quad \theta_2 = \frac{\sigma_2 - \sigma_1}{\sigma_3 - \sigma_1}. \tag{19}$$

Since $\theta_1 < 1$ and $\theta_2 < 1$ we have

$$\big|\log F(s_0)\big| \le C_3 D\left(\frac{\log T}{\log\log T}\right)^{\theta_1} \tag{20}$$

where $\theta_1 \le 1 - C_4\big(D + O(1)\big)(\log\log T)^{-1}$. Thus the first part of the lemma is proved. The second part follows by a similar application of the maximum modulus principle.

Collecting, we have,

Lemma 8. *Let* $F(s) \ne 0$ *in* $\big(\sigma \ge \alpha - (10D)/(\log\log T), T_1 - H \le t \le T_1 + H\big)$ *and let* A_8 *be any positive constant independent of* D. *Then, we have, in* $\big(\sigma \ge \sigma_2 = \alpha - A_8(\log\log T)^{-1}, T_1 - H/6 \le t \le T_1 + H/6\big)$ *the inequality*

$$\big|\log F(s)\big| \le \frac{\varepsilon^2\log T}{\log\log T} \tag{21}$$

where ε *is an arbitrary constant subject to* $0 < \varepsilon < 1$, *provided* $D = D(\varepsilon)$.

Lemma 9. *Let $H = T^{\varepsilon}$ and $\alpha = \frac{1}{2}$. Then for $\sigma = \sigma_2$, we have,*

$$\frac{1}{H}\int_{T_1-H/6}^{T_1-H/6} |F(s)|^2\,dt \geq c_5 \sum_{n\leq c_6 H} |a_n|^2 n^{-2\sigma}. \tag{22}$$

Proof. It is possible to prove what we want i.e. a result weaker than (22) in a simpler way but in order to reduce the length of the present paper we refer the reader to the third main theorem in $[\mathrm{BR}]_1$.

Lemma 10. *Choose $A_8 = 2d_2$ then*

$$\max_{|t-T_1|\leq H/6} |F(\sigma_2+it)| \gg \operatorname{Exp}\left(d_2\varepsilon\frac{\log T}{\log\log T}\right). \tag{23}$$

Proof. Trivially by the condition (4), the square root of the right hand side of (22) is

$$\gg \operatorname{Exp}\left(\frac{A_8\log H}{\log\log T} - \frac{1}{2}d_2\frac{\log H}{\log\log H}\right) \gg \operatorname{Exp}\left(d_2\frac{\log H}{\log\log T}\right) \gg \operatorname{Exp}\left(d_2\varepsilon\frac{\log T}{\log\log T}\right).$$

The lemma is completely proved.

Now (21) and (23) contradict each other if ε is a small positive constant. This proves that if $H = T^{\varepsilon}$ every rectangle $(\sigma \geq \frac{1}{2} - 10D/(\log\log T), |t-T_1| \leq H)$ contains a zero of $F(s)$. This proves Theorem 2 completely.

Added in proof. (i) R. Balasubramanian and myself have proved some general results regarding the zeros of a class of generalised Dirichlet series in the rectangle $\left(\sigma \geq \frac{1}{2} - C_0(\log\log T)^{3/2}(\log T)^{-1/2}, T \leq t \leq 2T\right)$ where C_0 is a large positive constant depending on the Dirichlet series. The results are not too general; but they cover functions like those referred to in Theorem 1. The lower bound for the number of zeros is $\gg T(\log\log T)^{-1}$.

(ii) We have also proved (using the expression (1) for $\zeta(s)$ and also the Euler product for $\zeta(s)$) that $\zeta(s)$ has $\gg T(\log\log\log T)^{-1}$ zeros in $\left(|\sigma - \frac{1}{2}| \leq C_0(\log\log T)(\log T)^{-1}, T \leq t \leq 2T\right)$.

For the results (i) and (ii) see our papers VIII and IX with the same title as the present one in pages 21–33 and 33–43 of Hardy-Ramanujan Journal 14, 1991.

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Received 4 February 1991