LOCAL AND GLOBAL INTEGRABILITY OF
GRADIENTS IN OBSTACLE PROBLEMS

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Abstract. We establish local and global higher integrability results for the derivatives of the solutions to obstacle problems associated with the second order degenerate elliptic partial differential equation $\text{div } A(x, \nabla u(x)) = 0$, where $|A(x, \xi)| \approx |\xi|^{p-1}$, $p > 1$.

1. Introduction

In this paper we consider the obstacle problem associated with the second order degenerate elliptic equation

$$\text{div } A(x, \nabla u(x)) = 0$$

with $|A(x, \xi)| \leq \beta |\xi|^{p-1}$ and $A(x, \xi) \cdot \xi \geq \alpha |\xi|^p$ for some $0 < \alpha \leq \beta < \infty$ and $p > 1$, see 2.1. The prototype of equation (1.1) is the $p$-harmonic equation

$$\text{div } (|\nabla u|^{p-2} \nabla u) = 0.$$  

Suppose that $\Omega$ is a bounded open set in $\mathbb{R}^n$, that $\psi$ is any function in $\Omega$ with values in $\mathbb{R} \cup \{-\infty, \infty\}$, and that $\theta \in W^{1,p}(\Omega)$. The function $\psi$ is an obstacle and $\theta$ determines the boundary values. Let

$$\mathcal{K}_{\psi, \theta} = \{ v \in W^{1,p}(\Omega) : v \geq \psi \text{ a.e. and } v - \theta \in W^{1,p}_0(\Omega) \}.$$ 

A solution to the $\mathcal{K}_{\psi, \theta}$-obstacle problem is a function $u \in \mathcal{K}_{\psi, \theta}$ such that

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla (v - u) \, dx \geq 0$$

whenever $v \in \mathcal{K}_{\psi, \theta}$.

For solutions $u$ of equation (1.1) it is known ([GM], [Str 1–2], [I], [RZ]) that $u \in W^{1,q}_{\text{loc}}(\Omega)$ where $q = q(p, n, \alpha/\beta) > p$. Our first result generalizes this to the solution of the $\mathcal{K}_{\psi, \theta}$-obstacle problem.

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Theorem A. Suppose that $\psi \in W^{1,s}_{\text{loc}}(\Omega)$, $s > p$. Then a solution $u$ to the $\mathcal{H}_{\psi,\theta}$-obstacle problem belongs to $W^{1,q}_{\text{loc}}(\Omega)$ where $q = q(p, s, n, \alpha/\beta) > p$.

For variational extremals the global higher integrability of the derivative $\nabla u$ has been studied by S. Granlund [G] in the case $p = n$. For this it seems necessary to impose a regularity condition for $\partial \Omega$. We say that $\partial \Omega$ is $p$-Poincaré thick if there is $\gamma < \infty$ such that for all open cubes $Q(r) \subset \mathbb{R}^n$ with side length $r > 0$ it holds

\[
\left( \int_{Q(2r)} |u|^p \, dx \right)^{1/p} \leq \gamma \left( \int_{Q(2r)} |\nabla u|^{pn/(p+n)} \, dx \right)^{(p+n)/pn}
\]

whenever $u \in W^{1,p}(Q(2r))$, $u = 0$ a.e. on $(\mathbb{R}^n \setminus \Omega) \cap Q(2r)$, and $Q(3r) \setminus \overline{\Omega} \neq \emptyset$; here, and in the following, $Q(\lambda r)$, $\lambda > 0$, means a cube parallel to $Q(r)$ with the same center as $Q(r)$ and with side length $\lambda r$. Theorem 2.3 and Corollary 2.7 below give simple sufficient conditions such that (1.4) holds for $p \geq n/(n-1)$.

Theorem B. Suppose that a bounded domain $\Omega$ has a $p$-Poincaré thick boundary and that $p \geq n/(n-1)$. Let $\theta$ and $\psi$ belong to $W^{1,s}_{\text{loc}}(\Omega)$, $s > p$. Then a solution $u$ to the $\mathcal{H}_{\psi,\theta}$-obstacle problem belongs to $W^{1,q}(\Omega)$ where $q = q(p, s, n, \alpha/\beta, \gamma) > p$ and $\gamma$ is the constant of (1.4).

In Section 2 the assumptions on $A$ together with some preliminary lemmas are presented. Section 3 is devoted to the proofs of Theorems A and B. In Remark 3.14 some variants of Theorems A and B are discussed. In particular, local and global higher integrability for the derivatives of solutions of (1.1) is a consequence of Theorems A and B, respectively. Theorems A and B also imply the corresponding results for variational obstacle problems.

The higher integrability of solutions of (1.1) were first considered by Meyers and Elcrat [ME] in 1975. See also [Str 1–2]. For obstacle problems and for differential and variational inequalities most of the regularity studies have been devoted to prove the Hölder continuity of the solutions $u$ to the $\mathcal{H}_{\psi,\theta}$-obstacle problem for Hölder continuous obstacles $\psi$ [Gi]. Michael and Ziemer [MZ] proved the continuity of $u$ if $\psi$ is just continuous. For $p$-harmonic equations (1.2) the higher regularity, i.e. the $C^{1,\alpha}$-regularity, has been much studied, see [L]. For equations (1.1) the Hölder continuity and higher integrability of the derivatives are different aspects of regularity, although for $p \geq n$ there is an obvious connection via the Sobolev imbedding theorem.

When our work was completed, T. Kilpeläinen and P. Koskela [KK] replaced the Poincaré thickness in Theorem B by a capacitory condition on $\partial \Omega$. 
2. Equation (1.1) and preliminary results

2.1. Equation \( \text{div} \mathcal{A}(x, \nabla u) = 0 \). We consider mappings \( \mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) which satisfy the following assumptions for some \( p > 1 \) and \( 0 < \alpha \leq \beta \):

(a) the mapping \( x \mapsto \mathcal{A}(x, \xi) \) is measurable for all \( \xi \in \mathbb{R}^n \) and the mapping \( \xi \mapsto \mathcal{A}(x, \xi) \) is continuous for a.e. \( x \in \mathbb{R}^n \);

(b) \( \mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p \) and

(c) \( |\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1} \).

The constant \( p \) is always associated with \( \mathcal{A} \) as in (b) and (c).

The assumptions (a)–(c) are not strong enough to give a unique solution to the \( \mathcal{K}_{\psi, \theta} \)-obstacle problem. However, if \( \mathcal{A} \) satisfies the monotonicity condition

\[
(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0, \quad \xi_1 \neq \xi_2,
\]

for a.e. \( x \in \mathbb{R}^n \), then it can be shown that the \( \mathcal{K}_{\psi, \theta} \)-obstacle problem has a unique solution provided that \( \mathcal{K}_{\psi, \theta} \neq \emptyset \). For this result see [HKM].

In [HKM] the relation between \( \mathcal{K}_{\psi, \theta} \)-obstacle problems and variational obstacle problems is explained.

2.2. A sufficient condition for (1.4). Here we show that condition (1.4) follows from a measure-theoretic property of \( \mathcal{C} \Omega \); this observation is due to Granlund [G] for \( p = n \).

2.3. Theorem. Suppose that there is \( \mu > 0 \) such that each cube \( Q(r) \) with \( Q(\frac{3}{2}r) \cap \mathcal{C} \Omega \neq \emptyset \) satisfies

\[
(2.4) \quad m((\mathbb{R}^n \setminus \Omega) \cap Q(2r)) \geq \mu m(Q(2r)).
\]

Then \( \mathcal{C} \Omega \) is \( p \)-Poincaré thick for each \( p \geq n/(n-1) \) and the constant \( \gamma \) in (1.4) depends only on \( n, p, \) and \( \mu \).

Proof. Let \( Q(r) \) be a cube with \( Q(r) \cap \partial \Omega = \emptyset \) and let \( u \in W^{1,p}(Q(2r)) \) satisfy \( u = 0 \) on \( Q(2r) \setminus (\mathbb{R}^n \setminus \Omega) \). By [Mor, Theorem 3.6.5, p. 83] we have for each \( q > 1 \)

\[
(2.5) \quad \int_{Q(2r)} |u|^q \, dx \leq c_1(n, q, \mu) r^q \int_{Q(2r)} |\nabla u|^q \, dx.
\]

Next let

\[
c_u = \frac{1}{m(Q(2r))} \int_{Q(2r)} u \, dx
\]
be the mean value of $u$ in $Q(2r)$. For the following Sobolev–Poincaré inequality
\[
(\int_{Q(2r)} |u - c_u|^p \, dx)^{1/p} \leq c_2(n, p) \left( \int_{Q(2r)} |\nabla u|^{pn/(p+n)} \, dx \right)^{(p+n)/pn}
\]
see [GT, p. 174]; note that $p$ is the Sobolev conjugate exponent of $q = pn/(p+n)$ and that $q < n$.

Combining (2.5) and (2.6) we obtain
\[
\left( \int_{Q(2r)} |u|^p \, dx \right)^{1/p} \leq \left( \int_{Q(2r)} |u - c_u|^p \, dx \right)^{1/p} + m(Q(2r))^{1/p} |c_u| + \frac{1}{p} \int_{Q(2r)} |\nabla u|^q \, dx \right)^{1/q}
\]
\[
\leq c_2 \left( \int_{Q(2r)} |\nabla u|^q \, dx \right)^{1/q} + m(Q(2r))^{1/a} \left( \int_{Q(2r)} |u|^q \, dx \right)^{1/q}
\]
\[
\leq c_2 \left( \int_{Q(2r)} |\nabla u|^q \, dx \right)^{1/q} + c'(Q(2r))^{1/q} \left( \int_{Q(2r)} |\nabla u|^q \, dx \right)^{1/q}
\]
\[
\leq \gamma \left( \int_{Q(2r)} |\nabla u|^q \, dx \right)^{1/q}
\]
where the Minkowski and Hölder inequality has also been used. The theorem follows.

An open set $\Omega$ is $c$-coplump, $c \geq 1$, if for each $x \in \mathbb{R}^n \setminus \Omega$ and $r > 0$ there is $z \in B(x, r)$ such that $B(z, r/c) \cap \Omega = \emptyset$.

If $\Omega$ is $c$-coplump, then $\Omega$ clearly satisfies condition (2.4) for some $\mu = \mu(c) > 0$. Hence we obtain from Lemma 2.3

**2.7. Corollary.** If $\Omega$ is $c$-coplump, then $\mathring{\Omega}$ is $p$-Poincaré thick for all $p \geq n/(n-1)$.

**2.8. Reverse Hölder inequality.** To obtain the higher integrability we use the following semilocal reverse Hölder inequality due to Giaquinta and Modica [GM, p. 164]; a new and rather simple proof for Lemma 2.9 can be derived from the work of Kinnunen [K].

**2.9. Lemma.** Suppose that $q > 1$ and that $g \in L^q(Q(2r_0))$ and $f \in L^s(Q(2r_0))$, $s > q$. If for every $x \in Q(2r_0)$ and $r < \frac{1}{2}d(x, \partial Q(2r_0))$ we have the estimate
\[
\frac{1}{q} \int_{Q(r)} |g|^q \, dx \leq c \left[ \left( \frac{1}{q} \int_{Q(2r)} |g| \, dx \right)^q + \frac{1}{T} \int_{Q(2r)} |f|^q \, dx \right]
\]
for some $c > 0$ independent of the cube $Q(r)$ with center at $x$, then $g \in L^t_{loc}(Q(2r_0))$ for some $t = t(n, q, s, c) > q$. 
3. Proofs for Theorems A and B

3.1. Proof for Theorem A. Let $u$ be a solution to the $K_{\psi, \theta}$-obstacle problem and let $Q(2r) \subset \Omega$ be a cube. Fix a cutoff function $\varphi \in C_0^\infty(Q(2r))$ such that $0 \leq \varphi \leq 1$, $|\nabla \varphi| \leq c/r$, and $\varphi = 1$ on $Q(r)$. Consider the function

$$v = u - c_u - \varphi^p (u - c_u - (\psi - c_\psi));$$

here $c_u$ and $c_\psi$ denote the mean values of the functions $u$ and $\psi$, respectively, in $Q(2r)$, i.e.

$$c_u = \frac{1}{m(Q(2r))} \int_{Q(2r)} u \, dx.$$  

Now $v \in K_{\psi - c_u, \theta - c_u}$; indeed, $v - (\theta - c_u) \in W^{1,p}_0(\Omega)$ because $\varphi \in C_0^\infty(\Omega)$ and since $c_u \geq c_\psi$, we obtain

$$v = (1 - \varphi^p)(u - c_u) + \varphi^p(\psi - c_\psi) \geq (1 - \varphi^p)(u - c_u) + \varphi^p(\psi - c_u)$$

$$\geq (1 - \varphi^p)(\psi - c_u) + \varphi^p(\psi - c_u) = \psi - c_u$$

a.e. in $\Omega$. Since

$$\nabla v = (1 - \varphi^p) \nabla (u - c_u) + \varphi^p \nabla (\psi - c_\psi) + p \varphi^{p-1} \nabla \varphi [(\psi - c_\psi) - (u - c_u)]$$

and since $u - c_u$ is a solution to the $K_{\psi - c_u, \theta - c_u}$-obstacle problem, we have

$$\int_\Omega A(x, \nabla u) \cdot \nabla u \, dx \leq \int_\Omega A(x, \nabla u) \cdot \nabla v \, dx$$

$$\leq \int_\Omega (1 - \varphi^p) A(x, \nabla u) \cdot \nabla u \, dx + \int_\Omega \varphi^p A(x, \nabla u) \cdot \nabla \psi \, dx$$

$$+ p\beta \int_\Omega |\nabla u|^{p-1} \varphi^{p-1} |\nabla \varphi| (|\psi - c_\psi| + |u - c_u|) \, dx$$

where we have also used assumption (c). Using (b) and (c) again we obtain from the above inequality

$$\alpha \int_\Omega \varphi^p |\nabla u|^p \, dx \leq \int_\Omega \varphi^p A(x, \nabla u) \cdot \nabla u \, dx$$

$$\leq \beta \int_\Omega \varphi^p |\nabla u|^{p-1} |\nabla \psi| \, dx + p\beta \int_\Omega |\nabla u|^{p-1} \varphi^{p-1}$$

$$\times |\nabla \varphi| (|\psi - c_\psi| + |u - c_u|) \, dx.$$  

Next we use Young’s inequality

$$ab \leq \varepsilon a^{p'} + C(\varepsilon, p)b^p,$$

$$\frac{1}{p} + \frac{1}{p'} = 1,$$
valid for all \( a, b \geq 0, \varepsilon > 0, \) and \( p > 1 \). Now (3.2) yields
\[
\alpha \int_{\Omega} \varphi^p |\nabla u|^p dx \leq \varepsilon \beta \int_{\Omega} \varphi^p |\nabla \psi|^p dx + C(\varepsilon, p) \beta \int_{\Omega} \varphi^p |\nabla \psi|^p dx + p \beta \varepsilon \int_{\Omega} |\nabla u|^p \varphi^p dx + 2^p C(\varepsilon, p) \beta \int_{\Omega} |\nabla \varphi|^p (|\psi - c_\psi|^p + |u - c_u|^p) dx
\]
and choosing
\[
\varepsilon = \frac{\alpha}{2\beta(1 + p)}
\]
we obtain from the above inequality
\[
(3.4) \quad \int_{\Omega} \varphi^p |\nabla u|^p dx \leq c \left[ \int_{\Omega} \varphi^p |\nabla \psi|^p dx + \int_{\Omega} |\nabla \varphi|^p (|\psi - c_\psi|^p + |u - c_u|^p) dx \right]
\]
where \( c \) is a (generic) constant which depends only on \( n, p, \) and \( \alpha/\beta \). Next we estimate the last integral in (3.4) using the ordinary Poincaré inequality \([GT, 7.45, \text{p. } 164]\]
\[
(3.5) \quad \int_{Q(2r)} |v - c_v|^p dx \leq c r^p \int_{Q(2r)} |\nabla v|^p dx
\]
valid for all functions \( v \in W^{1,p}(Q(2r)) \) and the Sobolev–Poincaré inequality (2.6). Together with \( |\nabla \varphi| \leq c/r \) these give
\[
\int_{\Omega} |\nabla \varphi|^p (|\psi - c_\psi|^p + |u - c_u|^p) dx \\
\leq c \int_{Q(2r)} |\nabla \psi|^p dx + \frac{c}{r^p} \left( \int_{Q(2r)} |\nabla u|^{pn/(p+n)} dx \right)^{(p+n)/n}
\]
and hence we obtain from (3.4) the estimate
\[
\int_{Q(r)} |\nabla u|^p dx \leq \int_{\Omega} \varphi^p |\nabla u|^p dx \\
\leq c \int_{Q(2r)} |\nabla \psi|^p dx + \frac{c}{r^p} \left( \int_{Q(2r)} |\nabla u|^{pn/(p+n)} dx \right)^{(p+n)/n}.
\]
This implies
\[
\int_{Q(r)} |\nabla u|^p dx \leq c \left( \int_{Q(2r)} |\nabla u|^{pn/(p+n)} dx \right)^{(p+n)/n} + c \int_{Q(2r)} |\nabla \psi|^p dx.
\]
Setting \( g \equiv |\nabla u|^{pn/(p+n)}, f = |\nabla \psi|^{pn/(p+n)}, \) and \( q = (p + n)/n \) we obtain from Lemma 2.9 that \( |\nabla u| \in L^t_{\text{loc}}(\Omega) \) for some \( t = t(p, s, n, \alpha/\beta) > p \).

The Sobolev imbedding theorem \([GT, \text{p. } 164]\) yields \( u \in L^{np/(n-p)}_{\text{loc}}(\Omega) \) if \( p < n, u \in L^q_{\text{loc}}(\Omega) \) for all \( q > 1 \) if \( p = n, \) and \( u \in L^\infty_{\text{loc}}(\Omega) \) if \( p > n \). Hence \( u \in L^{t'}_{\text{loc}}(\Omega), t' = t'(p, n) > p, \) and choosing \( q = \min(t, t') > p \) we have proved Theorem A.
3.6. Proof for Theorem B. Since $\Omega$ is bounded, we can choose a cube $Q_0 = Q(2r_0)$ such that $\Omega \subset Q(r_0)$. Next let $Q(2r) \subset Q_0$. There are two possibilities: (i) $Q\left(\frac{3}{2}r\right) \subset \Omega$ or (ii) $Q\left(\frac{3}{2}r\right) \cap \Omega \neq \emptyset$. In the case (i) we can follow the proof for Theorem A to obtain the estimate

$$\int_{Q(r)} |\nabla u|^p dx \leq c \left[ \left( \int_{Q\left(\frac{3}{2}r\right)} |\nabla u|^{np/(p+n)} dx \right)^{(p+n)/n} + \int_{Q\left(\frac{3}{2}r\right)} |\nabla \psi|^p dx \right]$$

and then choosing $g = |\nabla u|^{np/(p+n)}$, $f = |\nabla \psi|^{np/(p+n)}$ in $Q\left(\frac{3}{2}r\right)$ and $g = f = 0$ in $Q(2r) \setminus Q\left(\frac{3}{2}r\right)$ with $q = (p+n)/p$ we arrive at the inequality

$$\int_{Q(r)} g^q dx \leq c \left[ \left( \int_{Q(2r)} g dx \right)^q + \int_{Q(2r)} f^q dx \right]$$

where $c = c(p, s, n, \alpha/\beta) < \infty$.

In the case (ii) note that replacing $\theta$ by $\theta_1 = \max(\theta, \psi)$ we may assume that the boundary function $\theta$ satisfies $\theta \geq \psi$ in $\Omega$. Indeed, $\theta_1 = (\psi - \theta)^+ + \theta$ and since

$$0 \leq (\psi - \theta)^+ \leq (u - \theta)^+ \in W^{1,p}_0(\Omega),$$

the function $(\psi - \theta)^+$, and hence $u - \theta_1$, belongs to $W^{1,p}_0(\Omega)$. Next let

$$v = u - \varphi^p(u - \theta)$$

in $\Omega$ where $\varphi \in C^\infty_c(Q(2r))$ is a similar cutoff function as in the proof of Theorem A. Now $v \in K_{\psi, \theta}$ because $v - \theta \in W^{1,p}_0(\Omega)$ and $u \geq \psi$, $\theta \geq \psi$ a.e. yields

$$v = (1 - \varphi^p)u + \varphi^p\theta \geq (1 - \varphi^p)\psi + \varphi^p\psi = \psi$$

a.e. Since

$$\nabla v = (1 - \varphi^p) \nabla u + \varphi^p \nabla \theta + p\varphi^{p-1}(\theta - u) \nabla \varphi,$$

we have the estimate

$$\int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla u dx \leq \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla v dx$$

$$\leq \int_{\Omega} (1 - \varphi^p) \mathcal{A}(x, \nabla u) \cdot \nabla u + \beta \int_{\Omega} |\nabla u|^{p-1} \varphi^p |\nabla \theta| dx$$

$$+ \beta p \int_{\Omega} |\nabla u|^{p-1} \varphi^{p-1} |\theta - u| |\nabla \varphi| dx$$
where assumption (c) has also been used. From this and from (b) we obtain

\[ \alpha \int_{\Omega} \varphi^p |\nabla u|^p \ dx \leq \int_{\Omega} \varphi^p A(x, \nabla u) \cdot \nabla u \ dx \]
\[ \leq \beta \int_{\Omega} |\nabla u|^{p-1} \varphi^{p-1} |\nabla \theta| \ dx + \beta p \int_{\Omega} |\nabla u|^{p-1} |\theta - u| |\nabla \varphi| \ dx \]
\[ \leq \beta \varepsilon \int_{\Omega} \varphi^p |\nabla u|^p \ dx + \beta C(\varepsilon, \beta) \int_{\Omega} \varphi^p |\nabla \theta|^p \ dx \]
\[ + \beta \varepsilon \int_{\Omega} \varphi^p |\nabla u|^p \ dx + \beta p C(\varepsilon, p) \int_{\Omega} |\nabla \varphi|^p |\theta - u|^p \ dx; \]

here we have also used Young’s inequality (3.3) twice. Now we choose

\[ \varepsilon = \frac{\alpha}{2\beta(1 + p)}. \]

Then the above inequality yields

\[ (3.8) \quad \int_{\Omega} \varphi^p |\nabla u|^p \ dx \leq c \left[ \int_{\Omega} \varphi^p |\nabla \theta|^p \ dx + \int_{\Omega} \varphi^p |\nabla \theta - u|^p \ dx \right] \]

where \( c \) is a (generic) constant depending only on \( p, s, \alpha/\beta, n, \) and \( \gamma. \)

To estimate the last integral in (3.8) we use the \( p \)-Poincaré thickness of \( \partial \Omega. \) Indeed, the function \( \theta - u \) can be continued as 0 to \( \bar{\Omega} \) and hence (1.4) implies

\[ (3.9) \quad \int_{\Omega} |\nabla \varphi|^p |\theta - u|^p \ dx \leq cr^{-p} \left( \int_{Q(2r) \cap \Omega} |\nabla(\theta - u)|^{pn/(p+n)} \ dx \right)^{(p+n)/n} ; \]

note that \( \nabla(\theta - u) = 0 \) a.e. in \( \bar{\Omega}. \) The Minkowski and Hölder inequalities yield

\[ r^{-p} \left( \int_{Q(2r) \cap \Omega} |\nabla(\theta - u)|^{pn/(p+n)} \ dx \right)^{(p+n)/n} \]
\[ \leq r^{-p} \left[ \left( \int_{Q(2r) \cap \Omega} |\nabla \theta|^{pn/(p+n)} \ dx \right)^{(p+n)/pn} \right. \]
\[ + \left. \left( \int_{Q(2r) \cap \Omega} |\nabla u|^{pn/(p+n)} \ dx \right)^{(p+n)/pn} \right] \]
\[ \leq r^{-p} \left[ r \left( \int_{Q(2r) \cap \Omega} |\nabla \theta|^p \ dx \right)^{1/p} + \left( \int_{Q(2r) \cap \Omega} |\nabla u|^{pn/(p+n)} \ dx \right)^{(p+n)/pn} \right]^p \]
\[ \leq 2^p \left[ \int_{Q(2r) \cap \Omega} |\nabla \theta|^p \ dx + r^{-p} \left( \int_{Q(2r) \cap \Omega} |\nabla u|^{pn/(p+n)} \ dx \right)^{(p+n)/n} \right]. \]
From (3.8) and (3.9) we thus obtain

\[
\int_{\Omega} \varphi^{p} |\nabla u|^{p} \, dx \leq c \left[ \int_{Q(2r) \cap \Omega} |\nabla \theta|^{p} \, dx + r^{-p} \left( \int_{Q(2r) \cap \Omega} |\nabla u|^{pn/(p+n)} \, dx \right)^{(p+n)/n} \right].
\]  

If we now set \( g = |\nabla u|^{pn/(p+n)} \) and \( f = |\nabla \theta|^{pn/(p+n)} \) in \( \Omega \cap Q(2r) \), \( g = f = 0 \) in \( Q(2r) \setminus \Omega \), and \( q = (p + n)/n \), then (3.10) yields

\[
\int_{Q(r)} g^{q} \, dx \leq c \left[ \int_{Q(2r)} f^{q} \, dx + \left( \int_{Q(2r)} g \, dx \right)^{q} \right]
\]

where \( c = c(p, s, n, \alpha/\beta, \gamma) < \infty \).

Lemma 2.9 together with inequalities (3.7) and (3.11) implies that \( |\nabla u| \in L^{t}(\Omega) \) for some \( t = t(p, s, n, \alpha/\beta, \gamma) > p \).

It remains to show that \( u \in L^{\delta}(\Omega) \) for some \( \delta = \delta(n, p) > p \). Continuing \( u - \theta \) as 0 to \( \mathbb{R}^{n} \) we obtain from the ordinary Sobolev imbedding theorem that for \( p < n \), \( p^{*} = pn/(n-p) \),

\[
\left( \int_{\Omega} |u - \theta|^{p^{*}} \, dx \right)^{1/p^{*}} \leq c \left( \int_{\Omega} |\nabla (u - \theta)|^{p} \, dx \right)^{1/p} < \infty.
\]

If now \( \delta = \min(s, p^{*}) > p \), then by the Minkowski and Hölder inequalities

\[
\left( \int_{\Omega} |u|^{\delta} \, dx \right)^{\delta} \leq \left( \int_{\Omega} |\theta|^{\delta} \, dx \right)^{1/\delta} + \left( \int_{\Omega} |u - \theta|^{\delta} \, dx \right)^{1/\delta}
\]

\[
\leq \left( \int_{\Omega} |\theta|^{\delta} \, dx \right)^{1/\delta} + c_{1} \left( \int_{\Omega} |u - \theta|^{p^{*}} \, dx \right)^{1/p^{*}}
\]

where \( c_{1} \) depends on \( \text{diam } \Omega, p, \) and \( n \).

Since \( \theta \in L^{s}(\Omega) \), we obtain from (3.13) that \( u \in L^{\delta}(\Omega) \). Setting \( q = \min(t, \delta) > p \) we see that \( u \in W^{1,q}(\Omega) \) in the case \( p < n \). If \( p \geq n \), then we can apply the above reasoning for any \( p^{*} < \infty \) together with Hölder’s inequality to conclude that \( u \in L^{s}(\Omega) \) and hence \( u \in W^{1,q}(\Omega) \) with \( q = \min(t, s) > p \) in this case. The theorem follows.

3.14. Remarks. Here we present some variants of Theorems A and B.

(a) A slight modification of the proof of Theorem A shows that if \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is a solution of \( \text{div } A(x, \nabla u) = 0 \) in \( \Omega \), then \( u \in W^{1,q}_{\text{loc}}(\Omega), q = q(n, p, \alpha/\beta) > p \). This situation has already been considered in [GM], [Str 1–2], and [I]. In fact, this situation corresponds to the case \( \psi = -\infty \).
(b) If in Theorem B it is assumed that $\theta, \psi \in W^{1,p}(\Omega)$, $s > p \geq n/(n-1)$, then it follows from the proof of Theorem B that $\nabla u \in L^q(\Omega)$, $q = q(n, p, s, \alpha/\beta, \gamma) > p$. Granlund proved this result for variational obstacle problems in the case $p = n$ [G, Theorem 1.5].

(c) A simplified version of the proof for Theorem B shows that if $u$ is a solution of $\nabla \cdot A(x, \nabla u) = 0$ in $\Omega$ with $u - \theta \in W^{1,p}_0(\Omega)$ and if $\theta \in W^{1,s}(\Omega)$, $s > p \geq n/(n-1)$, then $u \in W^{1,q}(\Omega)$, $q = q(p, s, n, \alpha/\beta, \gamma) > p$.

References


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