A CONVERSE DEFECT RELATION FOR QUASIMEROMORPHIC MAPPINGS

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Abstract. Let \( f: \mathbb{R}^n \to \overline{\mathbb{R}}^n \) be a nonconstant \( K \)-quasimeromorphic map. We prove first that given \( C > 1 \), there exists \( \theta > 1 \), \( \theta \) depending only on \( n, K, C \), such that whenever \( a_1, \ldots, a_q \in \overline{\mathbb{R}}^n \) are distinct, we have \( n(r, a_j) \leq CA(\theta r) \) for \( j = 1, \ldots, q \) and \( r \in E \), where \( E = E(f, a_1, \ldots, a_q) \) has infinite logarithmic measure. This result is then used to obtain the following converse to the defect relation as established by S. Rickman. Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be a nonconstant \( K \)-quasimeromorphic map. Then there exist constants \( C_1 > 1 \) and \( \theta_1 > 1 \), depending only on \( n \) and \( K \) such that for \( a_1, \ldots, a_q \in \mathbb{R}^n \) any distinct points, we have

\[
\limsup_{r \to \infty} \frac{\sum_{j=1}^{q} n(r, a_j)}{(A(\theta_1 r))^{-1}} + \leq C_1
\]

where \( E \) can be taken to be the same set as above. Any improvement or enlargement of the set \( E \) for the first result is immediately valid for the second (main) result.

1. Introduction

Quasiregular (and quasimeromorphic) mappings form a natural generalization of analytic (and meromorphic) maps to real \( n \)-dimensions. We abbreviate these classes as \( qr \) and \( qm \). These functions retain some of the most important topological properties of analytic functions. A study of the value distribution theory of such maps has been a subject of interest for many years. For an overview of results in this area we refer to [R2].

Rickman has shown [R3] that a weak form of Picard’s theorem holds for these mappings. Moreover in [R2], [R6] he proved that for a nonconstant, real \( n \)-dimensional, \( n \geq 3 \), \( K \)-\( qm \) function \( f \), there exists a set \( E \subset [1, \infty) \) of finite logarithmic measure, and a constant \( C(n, K) < \infty \), depending only on \( n \) and \( K \) such that

\[
\limsup_{r \to \infty} \sum_{j=1}^{q} \left(1 - \frac{n(r, a_j)}{A(r)} \right) + \leq C(n, K),
\]

where \( a_1, \ldots, a_q \) are distinct points. For \( n = 3 \) this is qualitatively sharp, as can be seen from [R6, Theorem 1.7]. Thus Nevanlinna’s defect relation generalizes in qualitative form to \( qm \) maps.

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In this paper, we consider a converse inequality. For a nonconstant meromorphic function \( f \) in the plane, it was shown by J. Miles [Mi] that there exist absolute constants \( K < \infty \) and \( C \in (0, 1) \) and a set \( E = E(\theta) \subset [1, \infty) \) having lower logarithmic density at least \( C \) such that if \( a_1, \ldots, a_q \) are distinct elements of the Riemann sphere, then

\[
\limsup_{r \to \infty} \sum_{j=1}^{q} \left( \frac{n(r, a_j)}{A(r)} - 1 \right) \leq K.
\]

Here we extend the above result, for meromorphic functions in the plane, to \( qm \) maps and all dimensions.

The proof breaks up into two parts: Sections 3 and Section 4. In Section 3 we show that \( n(r, a_j) \leq CA(\theta r) \) for any given \( q \) points \( a_1, \ldots, a_q \) and \( r \) taking values in a set \( E \) of infinite logarithmic measure. This is an extension of [R1, 5.16], where the case \( q = 1 \) is considered. The proof is a slight modification of the proof of the same. In Section 4 we first obtain an estimate which holds for all except possibly one value \( a_j \). This estimate holds without the exceptional \( r \)-set, but the \( a_j \) chosen as exception does depend upon \( r \). For such an \( a_j \) we then use the bound obtained in Theorem 3-1. An important open problem is to get a result such as Theorem 3-1 off an exceptional set which does not depend on \( a \). The main analytic tool is path families, a natural generalization to space of extremal length.

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2. Notation and definitions

We denote by \( \mathbb{R}^n \) the real euclidean \( n \)-space, and by \( \bar{\mathbb{R}}^n \) the one-point compactification \( \bar{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\} \). Set

\[
B_r(x) = \{ y \in \mathbb{R}^n : |x - y| < r \}, \quad S(x, r) = \partial B_r(x),
\]

\[
B(r) = B_r(0), \quad S(r) = S(0, r), \quad \text{and} \quad S = S(1).
\]

The Lebesgue measure in \( \mathbb{R}^n \) is denoted by \( \mathcal{L}^n \) and the normalized \( k \)-dimensional Hausdorff measure in \( \mathbb{R}^n \) by \( \mathcal{H}^k \). We set \( \omega_{n-1} = \mathcal{H}^{n-1}(S) \). The Euclidean metric in \( \mathbb{R}^n \) is \( d \). If \( \gamma : \Delta \to \bar{\mathbb{R}}^n \) is a path, we denote its locus \( \gamma \Delta \) by \( |\gamma| \).

\( \bar{\mathbb{R}}^n \) is equipped with the spherical metric,

\[
d[x, y] = |x - y|/[(1 + |x|^2)(1 + |y|^2)]^{1/2}; \quad x, y \neq \infty
\]

\[
d[x, \infty] = 1/(1 + |x|^2)^{1/2}.
\]

Definition. Let \( n \geq 2 \), and let \( G \) be a domain in \( \mathbb{R}^n \). A continuous mapping \( f : G \to \mathbb{R}^n \) is called quasiregular if (1) \( f \) is in the local Sobolev space \( W^1_{n, \text{loc}}(G) \);
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i.e., \( f \) has distributional partial derivatives which are locally \( L^n \)-integrable, and

(2) there exists a constant \( K, 1 \leq K \leq \infty \), such that

\[
|f'(x)|^n \leq K J_f(x)
\]

holds for almost every \( x \in G \). Here \(|f'(x)|\) is the sup norm of the formal derivative \( f'(x) \) defined by means of partial derivatives and \( J_f(x) \) is the Jacobian determinant of \( f \) at \( x \). The smallest \( K \) in (2-1) is the outer dilatation \( K_O(f) \), and the smallest \( K, 1 \leq K \leq \infty \), for which

\[
J_f(x) \leq K \inf_{|h|=1} |f'(x)h|^n
\]

holds is the inner dilatation \( K_I(f) \) of \( f \). \( K(f) = \max(K_O(f), K_I(f)) \) is the maximal dilatation of \( f \). If \( f \) is quasiregular and \( K(f) \leq K \), it is called \( K \)-quasiregular.

Let \( G \subset \bar{\mathbb{R}}^n \) be a domain. A mapping \( f: G \to \bar{\mathbb{R}}^n \) is called quasimeromorphic if either \( fG = \{\infty\} \) or the set \( E = f^{-1}(\infty) \) is discrete and \( f_1 = f|G \setminus (E \cup \{\infty\}) \) is quasiregular. We set \( K(f) = K(f_1), K_O(f) = K_O(f_1), \) and \( K_I(f) = K_I(f_1) \).

For a definition of the modulus of a family of curves we refer to [Vu].

If \( f: \mathbb{R}^n \to \mathbb{R}^n \) is nonconstant and \( qm \), the counting function \( n(r, y) \) is defined for \( r > 0, y \in \mathbb{R}^n \), by

\[
n(r, y) = \sum_{x \in f^{-1}(y) \cap B(r)} i(x, f),
\]

where \( i(x, f) \) is the local topological index; see [MRV1].

\( A(r) \) is the average of \( n(r, y) \) over \( \bar{\mathbb{R}}^n \) with respect to the spherical metric. If \( r, t > 0 \), \( \nu(r, S(a, t)) \) is the average of the counting function over the sphere \( S(a, t) \) with respect to \( \mathcal{H}^{n-1} \),

\[
\nu(r, S(a, t)) = \frac{1}{\omega_{n-1}} \int_S n(r, a + ty) \, d\mathcal{H}^{n-1}(y),
\]

\[
A(r) = \frac{2^n}{\omega_n} \int_{\mathbb{R}^n} \frac{n(r, y)}{(1 + |y|^2)^n} \, dy.
\]

In particular, when \( S(a, t) = S(t) \), we set \( \nu(r, S(t)) = \nu(r, t) \), and also \( \nu(r, 1) = \nu(r) \).

Let \( f: G \to \bar{\mathbb{R}}^n \) be \( qm \). A domain \( D \) such that \( \partial D \subset G \) is called a normal domain if \( \partial fD = \partial fD \). If \( x \in G \) and \( U \) is a normal domain such that \( U \cap f^{-1}(f(x)) = \{x\} \), then \( U \) is called a normal neighbourhood of \( x \). By [MRV1, 2.10], every point in \( G \) has arbitrarily small normal neighbourhoods.
We repeatedly use the following result [R4, p. 228, 2.1]. If $\theta > 1$ and $r, s, t > 0$, then

\[
\nu(\theta r, t) \geq \nu(r, s) - \frac{K_I|\log(t/s)|^{n-1}}{(\log \theta)^{n-1}}.
\]

We also need a comparison between averages on non-concentric spheres, $S$ and $S(a, t) \subset B(1/2)$, for $t$ small enough, say $t < 1/4$. This can be obtained by applying the above result to the map $\phi \circ f$, where $\phi$ is a quasiconformal map of $\mathbb{R}^n$ onto $\mathbb{R}^n$, which is the translation $x \mapsto x - a$ in $B_t(a)$ and it is the identity map outside $B(1)$. $\phi$ can be taken to be $4$-bilipschitz. Thus we get,

\[
\nu(r, S(a, t)) \leq \nu(2r) + c_1 (\log(1/t))^{n-1},
\]

where we may take $c_1 = 4^{2n-2}K/((\log 2)^{n-1})$, since $\phi$ is $4^{2n-2}$-quasiconformal.

3. An upper bound on $n(r, a)/A(\theta r)$

**Theorem 3-1.** Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a nonconstant $K$-quasimeromorphic map. Then for each $C > 1$, there exists $\theta > 1$, $\theta = \theta(C, n, K)$, such that for every $a_1, \ldots, a_q \in \mathbb{R}^n$, there exists a set $E = E(a_1, \ldots, a_q) \subset [1, \infty)$, with $\int_E d\lambda/\lambda = \infty$, such that

\[
n(r, a_j) \leq CA(\theta r) \quad \text{for } j = 1, \ldots, q, \quad r \in E.
\]

Note that here the role of $E$ is different from that in [R4]. We begin with an adaptation of [R1, 5.4] to the case that $a \neq 0$. It is a quantification of the fact that a nonconstant $qm$ map is light.

**Lemma 3-3.** Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a nonconstant $K$-quasimeromorphic map. Choose $1 < u < v$, $t > 0$ and $r > 0$. Let $a \in \mathbb{R}^n$ be given. Set

\[
H_{a,f}(r, t) = \{ \lambda \in [r, ur]: S(\lambda) \cap f^{-1}(B_t(a^c)) \neq \emptyset \},
\]

\[
\phi_{a,f}(r, t) = \int_{H_{a,f}(r, t)} \frac{d\lambda}{\lambda}.
\]

Then,

\[
\nu(vr, S(a, t)) \geq \left[ 1 - \frac{2\omega_{n-1}K_IC_0}{c_n \phi_{a,f}(r, t)(\log v/u)^{n-1}} \right] n(r, a)
\]

where $c_n > 0$ is the constant in [V1, 10.11] which depends only on $n$. 
Proof. Using (2-2), we may obtain [R1, 5.5] without the constant $c'$, as

$$\nu(vr, t) \geq \left[ 1 - \frac{2K_tK_O\omega_{n-1}}{c_n\phi(r, t)(\log v/u)^{n-1}} \right] n(r, 0).$$

Let $g(z) = f(z) - a$. Then $\nu_g(vr, t) \equiv \nu_f(vr, S(a, t))$ and $n_g(r, 0) = n_f(r, a)$. Let $\zeta = w - a$, so that $g(z) = \zeta f(z)$, and also $S(\lambda) \cap g^{-1}(B(t)c) = S(\lambda) \cap f^{-1}(B_t(a)c)$. Hence $H_{0, g}(r, t) = H_{a, f}(r, t)$ and $\phi_{0, g}(r, t) = \phi_{a, f}(r, t)$. Now (3-6) applied to $g$ gives (3-5).

Proof of Theorem 3-1. We divide the proof into three steps. The second step proves the theorem under the normalization $a_1, \ldots, a_q \in B(1/2)$. The first and third steps are merely to facilitate this normalization.

**Step I:** Let $C > 1$ be given. Let $a_1, \ldots, a_q \in \mathbb{R}^n$. By a rotation of the sphere we may assume that $a_1, \ldots, a_q \in B(\tau/2)$ for some $\tau \geq 1$. Let $\sigma > 0$ be such that the balls $\{B_\sigma(a_j)\}$ are disjoint and $\{B_\sigma(a_j)\} \subset B(\tau/2)$ for all $j$. We claim that for given $r_0 > 0$, there exists $r_1 \geq r_0$ such that for all $r \in [r_1, u^{1/4}r_1]$, 

$$n(r, a_j) \leq CA(\theta r) \quad \text{for } j = 1, \ldots, q,$$

where $u > 1$ is defined in (3-11). By repeating this argument, we obtain our set $E = \bigcup_{i=1}^\infty [r_i, u^{1/4}r_i]$, so that $E$ has infinite logarithmic measure. We may assume that $n(r_0, a_j) \geq 1$ for all $j$, since the $j$’s for which $n(r, a_j) = 0$ for all $r$ satisfy the claim. Let

$$C' = C^{1/4} > 1.$$ 

By [R1, 4.10] we choose $r_0$ so that for $r \geq r_0$,

$$\nu(r) < C'A(2r).$$

We assume $\infty$ is an essential singularity (i.e. $f$ has no limit in $\mathbb{R}^n$ as we approach $\infty$), for otherwise $f$ extends to $\mathbb{R}^n$ as a qm map and it has finite degree [MRV2], [MS]. By [R1, 3.1] we then have that $A(r) \to \infty$. So we may choose $r_0$ such that for $r \geq r_0$

$$C'^2K_I\left(\frac{\log \tau}{\log 2}\right)^{n-1} + C'c_1\left(\log \frac{1}{\sigma}\right)^{n-1} < (C'^4 - C'^3)A(r).$$

**Step II:** In this step we replace $f$ by $f/\tau$ and $a_1, \ldots, a_q$ by $a_1/\tau, \ldots, a_q/\tau$. However, for convenience of notation, we still call them $f$ and $a_1, \ldots, a_q$. Note that we are now in the situation $a_1, \ldots, a_q \in B(1/2)$, $\{\hat{B}_\sigma(a_j)\}$ disjoint and each $\hat{B}_\sigma(a_j) \subset B(1/2)$. In order to apply Lemma 3-3 we define $u > 1$ by

$$\frac{1}{C'} = 1 - \frac{4\omega_{n-1}K_OK_I}{c_n(\log u)^n}.$$
where \( c_n > 0 \) is as in [V1, 10.11].

For \( u > 1 \), as above and \( t, r > 0 \), let \( \phi_j(r, t) \equiv \phi_{a_j, f}(r, t) \) be as in Lemma 3-3, and let

\[
(3-12) \quad \Psi(t) = \sup_{r \geq r_0} \min_{1 \leq j \leq q} \phi_j(r, t).
\]

Then \( \Psi \) is decreasing in \( t \).

Case (i): \( \Psi(\sigma) \geq (7/8) \log u \).

Then, by the definition of \( \Psi(\sigma) \), there exists \( r_1 \geq r_0 \) such that \( \min_j \phi_j(r_1, \sigma) \geq (3/4) \log u \); i.e.

\[
(3-13) \quad \phi_j(r_1, \sigma) \geq (3/4) \log u, \quad 1 \leq j \leq q.
\]

From the definition of \( \phi_j(r_1, \sigma) \), we note that

\[
\phi_j(r_1, \sigma) = \int_{H_j(r_1, \sigma)} \frac{d\lambda}{\lambda} = \int_{H_j(r_1, \sigma) \cap [r_1, u^{1/4}r_1]} \frac{d\lambda}{\lambda} + \int_{H_j(r_1, \sigma) \cap [u^{1/4}r_1, ur_1]} \frac{d\lambda}{\lambda}.
\]

From this and (3-13) we obtain for \( r \in [r_1, u^{1/4}r_1] \) and for all \( j = 1, \ldots, q \),

\[
(3-14) \quad \phi_j(r, \sigma) \geq \int_{H_j(r_1, \sigma) \cap [u^{1/4}r_1, ur_1]} \frac{d\lambda}{\lambda} \geq \frac{1}{2} \log u.
\]

We now apply Lemma 3-3 with \( a = a_j, t = \sigma, r \in [r_1, u^{1/4}r_1], v = u^2 \) along with (3-14) and (3-11) to obtain

\[
(3-15) \quad \nu(vr, S(a_j, \sigma)) \geq \left[ 1 - \frac{2\omega_{n-1}K_O}{c_n \phi_j(r, \sigma) (\log u)^{n-1}} \right] n(r, a_j)
\]

\[
= \frac{1}{C'} n(r, a_j) \quad j = 1, \ldots, q.
\]

Now using (2-3) with \( t = \sigma \) and (3-15), we get for \( r \in [r_1, u^{1/4}r_1] \) and \( j = 1, \ldots, q \), that

\[
(3-16) \quad n(r, a_j) \leq C' \nu(vr, S(a_j, \sigma)) \leq C' \nu(2vr) + C'c_1(\log 1/\sigma)^{n-1}.
\]

Case (ii): \( \Psi(\sigma) < (7/8) \log u \).
Since $f$ is discrete, for each fixed $r$, $\phi_j(r, t) \to \log u$ as $t \to 0$. Let $t_0 = \inf \{ t : t \leq \sigma, \Psi(t) \leq (7/6) \log u \}$. One checks that $t_0 > 0$. We may assume $t_0 < \sigma$. Let $\delta$ be so small that

\[
0 < \delta < \min \{ \frac{1}{2} t_0, \sigma - t_0 \}, \quad \frac{4\delta}{t_0} < (\log 2) \left( \frac{C' - 1}{K_I C''} \right)^{1/(n-1)}
\]

and let

\[
(3-18) \quad t_1 = t_0 - \delta, \quad t'_1 = t_0 + \delta.
\]

Since $\Psi(t_1) > \frac{7}{8} \log u$, there exists $r_1 \geq r_0$ with $\min_j \phi_j(r_1, t_1) \geq \frac{3}{4} \log u$; i.e.

\[
\phi_j(r_1, t_1) \geq \frac{3}{4} \log u, \quad j = 1, \ldots, q.
\]

From this we may conclude, exactly as in Case (i), that for $r \in [r_1, u^{1/4} r_1]$,

\[
(3-19) \quad \phi_j(r, t_1) \geq \frac{1}{2} \log u, \quad j = 1, \ldots, q.
\]

Now we apply Lemma 3-3 with $r \in [r_1, u^{1/4} r_1]$, $t = t_1$, $a = a_j$, $v = u^2$, along with (3-19) and (3-11), to obtain

\[
\nu(vr, S(a_j, t_1)) \geq \left[ 1 - \frac{2\omega_{n-1} K_i K_O}{c_n \phi_j(r, t_1) (\log u)^{n-1}} \right] n(r, a_j)
\]

\[
\geq \left[ 1 - \frac{4\omega_{n-1} K_i K_O}{c_n (\log u)^n} \right] n(r, a_j)
\]

\[
\geq \frac{1}{C'} n(r, a_j), \quad 1 \leq j \leq q.
\]

Let $t_0 < t < t'_1$. By (3-12), $\Psi(t) \equiv \sup_{r \geq r_0} \min_j \phi_j(r, t) \leq (7/6) \log u$, and since $2vr \geq r \geq r_0$, we find for an appropriate $1 \leq l \leq q$, that $\phi_l(2vr, t) \equiv \min_j \phi_j(2vr, t) \leq (7/6) \log u$. Then by the definition of $\phi_l(2vr, t)$ there exists $\varrho \in [2vr, 2vur]$ such that $S(\varrho) \cap f^{-1}(B_l(a_l)^c) = \emptyset$. The analysis of [MRV1, 2.5], which is stated only for $qr$ maps but applies as well to $qm$ maps, shows that every component of $f^{-1}(B_l(a_l)^c)$ which meets $\overline{B}(\varrho)$ is a normal domain contained in $B(\varrho)$. Hence

\[
(3-21) \quad n(\varrho, y) = n(\varrho, z) \quad \text{for all } y, z \in \overline{B}(a_l)^c.
\]

In particular, since $t < t'_1 < \sigma$ and the $\{ \overline{B}(a_j, \sigma) \}$ are disjoint, we have for $j \neq l$, $n(\varrho, y) = n(\varrho, a_j + t_1 y)$ for all $y \in S$. And so on averaging,

\[
(3-22) \quad \nu(\varrho) = \nu(\varrho, S(a_j, t_1)) \quad j \neq l.
\]
For \( j = l \), since \( t < t'_1 \), we note from (3-21) that \( n(\rho, y) = n(\rho, a_t + t'_1 y) \) for all \( y \in S \). So again on averaging,

\[
(3-23) \quad \nu(\rho) = \nu(\rho, S(a_l, t'_1)).
\]

We now replace \( \nu(\rho, S(a_l, t'_1)) \) by \( \nu(\rho, S(a_l, t'_1)) \) with controllable error. Letting \( \theta = 2, s = t_1, t = t'_1, r = vr \), we obtain from (2-2) that

\[
(3-24) \quad \nu(vr, S(a_l, t_1)) \leq \nu(2vr, S(a_l, t'_1)) + \frac{K_I (\log(t'_1/t_1))^{n-1}}{(\log 2)^{n-1}}.
\]

Now we find, using (3-18) and (3-17), that

\[
\log \frac{t'_1}{t_1} = \log \left(1 + \frac{2\delta}{t_0 - \delta}\right) < \frac{2\delta}{t_0 - \delta} < \frac{4\delta}{t_0} < (\log 2) \left(\frac{C - 1}{K_I C^2}\right)^{(n-1)}.
\]

Hence, from (3-24),

\[
(3-25) \quad \nu(vr, S(a_l, t_1)) \leq \nu(2vr, S(a_l, t'_1)) + (C' - 1)/C'^2.
\]

Since \( n(r, a_l) \geq n(r, a_l) \geq 1 \) as stated in Step I, we have from (3-20) that \( \nu(vr, S(a_l, t'_1)) \geq 1/C' \). Substituting this inequality on the right hand side of (3-25) and unraveling, we obtain,

\[
\nu(vr, S(a_l, t'_1)) \leq C' \nu(2vr, S(a_l, t'_1)).
\]

But since \( 2vr \leq \rho \leq 2uvr \), the last inequality, together with (3-20) and (3-23) gives for \( r \in [r_1, u^{1/4}r_1] \),

\[
(3-26) \quad n(r, a_l) \leq C'^2 \nu(\rho, S(a_l, t'_1)) = C'^2 \nu(\rho).
\]

And again using the fact that \( 2vr \leq \rho \) along with (3-20) and (3-22), we find for \( j \neq l, r \in [r_1, u^{1/4}r_1] \)

\[
(3-27) \quad n(r, a_j) \leq C' \nu(\rho, S(a_j, t_1)) = C' \nu(\rho).
\]

Using the inequality \( 2vr \leq \rho \), we conclude in both cases, from (3-26), (3-27) and (3-16) that, for \( j = 1, \ldots, q, r \in [r_1, u^{1/4}r_1] \),

\[
(3-28) \quad n(r, a_j) \leq C'^2 \nu(\rho) + C'c_1 (\log 1/\sigma)^{n-1}.
\]

Finally, we recall the change of scale we made in the beginning of Step II, and conclude from (3-28) that for \( r \in [r_1, u^{1/4}r_1] \),

\[
(3-29) \quad n(r, a_j) \leq C'^2 \nu(\rho, \tau) + C'c_1 (\log 1/\sigma)^{n-1}
\]

for the original \( f \) and \( a_1, \ldots, a_q \).

**Step III:** First we use (2-2) to replace \( \nu(\rho, \tau) \) by \( \nu(2\rho) \) in (3-29) and get

\[
n(r, a_j) \leq C'^2 \nu(2\rho) + C'^2 K_I \left(\frac{\log \tau}{\log 2}\right)^{n-1} + C'c_1 (\log 1/\sigma)^{n-1}.
\]

Using (3-9), (3-10) (3-8) and \( \rho \leq 2uvr \) we now get for \( r \in [r_1, u^{1/4}r_1] \) and \( j = 1, \ldots, q \),

\[
n(r, a_j) \leq C'^3 A(4\rho) + (C'^4 - C'^3) A(4\rho) \leq CA(\theta r),
\]

where \( \theta = 8uv = 8u^3 \). This proves the theorem.
4. The main result

We first prove an intermediate result, i.e., the estimate (4-2). This is an essential fact needed for the main theorem.

Theorem 4-1. Let \( n \geq 2 \) and \( K \geq 1 \). There exist positive constants \( \theta_0 = \theta_0(n, K) \), \( b = b(n, K) \) such that if \( f: \mathbb{R}^n \to \mathbb{R}^n \) is a nonconstant \( K \)-qm map and \( a_1, \ldots, a_q \in \mathbb{R}^n \), are any distinct points, with \( q > 1 \), then there exist \( r_0 = r_0(a_1, \ldots, a_q, f) > 0 \) such that for each \( r \geq r_0 \), we have

\[
\sum_{j=1}^{q} n(r, a_j) \leq \left[ q + \frac{4K_Ib}{(\log 2)^n-1} + 2 \right] A(16\theta_0 r),
\]

for some \( J(r) \in \{1, \ldots, q\} \). The constants \( \theta_0 \) and \( b \) are given by

\[
\log \theta_0 = \frac{\omega_n \log c_1}{2n-4c_n n}, \quad b = \frac{2K_O \omega_n}{c_n \log \theta_0}
\]

with \( c_1 \) and \( c_n \) as in (2-3) and (3-5) respectively.

Observe that there is no exceptional set for the \( r \)-values here. However, the estimate obtained is close to what we want, save for one \( a_{J(r)} \). For this \( a_{J(r)} \) we use Theorem 3-1. We thus obtain our main result, Theorem 4-26, on the same exceptional set of \( r \)-values as that obtained in Theorem 3-1. It is worth noting that any enlargement or improvement of the set \( E \) of Theorem 3-1, is also valid for Theorem 4-26.

Proof of Theorem 4-1. Again we divide the proof into three steps with main body of the proof being in the second step.

Step I: We may assume, as in the proof of Theorem 3-1, that \( \infty \) is an essential singularity, so that \( A(r) \to \infty \) as \( r \to \infty \). By a rotation we assume that \( a_1, \ldots, a_q \in \mathbb{R}^n \). Let \( \tau \geq 1 \) and \( \sigma > 0 \) be such that \( B_{\sigma \tau}(a_j) \subset B(\tau/2) \), and the \( \{\bar{B}_{\sigma \tau}(a_j)\} \) are disjoint. We set \( r_0 = \max(r_1, r_2) \), where \( r_1 \) and \( r_2 \) are obtained below. Choose \( r_1 = r_1(\tau, q, f) > 0 \) such that for \( r \geq r_1 \),

\[
(i) \quad \left[ q + \frac{K_IB}{(\log 2)^{n-1}} \right] K_I \left( \frac{\log \tau}{\log 2} \right)^{n-1} \leq \frac{K_IB}{(\log 2)^{n-1}} \nu(r)
\]

\[
(ii) \quad \nu(r) < \frac{q}{q-1} A(2r) \quad \text{by [R1, 4.10]}
\]

Step II: Again by replacing \( f \) by \( f/\tau \) we reduce to the case \( \tau = 1 \). Since
\( \nu(r) \to \infty \) as \( r \to \infty \), we can choose \( r_2 = r_2(\sigma, q, f) > 0 \) such that for \( r \geq r_2 \),

\[
\begin{align*}
(i) & \quad [bv(2\theta_0r)]^{1/n} + 1 < [2bv(2\theta_0r)]^{1/n}, \\
(ii) & \quad \log 2 < (bv(2\theta_0r))^{1/(n-1)} - (bv(2\theta_0r))^{1/n}, \\
(iii) & \quad \frac{1}{1 + (\log(\sigma/2))/(bv(2\theta_0r))^{1/(n-1)}} < 2^{1/n}, \\
(iv) & \quad 2^{1/n} < (bv(2\theta_0r))^{1/n}, \\
(v) & \quad c_1gb < (bv(2\theta_0r))^{1/n}.
\end{align*}
\]

(4-5)

Fix \( r \geq r_2 \). Since \( f \) is \( qn \), \( H^n(\partial B(\theta_0r)) = 0 \) implies \( H^n(f(\partial B(\theta_0r))) = 0 \), by [Vu, 10.5(3)]. From this and Fubini’s theorem it follows that

\[ \sigma \in \left\{ \exp \left\{ -(bv(2\theta_0r))^{1/(n-1)} \right\}, 2 \exp \left\{ -(bv(2\theta_0r))^{1/(n-1)} \right\} \right\}, \]

for each \( j = 1, \ldots, q \). Hence there exists \( \varepsilon_1 \in [1, 2] \) such that for

\[ \sigma_1 = \varepsilon_1 \exp \left\{ -(bv(2\theta_0r))^{1/(n-1)} \right\} \]

(4-6)

(4-7) \( H^{n-1}(f(\partial B(\theta_0r)) \cap S(a_j, \sigma_1)) = 0 \) for all \( j = 1, \ldots, q \).

Then by (4-6) and (4-5) (ii) we have

\[ \sigma_1 \leq 2 \exp \left\{ -(bv(2\theta_0r))^{1/(n-1)} \right\} < \exp \left\{ -(bv(2\theta_0r))^{1/n} \right\} = \sigma_2 \]

and by (4-5) (iv),

\[ \sigma_2 = \exp \left\{ -(bv(2\theta_0r))^{1/n} \right\} < \sigma. \]

Let \( \alpha_j \) and \( \beta_j \) be the maps of \( S \) onto \( S(a_j, \sigma_1) \) and \( S(a_j, \sigma_2) \) respectively given by \( \alpha_j(y) = a_j + \sigma_1y \), \( \beta_j(y) = a_j + \sigma_2y \).

For \( y \in S \), let \( \gamma^j_y : [0, 1] \to \mathbf{R}^n \) be the line segment joining \( a_j \) to \( \beta_j(y) \), parametrized so that \( \gamma^j_y : [0, 1/2] \) joins \( a_j \) to \( \alpha_j(y) \in S(a_j, \sigma_1) \), \( \gamma^j_y : [1/2, 1] \) joins \( \alpha_j(y) \) to \( \beta_j(y) \in S(a_j, \sigma_2) \).

Comparison of \( n(r, a_j) \) with \( n(\theta_0r, \alpha_j(y)) \): Let \( f \mid X \) denote \( f \) restricted to \( X \) and let \( \Lambda^j_y = \{ \lambda_1, \ldots, \lambda_h \} \) be a maximal sequence of \( f \mid B(4\theta_0r + 1) \)-liftings of \( \gamma^j_y \mid [0, 1/2] \) starting at points of \( f^{-1}(a_j) \cap B(r) \), as defined in [R1]. Then necessarily \( h = n(r, a_j) \). The following crucial lemma has been inspired by the proof of [R2, 3.2].
Lemma 4-9. The family of curves
\[ \mathcal{F}_j = \bigcup_{y \in S} \Lambda^j_y \]
lies completely in \( B(\theta_0 r) \), except perhaps for one \( j = J(r) \in \{1, \ldots, q\} \).

Proof. Note that by definition, all paths in \( \mathcal{F}_j \) start at preimages of \( a_j \) in \( B(r) \). We prove the lemma by contradiction. Suppose there exist \( j \neq k \) and \( \eta_j \in \mathcal{F}_j \), \( \eta_k \in \mathcal{F}_k \), such that \( \eta_j, \eta_k \notin B(\theta_0 r) \). Let \( \Gamma \) be the family of paths in \( B(\theta_0 r) \setminus B(r) \) joining the loci \( |\eta_j| \) and \( |\eta_k| \). Note that \( |f(\eta_j)| \) and \( |f(\eta_k)| \) are line segments starting at \( a_j \) and \( a_k \) and contained in \( B(a_j, \sigma_1) \) and \( B(a_k, \sigma_1) \) respectively. Hence each path in \( f\Gamma \) contains sub-paths which join \( S(a_j, \sigma) \) to \( S(a_j, \sigma) \) and \( S(a_k, \sigma) \) to \( S(a_k, \sigma) \). Set
\[ \varrho(z) = \begin{cases} (2 \log(\sigma/\sigma_1)|z - a_j|)^{-1}, & \sigma_1 < |z - a_j| < \sigma \\ (2 \log(\sigma/\sigma_1)|z - a_k|)^{-1}, & \sigma_1 < |z - a_k| < \sigma \\ 0, & \text{otherwise}. \end{cases} \]
Then \( \varrho \) is well-defined by the choice of \( \sigma \). Also, \( \varrho \) is admissible for the family \( f\Gamma \), and by [MRV1, 3.2] we obtain
\[
M(\Gamma) \leq K_O \int_{R^n} \varrho(z)^n n(\theta_0 r, z) d\mathcal{L}^n(z) \\
= \frac{K_O}{(2 \log(\sigma/\sigma_1))^{n}} \int_{\{\sigma_1 < |z - a_j| < \sigma\}} n(\theta_0 r, z)|z - a_j|^{-n} d\mathcal{L}^n(z) \\
+ \frac{K_O}{(2 \log(\sigma/\sigma_1))^{n}} \int_{\{\sigma_1 < |z - a_k| < \sigma\}} n(\theta_0 r, z)|z - a_k|^{-n} d\mathcal{L}^n(z) \\
= I + II.
\]
We obtain an estimate for \( I \). Exactly the same estimate holds for \( II \) as well. By transferring the integral of (4-10) into polar coordinates, we find that,
\[
I = K_O (2 \log(\sigma/\sigma_1))^{-n} \int_{\sigma_1} \int_{S} n(\theta_0 r, a_j + \tau y) d\mathcal{H}^{n-1}(y) \tau^{-1} d\tau \\
= K_O \omega_{n-1} (2 \log(\sigma/\sigma_1))^{-n} \int_{\sigma_1} \nu(\theta_0 r, S(a_j, \tau)) \tau^{-1} d\tau.
\]
Using (2-3), with \( \theta = \theta_0 \),
\[
I \leq \frac{K_O \omega_{n-1}}{(2 \log(\sigma/\sigma_1))^{n}} \int_{\sigma_1} \left\{ \nu(2\theta_0 r) + c_1 (\log(1/\tau))^{n} \right\} \tau^{-1} d\tau \\
\leq \frac{K_O \omega_{n-1}}{(2 \log(\sigma/\sigma_1))^{n}} \left[ \nu(2\theta_0 r) \log(\sigma/\sigma_1) + c_1 \left( \frac{\log(1/\sigma_1)}{n} \right)^n \right] \\
\leq \frac{K_O \omega_{n-1}}{2^n} \left[ \frac{\nu(2\theta_0 r)}{\left( \log(\sigma/\sigma_1) \right)^{n-1}} + c_1 \left( \log \frac{1}{\sigma_1} \right)^n \right]
\]
Now using (4-6), the fact that \( \varepsilon_1 \in [1, 2] \), and (4-5) (iii), we find that

\[
\frac{\log(1/\sigma_1)}{\log(\sigma/\sigma_1)} = \frac{\log(1/\varepsilon_1) + (b\nu(2\theta_0r))^{1/(n-1)}}{\log \sigma + \log(1/\varepsilon_1) + (b\nu(2\theta_0r))^{1/(n-1)}} \\
\leq \frac{(b\nu(2\theta_0r))^{1/(n-1)}}{\log \sigma + \log(1/2) + (b\nu(2\theta_0r))^{1/(n-1)}} \\
= \frac{1}{1 + (\log \sigma/2)/(b\nu(2\theta_0r))^{1/(n-1)}} \leq 2^{1/n}.
\]

Also, since \( \varepsilon_1 < 2 \), (4-6) and (4-5) (iv) yield that

\[
\frac{\sigma}{\sigma_1} > \frac{2\exp\left\{-\frac{1}{2}(b\nu(2\theta_0r))^{1/(n-1)}\right\}}{\varepsilon_1 \exp\left\{-(b\nu(2\theta_0r))^{1/(n-1)}\right\}} > \exp\left\{\frac{1}{2}(b\nu(2\theta_0r))^{1/(n-1)}\right\},
\]

and hence

\[
(4-13) \quad \left(\log \frac{\sigma}{\sigma_1}\right)^{n-1} > \frac{b\nu(2\theta_0r)}{2^{n-1}}.
\]

Substituting (4-12) and (4-13) into (4-11) we get

\[
I \leq K_O \omega_{n-1} 2^{-n} \left[ \frac{2^{n-1}}{b} + \frac{2c_1}{n} \right] \leq \frac{K_O \omega_{n-1}}{2b} + \frac{\omega_{n-1} K_O c_1}{2^{n-1} n}.
\]

The same estimate holds for II. Substituting these and the value of \( b \) from (4-3) into (4-10) we obtain

\[
M(\Gamma) \leq \frac{K_O \omega_{n-1}}{b} + \frac{\omega_{n-1} K_O c_1}{2^{n-1} n} = \frac{c_n \log \theta_0}{2} + \frac{\omega_{n-1} K_O c_1}{2^{n-1} n}.
\]

Further by [V1, (10.12)], \( M(\Gamma) \geq c_n \log \theta_0 \) so that

\[
\frac{c_n \log \theta_0}{2} \leq \frac{\omega_{n-1} K_O c_1}{2^{n-2} n}.
\]

But this contradicts our choice of \( \theta_0 \) in (4-3). This proves the lemma.

From this lemma, we find that for \( j \neq J(r) \), \( \mathcal{F}_j \subset B(\theta_0r) \). If \( J(r) \) does not exist, so that \( \mathcal{F}_j \subset B(\theta_0r) \) for all \( j \), we then set \( J(r) = q \). Fix \( j \neq J \), and \( y \in S \). Then \( \Lambda_j = \{\lambda_1, \ldots, \lambda_k\} \subset B(\theta_0r) \), and since \( \Lambda_j \) is a maximal sequence of \( f|B(4\theta_0r+1) \) lifts of \( \gamma_j^j |[0, 1/2] \) we have, for all \( j \neq J \), \( y \in S \),

\[
(4-14) \quad h = n(r, a_j) \leq n(\theta_0r, \alpha_j(y)).
\]
Now set

\[ A_j = S(a_j, \sigma_1) \cap \{ f(B_f \cap \overline{B}(\theta_0r)) \cup f(\partial B(\theta_0r)) \} \]

where \( B_f \) is the branch set, i.e. the set of points where \( f \) is not a local homeomorphism. From [MR, 3.1] we note that for all \( j = 1, \ldots, q \),

\[ \mathcal{H}^{n-1}(S(a_j, \sigma_1) \cap f(B_f \cap \overline{B}(\theta_0r))) = 0. \]

This along with (4-7) implies that \( \mathcal{H}^{n-1}(A_j) = 0 \) for all \( j \). Further, we have that \( \mathcal{H}^{n-1}(\alpha_j^{-1}(A_j)) = 0 \) for all \( j \).

Set

\[ S' = S \setminus \bigcup_{j=1}^q \alpha_j^{-1}(A_j). \]

Comparison of \( n(\theta_0r, \alpha_j(y)) \) with \( n(2\theta_0r, \beta_j(y)) \). For any \( y \in S \), we redefine \( \Lambda^j_y = \{ \lambda_1, \ldots, \lambda_g \} \) to be a maximal sequence of \( f|B(4\theta_0r + 1) \)-liftings of \( \gamma_j^y | [1/2, 1] \), starting at points of \( f^{-1}(\alpha_j(y)) \cap \overline{B}(\theta_0r) \), where \( g = n(\theta_0r, \alpha_j(y)) \).

Let the set of such sequences be \( \Omega^j_y \). For \( \Lambda^j_y \in \Omega^j_y \) we set

\[ N(\Lambda^j_y) = \text{card} \{ \nu : |\lambda_\nu| \subset \overline{B}(2\theta_0r) \} \]

and define

\[ p_j(y) = \sup_{\Lambda^j_y \in \Omega^j_y} N(\Lambda^j_y). \]

Fix an extremal sequence \( \hat{\Lambda}^j_y \in \Omega^j_y \); i.e. \( N(\hat{\Lambda}^j_y) = p_j(y) \). Then by the definition of a maximal sequence of \( f \)-liftings, we have,

\[ p_j(y) \leq n(2\theta_0r, \beta_j(y)). \]

We shall integrate \( n(\theta_0r, \alpha_j(y)) - p_j(y) \) on \( S \) and for this we need the following lemma, which is almost entirely an imitation of [R4, 4.1].

**Lemma 4-19.** Let \( S' \) and \( p_j \) be as in (4-16) and (4-17), then \( p_j \) is upper semi-continuous on \( S' \).

**Proof.** Let \( y_0 \in S' \), then by (4-16) and (4-15), \( \alpha_j(y_0) \notin f(B_f \cap \overline{B}(\theta_0r)) \cup f(\partial B(\theta_0r)) \). So if \( f^{-1}(\alpha_j(y)) \cap \overline{B}(\theta_0r) = \{ x_1, \ldots, x_g \} \), with \( g = n(\theta_0r, \alpha_j(y_0)) \), then \( \{ x_1, \ldots, x_g \} \subset B(\theta_0r) \). Let \( y_1, y_2, \ldots \) be a sequence in \( S' \) such that \( y_h \to y_0 \). The lemma asserts that

\[ \limsup_{h \to \infty} p_j(y_h) \leq p_j(y_0). \]
By choosing a subsequence we may assume that for some integer \( m \), \( p_j(y_h) \equiv m \) holds for all \( h \geq 1 \). Also \( n(\theta_0r, \alpha_j(y)) \) is upper semi-continuous in \( y \) because \( n(r, y) \) is. Hence if \( g_h = n(\theta_0r, \alpha_j(y_h)) \), then \( \limsup_{h \to \infty} g_h \leq g \). We choose and fix the following:

(i) Normal neighbourhoods \( V_1, \ldots, V_g \subset B(\theta_0r) \) of the points \( x_1, \ldots, x_g \), respectively, such that \( \alpha_j(y_h) \in \bigcap_{\nu=1}^g f(V_\nu), h \geq 1 \). (This then implies \( f^{-1}(\alpha_j(y_h)) \cap V_\nu \neq \emptyset \) for all \( \nu \), so that \( g_h \geq g \); i.e. \( g_h = g \).

(ii) For each \( h \geq 1 \) a maximal sequence \( \hat{\Lambda}_y \) such that \( \lambda_{h,1}, \ldots, \lambda_{h,g} \) are \( \Omega_y \) such that \( \lambda_{h,\nu} \) starts at a point \( \zeta_{h,\nu} \in f^{-1}(\alpha_j(y_h)) \cap V_\nu \) for \( \nu = 1, \ldots, g \), and \( \lambda_{h,\nu} \subset B(\theta_0r) \) for \( \nu = 1, \ldots, m \) (since \( p_j(y_h) \equiv m \)).

We divide the \( \nu \)'s, \( 1 \leq \nu \leq g \), into two groups. First let \( \nu \in \{1, \ldots, m\} \) be fixed. We claim that the family \( \{\lambda_{h,\nu} : h = 1, 2, \ldots\} \) is equicontinuous on \( 1/2 \leq t \leq 1 \). Indeed, choose \( \varepsilon > 0 \). For \( t \in [1/2, 1] \) there exists \( \delta_t > 0 \) such that \( U(\xi, f, \varrho) \) is a normal neighbourhood of \( \xi \) with \( d(U(\xi, f, \varrho)) < \varepsilon \) for each \( \xi \in f^{-1}(\gamma_{y_0}(t)) \cap B(\theta_0r) \), and

\[
(4-20) \quad B(\theta_0r) \cap f^{-1}(B(\gamma_{y_0}(t), \theta)) \subset \bigcup_{\xi} U(\xi, f, \varrho): \xi \in f^{-1}(\gamma_{y_0}(t)) \cap B(\theta_0r)
\]

whenever \( 0 < \varrho < \theta \). We cover \( \gamma_{y_0}([1/2, 1]) \) with a finite number of balls \( B(\gamma_{y_0}(t), \delta_t/2) \), say \( B(\eta_u, \varrho_u), u = 1, \ldots, v \). Again by taking a subsequence of the \( \{y_h\} \) we have \( \gamma_{y_0}([1/2, 1]) \subset \bigcup_{u=1}^v B(\eta_u, \varrho_u) \), and \( |\alpha_j(y_h) - \alpha_j(y_0)| \leq \delta = \min_{1 \leq u \leq v} \{|\varrho_u/8\} \), \( |\beta_j(y_h) - \beta_j(y_0)| \leq \delta \) for all \( h \geq 1 \). Fix \( t \in [1/2, 1] \). Since \( \gamma \) is continuous there exists \( u \) such that for any \( h \geq 1 \)

\[
\gamma_{y_h}(t') \in B(\eta_u, 2\varrho_u) \quad \text{for} \quad |t' - t| < \delta.
\]

For each such \( h \) there exists then \( \xi \in f^{-1}(\eta_u) \cap B(\theta_0r) \) such that, by (4-20)

\[
|\lambda_{h,\nu}(t')| \subset U(\xi, f, 2\varrho_u) \quad \text{for} \quad |t' - t| < \delta.
\]

And since \( d(U(\xi, f, 2\varrho_u)) < \varepsilon \) for all \( h \geq 1 \), the family \( \{\lambda_{h,\nu} : h \geq 1\} \) is equicontinuous. By Ascoli’s theorem we may conclude that \( \{\lambda_{h,\nu} : h \geq 1\} \) converges uniformly to a path \( \lambda_- : [1/2, 1] \to B(\theta_0r) \). The path \( \lambda_- \) is a maximal \( f \mid B(4\theta_0r + 1) \)-lift of \( \gamma_{y_0} \mid [1/2, 1] \).

Next fix \( \nu \in \{m+1, \ldots, g\} \). Let the end-point of \( \lambda_{h,\nu} \), in \( B(4\theta_0r+1) \), occur at \( t = t_h < 1 \) and set \( t_0 = \lim sup_{h \to \infty} t_h \). We shall construct a maximal \( f \mid B(4\theta_0r + 1) \)-lift \( \lambda_- \) of \( \gamma_{y_0} \mid [1/2, 1] \) with end-point \( t_0 \) as follows. By taking subsequence of \( \{t_h\} \) again, we may assume \( t_0 = \lim_{h \to \infty} t_h \). As above we conclude that the paths \( \lambda_{h,\nu} \circ G_{t_h} \), where \( G_{t_h} \) maps \([1/2, t_0)\) affinely onto \([1/2, t_h)\), converges uniformly on compact subsets of \([1/2, t_0)\) to a path \( \lambda_- : [1/2, t_0) \to B(4\theta_0r + 1) \) which is then a lift of \( \gamma_{y_0} \mid [1/2, t_0) \). The path has an extension to a path \( \lambda_- : [1/2, t_0] \to B(4\theta_0r + 1) \), by [MRV3, 3.12]. If \( \Delta \subset [1/2, t_0] \) is the largest interval such that
$1/2 \in \Delta$ and $\tilde{\lambda}_\nu \Delta \subset \tilde{B}(4\theta_0 r + 1)$, then $\lambda_\nu = \tilde{\lambda}_\nu \mid \Delta$ is maximal $f \mid B(4\theta_0 r + 1)$-lift of $\gamma_{y_0}^j \mid [1/2, 1]$, and we have constructed paths $\lambda_1, \ldots, \lambda_g$, each of which is a maximal lift of $\gamma_{y_0}^j \mid [1/2, 1]$. Next we will show that $\Lambda_{y_0} = \{\lambda_1, \ldots, \lambda_g\} \in \Omega_{y_0}$; i.e. $\Lambda_{y_0}$ is a maximal sequence of $f \mid B(4\theta_0 r + 1)$-liftings of $\gamma_{y_0}^j \mid [1/2, 1]$, as defined in [R1]. We need only check that

$$\text{card} \{ \nu : \lambda_\nu(t) = x \} \leq i(x, f) \quad \text{for all } t \text{ and } x.$$ 

Let $A = \{ \nu : \lambda_\nu(t) = x \} \neq \emptyset$, and let $U(x, f, \varrho)$ be normal neighbourhood of $x$. There exists $h_0$ such that $|\lambda_{h_\nu} \mid \cap U \neq \emptyset$ for all $h \geq h_0$, $\nu \in A$. Let $h \geq h_0$. We may easily find a point $\eta = \gamma_{y_0}^j(t')$ in $\bigcap_{\nu \in A} \{ f(|\lambda_{h_\nu} \mid \cap U) \}$. Let $\xi_1, \ldots, \xi_w$ be the points in $\{ \lambda_{h_\nu}(t') : \nu \in A \} \subset f^{-1}(\eta) \cap U$. Since $\lambda_{h_1,1}, \ldots, \lambda_{h,g}$ is a maximal sequence, we have for $u = 1, \ldots, w,$

$$\theta_u = \text{card} \{ u : \lambda_{h_\nu}(t') = \xi_u \} \leq i(\xi_u, f).$$

Further, by the choice of $\eta$ and since $U$ is a normal neighbourhood of $x,$

$$\text{card} A = \sum_{u=1}^w \theta_u \leq \sum_{u=1}^w i(\xi_u, f) \leq n(U, \eta) = n(U, x) = i(x, f),$$

where the last inequality is true because $f^{-1}(f(x)) \cap U = \{x\}$. This proves that $\Lambda_{y_0} = \{\lambda_1, \ldots, \lambda_g\}$ obtained above is a maximal sequence of $f \mid B(4\theta_0 r + 1)$-liftings of $\gamma_{y_0}^j \mid [1/2, 1]$, such that $|\lambda_\nu| \subset \tilde{B}(2\theta_0 r)$ for $1 \leq \nu \leq m$. Thus $p_j(y_0) \geq N(\Lambda_{y_0}) = m$. This proves the lemma.

Set

$$q_j(y) = n(\theta_0 r, \alpha_j(y)) - p_j(y).$$

$q_j$, being the difference of two measurable functions, is measurable relative to $S'$. With $\hat{\Lambda}_y^j$ such that $p_j(y) = N(\hat{\Lambda}_y^j)$, for $k = 1, 2, \ldots,$ let

$$E_k^j = \{ y \in S' : q_j(y) = k \}, \quad E_{k'}^j = \{ y + a_j : y \in E_k^j \}$$

$$\Gamma_k^j = \{ \gamma_{y}^j \mid [1/2, 1] : y \in E_k^j \}$$

$$\Delta_k^j = \{ \lambda_{\nu} : \lambda_{\nu} \in \hat{\Lambda}_y^j, y \in E_k^j \mid |\lambda_{\nu|} \not\subset \tilde{B}(2\theta_0 r) \}.$$ 

Then $\mathcal{H}^{n-1}(E_k^j) = \mathcal{H}^{n-1}(E_{k'}^j)$ and by the definition of $E_k^j$ and the fact that $\mathcal{H}^{n-1}(S \setminus S') = 0,$ we have

$$\frac{1}{\omega_{n-1}} \int_S q_j(y) \, d\mathcal{H}^{n-1}(y) = \frac{1}{\omega_{n-1}} \sum_{k=1}^\infty k \mathcal{H}^{n-1}(E_k^j)$$

$$= \frac{1}{\omega_{n-1}} \sum_{k=1}^\infty k \mathcal{H}^{n-1}(E_{k'}^j).$$
We get $\mathcal{H}^{n-1}(E'_k) = (\log(\sigma_2/\sigma_1))^{n-1} M(\Gamma^j_k)$ using a standard estimate, [V1, 7.7]. Thus (4-22) becomes

$$\frac{1}{\omega_{n-1}} \int_S q_j(y) d\mathcal{H}^{n-1}(y) = \frac{1}{\omega_{n-1}} \sum_{k=1}^{\infty} kM(\Gamma^j_k)(\log(\sigma_2/\sigma_1))^{n-1}$$

$$= \frac{1}{\omega_{n-1}} (\log(\sigma_2/\sigma_1))^{n-1} \sum_{k=1}^{\infty} kM(\Gamma^j_k).$$

Further, Väisälä’s inequality [V2, 3.1] gives us $kM(\Gamma^j_k) \leq K_I M(\Delta^j_k)$. Also note that since the $\{\Gamma^j_k\}_{j,k}$ are disjoint, so are the $\{\Delta^j_k\}_{j,k}$, and by [V1, 6.7],

$$\sum_{k=1}^{\infty} \sum_{j \neq J} M(\Delta^j_k) \leq M\left(\bigcup_{j \neq J} \bigcup_{k=1}^{\infty} \Delta^j_k\right).$$

Using these two estimates, summing over $j \neq J$ and recalling $\sigma_1$ from (4-6) we get

$$\sum_{j \neq J} \frac{1}{\omega_{n-1}} \int_S q_j(y) d\mathcal{H}^{n-1}(y) \leq \frac{1}{\omega_{n-1}} (\log(\sigma_2/\sigma_1))^{n-1} K_I M\left(\bigcup_{j \neq J} \bigcup_{k=1}^{\infty} \Delta^j_k\right)$$

$$\leq \frac{1}{\omega_{n-1}} (\log(\sigma_2/\sigma_1))^{n-1} K_I \frac{\omega_{n-1}}{(\log 2)^{n-1}}$$

$$\leq \frac{K_I}{(\log 2)^{n-1}} b\nu(2\theta_0 r).$$

If $y \in S$, then by (4-14), (4-21) and (4-18) we have

$$n(r, a_j) \leq q_j(y) + n(2\theta_0 r, \beta_j(y)).$$

On integrating over $S$, and summing over $j \neq J$, we obtain using (4-23),

$$\sum_{j \neq J} n(r, a_j) \leq \frac{K_I b\nu(2\theta_0 r)}{(\log 2)^{n-1}} + \sum_{j \neq J} \nu(2\theta_0 r, S(a_j, \sigma_2)).$$

But from (2-3) and (4-8)

$$\nu(2\theta_0 r, S(a_j, \sigma_2)) \leq \nu(4\theta_0 r) + c_1 (b\nu(2\theta_0 r))^{1-1/n}.$$
Finally we use (4-5) (v) in the above inequality to get
\[ \sum_{j \neq J} n(r, a_j) \leq \frac{K_I b}{(\log 2)^{n-1}} \nu(2\theta_0 r) + (q - 1)\nu(4\theta_0 r) + \nu(2\theta_0 r) \]
\[ \leq \left[ q + \frac{K_I b}{(\log 2)^{n-1}} \right] \nu(4\theta_0 r). \]

In the situation when \( \tau > 1 \) this gives us
\[ \sum_{j \neq J} n(r, a_j) \leq \left[ q + \frac{K_I b}{(\log 2)^{n-1}} \nu(4\theta_0 r, \tau) \right]. \]

**Step III:** Recall \( r_0 = \max(r_1, r_2) \). Fix \( r \geq r_0 \), and use (2-2) to replace \( \nu(4\theta_0 r, \tau) \) by \( \nu(8\theta_0 r) \) to get,
\[ \sum_{j \neq J} n(r, a_j) \leq \left[ q + \frac{2K_I b}{(\log 2)^{n-1}} \right] \nu(8\theta_0 r), \]
and by (4-4) (ii) we obtain,
\[ \sum_{j \neq J} n(r, a_j) \leq \left[ q + \frac{4K_I b}{(\log 2)^{n-1}} + 2 \right] A(16\theta_0 r). \]

This proves Theorem 4-1.

**Theorem 4-26.** For \( n \geq 2 \), and \( K \geq 1 \), let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a nonconstant \( K \)-qm function. Then there exist constants \( C_1 = C_1(n, K) > 1, \theta_1 = \theta_1(n, K) > 1 \) such that for every \( a_1, \ldots, a_q \in \mathbb{R}^n, q > 1 \), there exists a set \( E \subset [1, \infty) \) with \( \int_E d\lambda/\lambda = \infty \) such that
\[ \limsup_{r \to \infty} \sum_{j=1}^q \left[ \frac{n(r, a_j)}{A(\theta_1 r)} - 1 \right]_+ \leq C_1. \]

**Proof of Theorem 4-26.** We first use Theorem 3-1 with some fixed value of \( C \), say \( C = 2 \), and obtain a corresponding \( \theta \) and a set \( E \subset [1, \infty) \) with \( \int_E d\lambda/\lambda = \infty \), such that for \( j = 1, \ldots, q, r \in E \),
\[ n(r, a_j) \leq 2A(\theta r). \]
We will then show that (4-27) holds with

\[ \theta_1 = \max(16\theta_0, \theta), \quad C_1 = 4 + \frac{4K_I b}{(\log 2)^{n-1}} \]

where \( b \) has been defined in (4-3). As in Theorem 4-1, we assume that \( a_1, \ldots, a_q \in B(\tau/2) \) and \( \sigma > 0 \) such that \( B_{\sigma \tau}(a_j) \subset B(\tau/2) \) and \( B_{\sigma \tau}(a_j) \) are disjoint.

Now apply Theorem 4-1 and obtain \( r_0 = r_0(\sigma, \tau, q, f) > 0 \). Fix \( r \in E \) such that \( r \geq r_0 \). If \( ((n(r, a_j)/A(\theta_1 r)) - 1) \leq 0 \) for \( (q - 1) \) values of \( j \), then by (4-28) there is nothing to prove. So let \( Q = \{1 \leq j \leq q : ((n(r, a_j)/A(\theta_1 r)) - 1) > 0\} \) for all \( j \in Q \). We assume \( \text{card}Q = q' \geq 2 \).

Again we apply Theorem 4-1, to the same function \( f \), but using the set \( \{a_j : j \in Q\} = \{a_j'\} \). Note that the same \( \sigma \) and \( \tau \), as for the \( \{a_j\} \), work for \( \{a_j'\} \). Theorem 4-1 yields \( r_0' = r_0'(\sigma, \tau, q', f) \). From (4-4) and (4-5) (v) we see that we may choose \( r_0'(\sigma, \tau, q', f) = r_0(\sigma, \tau, q, f) \); i.e. \( r_0' = r_0 \). So we have for \( r \in E \), \( r \geq r_0 = r_0' \), by (4-2),

\[
\sum_{\substack{j \in Q \setminus j \neq J}} n(r, a_j) \leq \left[q' + \frac{4K_I b}{(\log 2)^{n-1}} + 2\right] A(16\theta_0 r) \leq \left[q' + \frac{4K_I b}{(\log 2)^{n-1}} + 2\right] A(\theta_1 r);
\]

i.e.,

\[
\sum_{\substack{j \in Q \setminus j \neq J}} \left[\frac{n(r, a_j)}{A(\theta_1 r)} - 1\right] \leq \left[3 + \frac{4K_I b}{(\log 2)^{n-1}}\right].
\]

For \( j = J \), since \( r \in E \), we have from (4-28) that

\[
n(r, a_j) \leq 2A(\theta r) \leq 2A(\theta_1 r).
\]

Hence

\[
\sum_{j \in Q} \left[\frac{n(r, a_j)}{A(\theta_1 r)} - 1\right] \leq \left[4 + \frac{4K_I b}{(\log 2)^{n-1}}\right] = C_1.
\]

And by the definition of \( Q \),

\[
\sum_{j=1}^{q} \left[\frac{n(r, a_j)}{A(\theta_1 r)} - 1\right] + \leq C_1, \quad \text{where } r \in E, \quad r \geq r_0. \quad \text{The theorem is proved.}
\]
A converse defect relation for quasimeromorphic mappings

References


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