Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 21, 1996, 89–100

NORMAL FAMILIES, ORDERS OF ZEROS, AND OMITTED VALUES

Chen Huaihui and Peter Lappan

Nanjing Normal University, Department of Mathematics
Nanjing, Jiangsu 210024, Peoples Republic of China

Michigan State University, Department of Mathematics
East Lansing, Michigan 48824, U.S.A.

Abstract. For k a positive integer, Ω a region in the complex plane, and α a complex number, let $\mathscr{M}_k(\Omega,\alpha)$ denote the collection of all functions f meromorphic in Ω such that each zero of $f-\alpha$ has multiplicity at least k. Let D denote the unit disk in the complex plane. We give three conditions, each of which is sufficient for a subset $\mathscr{F}$ of $\mathscr{M}_k(D,\alpha)$ to be a normal family. These conditions are: (1) for each compact subset K of D and for some $\beta > 0$ there exists a constant $M_K(\beta)$ (depending on both K and β) such that, for each $f \in \mathscr{F}$,

$$\{z \in K : |f(z)| < \beta\} \subset \{z \in K : |f^{(k)}(z)| \le M_K(\beta)\};$$

(2) for $\lambda > 2/k$ and for each compact subset K of D there exists a constant $C_{K,\lambda}$ (depending on both K and λ) such that

$$\iint_{\{z\in K:|f(z)|<1\}} |f^{(k)}(z)|^{\lambda}\,dx\,dy < C_{K,\lambda}$$

for each $f \in \mathscr{F}$; (3) for each compact subset K of D there is a constant $M_k(K)$ such that the product of the spherical derivatives of two or three consecutive derivatives of f, up to the derivative of order $k-1$, is uniformly bounded by $M_k(K)$ for $z \in K$, $f \in \mathscr{F}$. These results are suggested by and build on previous results of the first author and Gu Yong-Xing, Rauno Aulaskari and the second author, and the second author alone.

1. Preliminaries

Let $\mathbf{C}$ denote the complex plane and let $D = \{z \in \mathbf{C} : |z| < 1\}$ be the unit disk. A family $\mathscr{F}$ of functions meromorphic in a region $\Omega \subset \mathbf{C}$ is called a *normal family* if each sequence in $\mathscr{F}$ contains a subsequence which converges uniformly on each compact subset of Ω. (Here, we allow the subsequence to converge uniformly on each compact subset to the function which is identically ∞ in the sense that the subsequence of reciprocals converges uniformly on each compact subset to the function which is identically zero.) It is well known that a family $\mathscr{F}$ of meromorphic functions is a normal family if and only if it is locally a normal family, that is, it is a normal family on a neighborhood of each point of Ω.

1991 Mathematics Subject Classification: Primary 30D45.

Thus, all theorems about whether a family of functions is a normal family or not can be formulated in terms of families defined on disks. Although some of our results are formulated in general regions, the proofs are all "localized" to disks.

Let $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$ denote the spherical derivative of f. A well known result due to F. Marty [8] says that a family $\mathscr{F}$ of functions meromorphic in D is a normal family if and only if for each compact subset K of D there exists a constant M_K such that $f^{\#}(z) \le M_K$ for each $f \in \mathscr{F}$ and each $z \in K$. A single function f meromorphic in D is called a *normal function* if $\sup\big\{(1-|z|^2)f^{\#}(z) : z \in D\big\} < \infty$ (see [6]).

We will make repeated use of the following results, due, repectively, to J. Lohwater and Ch. Pommerenke [7], L. Zalcman [11], X. Pang [9], and H.H. Chen and Y.X. Gu [3].

Theorem LP [7, Theorem 1]. *If a function f meromorphic in D is not a normal function, then there exist sequences $\{z_n\}$ and $\{\varrho_n\}$ such that $z_n \in D$, $|z_n| \to 1$, $\varrho_n > 0$, $\varrho_n/(1-|z_n|) \to 0$, and the sequence $\{g_n(t) = f(z_n + \varrho_n t)\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function g.*

Theorem Z [11, Lemma, p. 814]. *Let $\mathscr{F}$ be a family of functions meromorphic on D. If $\mathscr{F}$ is not a normal family, then there exist a compact subset K of D, a sequence of functions $\{f_n\}$ in $\mathscr{F}$, a sequence of points $\{z_n\}$ in K, and a sequence of positive real numbers $\{\varrho_n\}$, where $\varrho_n \to 0$, such that the sequence of functions $\{g_n(t) = f_n(z_n + \varrho_n t)\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function g.*

Theorem P [9]. *Let $\mathscr{F}$ be a family of functions meromorphic on D. If $\mathscr{F}$ is not a normal family and $-1 < \lambda < 1$, then there exist a compact subset K of D, a sequence of functions $\{f_n\}$ in $\mathscr{F}$, a sequence of points $\{z_n\}$ in K, and a sequence of positive real numbers $\{\varrho_n\}$, where $\varrho_n \to 0$, such that the sequence of functions $\{g_n(t) = (\varrho_n)^\lambda f_n(z_n + \varrho_n t)\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function g.*

Theorem CG [3, Theorem 2, p. 677]. *Let k be a positive number and let $\mathscr{F}$ be a family of meromorphic functions in D with the property that each function in $\mathscr{F}$ has only zeros of degree at least k. If $\mathscr{F}$ is not a normal family in each neighborhood of the point $z_0 \in D$, then, for any positive number α with $\alpha < k$, there exist a sequence of points $\{z_n\}$ in D, a sequence $\{\varrho_n\}$ of positive numbers, and a sequence of functions $\{f_n\}$ in $\mathscr{F}$ such that $z_n \to z_0$, $\varrho_n \to 0$, and the sequence of functions $\big\{(\varrho_n)^{-\alpha} f_n(z_n + \varrho_n t)\big\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function g.*

If Ω is an open set, k is a positive integer, and $\alpha \in \mathbf{C}$, we denote by $\mathscr{M}_k(\Omega, \alpha)$ the collection of all functions f meromorphic on Ω such that each zero of $f - \alpha$

has multiplicity at least k. We will make use of the following three properties of the class $\mathscr{M}_k(\Omega,\alpha)$:

(a) if f omits the value α on Ω then $f\in\mathscr{M}_k(\Omega,\alpha)$ for each positive integer k,

(b) if Ω is a region on the complex plane and $f\in M_k(\Omega,\alpha)$ then either $f^{(k)}$, the kth derivative of f, is not identically zero or $f\equiv\alpha$, and

(c) if $\{f_n\}\subset\mathscr{M}_k(\Omega,\alpha)$ is a sequence such that $\{f_n\}$ converges uniformly on each compact subset of the region Ω to a function f, then $f\in\mathscr{M}_k(\Omega,\alpha)$.

In Section 2 below, Theorem 1 gives a slight improvement of a result of the first author and Gu Yong-Xing [3, Corollary 1], using methods different from those used in [3]. Theorem 2 is a modification of Theorem 1 to normal functions. In Section 3, our Theorems 3 and 4 give some improvements to results of R. Aulaskari and the second author [1, 2]. Finally, in Section 4, our Theorems 5, 6, and 7 give some extensions to results by the second author [5].

2. Sufficient conditions for normal families and functions

We begin with a slight improvement of a result of the first author and Gu Yong-Xing [3, Corollary 1]. The method of proof here is quite different from that appearing in [3].

Theorem 1. *Let $\mathscr{F}$ be a family of functions meromorphic in D, let k be a positive integer, let $\alpha\in\mathbf{C}$, and let $\mathscr{F}\subset\mathscr{M}_k(D,\alpha)$. If, for each compact subset K in D and for some $\beta>0$ there exists a constant $M_K(\beta)$ such that*

$$\bigl\{z\in K:|f(z)|<\beta\bigr\}\subset\bigl\{z\in K:|f^{(k)}(z)|<M_K(\beta)\bigr\}\qquad\text{for each } f\in\mathscr{F},$$

then $\mathscr{F}$ is a normal family in D.

Proof. Suppose $\mathscr{F}$ is not a normal family in D. By Theorem Z there exist a sequence of functions $\{f_n\}$ in $\mathscr{F}$, a number r, $0<r<1$, a sequence of points $\{z_n\}$ in D satisfying $|z_n|<r$ for each n, and a sequence of positive numbers $\{\varrho_n\}$ for which $\varrho_n\to 0$ and the sequence $\{g_n(t)=f_n(z_n+\varrho_n t)\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function $g(t)$. The hypotheses of the theorem guarantee that all the zeros of $g-\alpha$, if any, are of order at least k. This means that $g^{(k)}$ is not identically zero.

Let $K=\bigl\{z\in D:|z|\le\frac{1}{2}(1+r)\bigr\}$. If $t\in\mathbf{C}$ and $|g(t)|<\beta$, then we have

$$|g^{(k)}(t)|=\lim_{n\to\infty}(\varrho_n)^k|f^{(k)}(z_n+\varrho_n t)|<\lim_{n\to\infty}(\varrho_n)^k M_K(\beta)\to 0,$$

and hence $g^{(k)}(t)=0$. But the set $\bigl\{t\in\mathbf{C}:|g(t)|<\beta\bigr\}$ is a non-empty open subset of $\mathbf{C}$, and so our reasoning in the previous sentence leads to the conclusion that $g^{(k)}\equiv 0$, in violation of the last sentence of the previous paragraph. Thus, the assumption that $\mathscr{F}$ is not a normal family leads to a contradiction, and the theorem is proved.

A similar result giving a sufficient condition for a function to be a normal function is the following.

Theorem 2. *Let f be a function meromorphic in D, let k be a positive integer, let $\alpha \in \mathbf{C}$, and let $f \in \mathscr{M}_k(D,\alpha)$. If there exists a number $\beta > 0$ and a constant M_β such that*

$$\big\{z \in D : |f(z)| < \beta\big\} \subset \big\{z \in D : |f^{(k)}(z)|(1-|z|)^k < M_\beta\big\},$$

then f is a normal function.

Proof. Suppose that f is not a normal function. By Theorem LP there exist a sequence of points $\{z_n\}$ in D, and a sequence of positive numbers $\{\varrho_n\}$, where $\varrho_n/(1-|z_n|) \to 0$, such that the sequence $\{g_n(t) = f(z_n + \varrho_n t)\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function $g(t)$. The hypotheses on f require that all the zeros of $g-\alpha$ are of order at least k. Thus $g^{(k)}(t)$ is not identically zero.

Now suppose that t is fixed and $|g(t)| < \beta$. This means that $|g_n(t)| < \beta$ for n sufficiently large, and thus,

$$\begin{aligned}
|g_n^{(k)}(t)| &= (\varrho_n)^k |f^{(k)}(z_n + \varrho_n t)| \\
&= \frac{(\varrho_n)^k}{(1-|z_n+\varrho_n t|)^k} |f^{(k)}(z_n + \varrho_n t)|(1-|z_n+\varrho_n t|)^k \\
&\le \frac{(\varrho_n)^k}{(1-|z_n|)^k} \frac{(1-|z_n|)^k}{(1-|z_n+\varrho_n t|)^k} |f^{(k)}(z_n + \varrho_n t)|(1-|z_n+\varrho_n t|)^k \\
&\le \Big(\frac{\varrho_n}{1-|z_n|}\Big)^k \Big(\frac{1-|z_n|}{1-|z_n+\varrho_n t|}\Big)^k M_\beta \to 0.
\end{aligned}$$

It follows that $g^{(k)}(t) = 0$ whenever $|g(t)| < \beta$, and this means that $g^{(k)}(t) \equiv 0$, contradicting the last sentence of the previous paragraph. Thus, it is not possible for f to be a non-normal function, and this proves the theorem.

3. Integral criteria

Our next result is suggested by a result due to R. Aulaskari and the second author [1, Theorem 1].

Theorem AL [1, Theorem 1, p. 29]. *Let $\mathscr{F}$ be a family of functions meromorphic in D. Then $\mathscr{F}$ is a normal family if and only if, for each R, $0 < R < 1$, and each $\lambda > 2$, there exists a constant $C_{\lambda,R}$ such that*

$$\iint_{|z|\le R} \big(f^{\#}(z)\big)^\lambda \, dx\, dy < C_{\lambda,R} \qquad \text{for each } f \in \mathscr{F}.$$

Theorem AL requires that the exponent λ be greater than 2. By imposing the condition that $\mathscr{F} \subset \mathscr{M}_k(D,\alpha)$ for some fixed k and α, we can change this restriction on λ and, at the same time, perform the integration over a potentially much smaller set, as follows.

Theorem 3. *Let k be a positive number and let $\alpha \in \mathbf{C}$. Let $\mathscr{F}$ be a family of functions meromorphic on D such that $\mathscr{F} \subset \mathscr{M}_k(D,\alpha)$. Let λ be a real number such that $\lambda > 2/k$. If for each compact subset K of D there exists a constant $C_{K,\lambda}$ such that $\iint_{K(f)} |f^{(k)}(z)|^\lambda \, dx\, dy < C_{K,\lambda}$ for each $f \in \mathscr{F}$, where $K(f) = \{z \in K : |f(z)| < 1\}$, then $\mathscr{F}$ is a normal family.*

We remark that the set $K(f)$ varies with the function f.

Proof. Suppose that $\mathscr{F}$ is not a normal family. By Theorem Z there exists a sequence $\{f_n\}$ in $\mathscr{F}$, a compact subset K_0 of D together with a sequence $\{z_n\}$ of points in K_0, and a sequence of positive numbers $\{\varrho_n\}$ where $\varrho_n \to 0$ such that the sequence of functions $\{g_n(t) = f_n(z_n + \varrho_n t)\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function $g(t)$. Fix $s > 0$ and let $z = z_n + \varrho_n t$. For $|t| \le s$ and n sufficiently large, we have that $z \in K_0$, so we may assume in what follows that $z \in K_0$. Also, we have $g_n^{(k)}(t) = (\varrho_n)^k f_n^{(k)}(z_n + \varrho_n t)$. Using the notation $z = x + iy = z_n + \varrho_n t$ and $t = u + iv$, we have $dx\, dy = (\varrho_n)^2 \, du\, dv$, so that if

$$G_{s,n} = \big\{t \in \mathbf{C} : |g_n(t)| < 1,\ |t| \le s\big\}$$

and

$$F_{s,n} = \big\{z = z_n + \varrho_n t : |t| \le s,\ |f_n(z)| < 1\big\},$$

we have

$$\begin{aligned}
\iint\limits_{G_{s,n}} |g_n^{(k)}(t)|^\lambda \, du\, dv &= \iint\limits_{G_{s,n}} (\varrho_n)^{\lambda k} |f_n^{(k)}(z_n + \varrho_n t)|^\lambda \, du\, dv \\
&= \iint\limits_{F_{s,n}} (\varrho_n)^{\lambda k - 2} |f_n^{(k)}(z)|^\lambda \, dx\, dy \\
&\le (\varrho_n)^{\lambda k - 2} \iint\limits_{K_0(f_n)} |f_n^{(k)}(z)|^\lambda \, dx\, dy \le (\varrho_n)^{\lambda k - 2} C_{K_0,\lambda} \to 0,
\end{aligned}$$

since $\varrho_n \to 0$ and $\lambda k - 2 > 0$. But, we also have that

$$\iint\limits_{G_{s,n}} |g_n^{(k)}(t)|^\lambda \, du\, dv \to \iint\limits_{K_s} |g^{(k)}(t)|^\lambda \, du\, dv$$

where $K_s = \{t \in \mathbf{C} : |g(t)| < 1,\ |t| \le s\}$. Thus, the integral $\iint_{K_s} |g^{(k)}(t)|^\lambda \, du\, dv$ must be zero for each choice of s, and it follows that $g^{(k)} \equiv 0$. But the hypotheses about the family $\mathscr{F}$ guarantee that $g^{(k)}$ cannot be identically zero. It follows that $\mathscr{F}$ must be a normal family.

Next, we give a version of the result of Theorem 3 for a normal function. This is an extension of both [1, Theorem 2, p. 32] and [2, Theorem, p. 438].

Theorem 4. *Let k be a positive integer, let $\alpha \in \mathbf{C}$, and let f be a function meromorphic in D. If $f \in \mathscr{M}_k(D,\alpha)$, if $\lambda > 2/k$, and if there exists a constant C_λ such that*

$$\iint_{F(f)} |f^{(k)}(z)|^\lambda (1-|z|)^{\lambda k-2}\, dx\, dy < C_\lambda,$$

where $F(f) = \{z \in D : |f(z)| < 1\}$, then f is a normal function.

Proof. Suppose that f is not a normal function. By Theorem LP there exists a sequence $\{z_n\}$ of points in D, and a sequence of positive numbers $\{\varrho_n\}$ where $\varrho_n/(1-|z_n|) \to 0$ such that the sequence of functions $\{g_n(t) = f(z_n + \varrho_n t)\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function $g(t)$. Fix $s > 0$ and let $z = z_n + \varrho_n t$. For $|t| \le s$ and n sufficiently large, we will have that $z \in D$, so we may assume in what follows that $z \in D$. Also, we have $g_n^{(k)}(t) = (\varrho_n)^k f^{(k)}(z_n + \varrho_n t)$. Using the notation $z = x + iy = z_n + \varrho_n t$ and $t = u + iv$, we have $dx\, dy = (\varrho_n)^2\, du\, dv$, so that if $G_{s,n} = \{t \in \mathbf{C} : |g_n(t)| < 1,\ |t| \le s\}$ and $F_{s,n} = \{z = z_n + \varrho_n t : |t| \le s,\ |f(z)| < 1\}$, we have

$$\begin{aligned}
\iint_{G_{s,n}} |g_n^{(k)}(t)|^\lambda\, du\, dv &= \iint_{G_{s,n}} (\varrho_n)^{\lambda k} |f_n^{(k)}(z_n + \varrho_n t)|^\lambda\, du\, dv \\
&= \iint_{G_{s,n}} \Big(\frac{\varrho_n}{1-|z_n+\varrho_n t|}\Big)^{\lambda k-2} |f^{(k)}(z_n+\varrho_n t)|^\lambda \\
&\quad \times (1-|z_n+\varrho_n t|)^{\lambda k-2} (\varrho_n)^2\, du\, dv \\
&\le \iint_{G_{s,n}} (a_n)^{\lambda k-2} |f^{(k)}(z_n+\varrho_n t)|^\lambda (1-|z_n+\varrho_n t|)^{\lambda k-2} (\varrho_n)^2\, du\, dv \\
&= (a_n)^{\lambda k-2} \iint_{F_{s,n}} |f_n^{(k)}(z)|^\lambda (1-|z|)^{\lambda k-2}\, dx\, dy \le (a_n)^{\lambda k-2} C_\lambda \to 0,
\end{aligned}$$

where

$$a_n = \sup\Big\{\frac{\varrho_n}{1-|z_n+\varrho_n t|} : |t| \le s\Big\}$$

(and $a_n \to 0$ since $\varrho_n/(1-|z_n|) \to 0$ and $(1-|z_n|)/(1-|z_n+\varrho_n t|) \to 1$ for $|t| \le s$). But, we also have that

$$\iint_{G_{s,n}} |g_n^{(k)}(t)|^\lambda\, du\, dv \to \iint_{K_s} |g^{(k)}(t)|^\lambda\, du\, dv$$

where $K_s = \{t \in \mathbf{C} : |g(t)| < 1,\ |t| \le s\}$. Thus, the integral $\iint_{K_s} |g^{(k)}(t)|^\lambda \, du\, dv$ must be zero for each choice of s, and it follows that $g^{(k)} \equiv 0$. But the hypotheses about the function f guarantee that $g^{(k)}$ cannot be identically zero. It follows that f must be a normal function.

4. Extensions of the spherical derivative

For a function meromorphic in D, define a product of spherical derivatives by

$$P_n(f)(z) = \prod_{j=0}^{n-1} (f^{(j)})^{\#}(z).$$

In [5], the second author proved that if $\mathscr{F}$ is a normal family of meromorphic functions in D, then for each positive integer n and each compact subset K of D, there exists a constant $M_n(K)$ such that $P_n(f)(z) \le M_n(K)$ for each $f \in \mathscr{F}$ and each $z \in K$. (The case $n = 2$ was suggested by Yamashita [10].) The converse of this result is not true as the example $\mathscr{F} = \{nz\}$ shows, since $P_2(f) \equiv 0$ for each $f \in \mathscr{F}$ but $\mathscr{F}$ is not a normal family. However, if we can guarantee that, for each function $f \in \mathscr{F}$, $P_n(f)(z)$ is not identically zero, then the converse has at least a chance of being valid. We show here that such a converse is valid for $n \le 3$, but is not valid for $n \ge 4$.

Theorem 5. *Let α be a complex number and let k be a positive integer. Suppose that $\mathscr{F}$ is a family of functions meromorphic in D such that $\mathscr{F} \subset \mathscr{M}_k(D, \alpha)$. Further, suppose that for each compact subset K of D there exists a constant $M_k(K)$ such that $P_k(f)(z) \le M_k(K)$ for each $f \in \mathscr{F}$ and each $z \in D$. If $k \le 3$, then $\mathscr{F}$ is a normal family. If $k \ge 4$, then there are examples for which $\mathscr{F}$ is not a normal family.*

Proof. Let $k = 2$ and suppose that $\mathscr{F}$ is not a normal family in D. By Theorem Z, there exists a sequence of functions $\{f_n\}$ in $\mathscr{F}$, a real number r, where $0 < r < 1$, a sequence of points $\{z_n\}$ in D such that $|z_n| < r$ for each n, and a sequence of positive real numbers $\{\varrho_n\}$, where $\varrho_n \to 0$, such that the sequence of functions $\{g_n(t) = f_n(z_n + \varrho_n t)\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function $g(t)$. The hypotheses imply that $g \in \mathscr{M}_2(\mathbf{C}, \alpha)$ and that g'' is not identically zero.

Fix $s > 0$. Since $\varrho_n \to 0$, we have that

$$\{w = z_n + \varrho_n t : |t| \le s\} \subset K_0 = \{z \in D : |z| \le \tfrac{1}{2}(1 + r)\}$$

for each n sufficiently large. Then

$$\begin{aligned} P_2(g)(t) &= \lim_{n\to\infty} P_2(g_n)(t) = \lim_{n\to\infty} \frac{\varrho_n |f_n'(z_n + \varrho_n t)|}{1 + |f_n(z_n + \varrho_n t)|^2} \cdot \frac{(\varrho_n)^2 |f_n''(z_n + \varrho_n t)|}{1 + |\varrho_n f_n'(z_n + \varrho_n t)|^2} \\ &= \lim_{n\to\infty} \varrho_n P_2(f_n)(z_n + \varrho_n t) \frac{(\varrho_n)^2 \big(1 + |f_n'(z_n + \varrho_n t)|^2\big)}{1 + |\varrho_n f_n'(z_n + \varrho_n t)|^2} = 0 \end{aligned}$$

because $\varrho_n \to 0$ implies that, for n sufficiently large,

$$\frac{(\varrho_n)^2\big(1+|f_n'(z_n+\varrho_n t)|^2\big)}{1+|\varrho_n f_n'(z_n+\varrho_n t)|^2} = \frac{(\varrho_n)^2+(\varrho_n)^2|f_n'(z_n+\varrho_n t)|^2}{1+|\varrho_n f_n'(z_n+\varrho_n t)|^2} < 1,$$

and $P_2(f_n)(z_n+\varrho_n t) \le M_2(K_0)$, by hypothesis. But this calculation says that $P_2(g) \equiv 0$, which implies $g'' \equiv 0$, so we have contradicted the last sentence of the previous paragraph. Thus, we must have that $\mathscr{F}$ is a normal function when $k=2$.

Now let $k=3$, and let $0<\lambda<1$. If $\mathscr{F}$ is not a normal family on D, then by Theorem P there exists a sequence of functions $\{f_n\}$ in $\mathscr{F}$, a sequence of points $\{z_n\}$ in D such that $z_n \to z_0 \in D$, and a sequence $\{\varrho_n\}$ of positive real numbers with $\varrho_n \to 0$ such that the sequence of functions $\{g_n(t) = (\varrho_n)^{-\lambda} f_n(z_n+\varrho_n t)\}$ converges uniformly on each compact subset of the complex plane to a non-constant meromorphic function $g(t)$. As in the previous case, $g \in \mathscr{M}_3(\mathbf{C},\alpha)$ and $g^{(3)}$ is not identically zero on $\mathbf{C}$.

Fix t_0 on the complex plane such that $g(t_0)$ is finite. Let r be such that $|z_0|<r<1$ and fix $s>|t_0|$. Since $\varrho_n \to 0$, we have that

$$K_{n,s} = \{w = z_n+\varrho_n t : |t| \le s\} \subset K_0 = \{z \in D : |z| \le \tfrac{1}{2}(1+r)\}$$

for n sufficiently large. Then, in some neighborhood of t_0, we have that the first three derivatives of g are all finite, $g^{(3)}$ is not identically zero, and $f_n(z_n+\varrho_n t) = (\varrho_n)^{\lambda} g_n(t) \to 0$, since $\varrho \to 0$ and $g_n(t)$ converges uniformly on compact subsets to $g(t)$. It follows that $\{F_n(t) = f_n(z_n+\varrho_n t)\}$ converges uniformly to zero in a neighborhood of t_0. Now, letting $u_n = z_n+\varrho_n t$ so that $g_n(t) = (\varrho_n)^{-\lambda} f_n(u_n)$, we have

$$\begin{aligned} P_3(g_n)(t) = (\varrho_n)^{6-3\lambda} P_3(f_n)(u_n) \times \frac{1+|f_n(u_n)|^2}{1+|(\varrho_n)^{-\lambda} f_n(u_n)|^2} \\ \times \frac{1+|f_n'(u_n)|^2}{1+|(\varrho_n)^{1-\lambda} f_n'(u_n)|^2} \times \frac{1+|f_n''(u_n)|^2}{1+|(\varrho_n)^{2-\lambda} f_n''(u_n)|^2}. \end{aligned}$$

Of these quantities, we have that $P_3\big(f_n(u_n)\big) \le M_3(K_0)$ by hypothesis, and $\{F_n(t) = f_n(u_n)\}$ converges to zero in a neighborhood of t_0. It follows that

$$H_n(t) = \frac{1+|f_n(u_n)|^2}{1+|(\varrho_n)^{-\lambda} f_n(u_n)|^2}$$

is uniformly bounded in a closed neighborhood of t_0. Thus, in a neighborhood of t_0,

$$P_3\big(g_n(t)\big) \le C(\varrho_n)^{6-3\lambda} \frac{1+|f_n'(u_n)|^2}{1+|(\varrho_n)^{1-\lambda} f_n'(u_n)|^2} \cdot \frac{1+|f_n''(u_n)|^2}{1+|(\varrho_n)^{2-\lambda} f_n''(u_n)|^2},$$

where C is a constant not greater than $M_3(K_0)$. But

$$(\varrho_n)^{6-3\lambda}\frac{1+|f_n'(u_n)|^2}{1+|(\varrho_n)^{1-\lambda}f_n'(u_n)|^2}\cdot\frac{1+|f_n''(u_n)|^2}{1+|(\varrho_n)^{2-\lambda}f_n''(u_n)|^2}$$
$$=(\varrho_n)^{\lambda}\frac{(\varrho_n)^{2-2\lambda}\big(1+|f_n'(u_n)|^2\big)}{1+|(\varrho_n)^{1-\lambda}f_n'(u_n)|^2}\cdot\frac{(\varrho_n)^{4-2\lambda}\big(1+|f_n''(u_n)|^2\big)}{1+|(\varrho_n)^{2-\lambda}f_n''(u_n)|^2}\to 0$$

since $\varrho_n \to 0$ and all of λ, $2-2\lambda$, and $4-2\lambda$ are all positive, which means that each of the last two factors is bounded when t is confined to a neighborhood of t_0. Thus, we conclude that $P_3(g_n)(t) \to P_3(g)(t) \equiv 0$ on a neighborhood of t_0, which means that g is a constant function on the complex plane, in violation of the supposition that g is a non-constant function. It follows that $\mathscr{F}$ must be a normal family.

To give a counterexample for the cases $k \ge 4$, we can use the family $\mathscr{F} = \{h_n(z) = e^{nz}\}$, which is not a normal family. Each function in $\mathscr{F}$ omits the value zero. If $f_n(z) = e^{nz}$, we claim that $P_k(f_n)$ is bounded for each $k \ge 4$. First suppose $k \ne 5$. Let T_k denote the kth triangular number, that is, $T = \frac{1}{2}k(k+1)$, let $q_k = [\frac{1}{2}k]$, where $[\cdot]$ denotes the greatest integer function, and let $N_k = \{2, 4, \ldots, 2k-2\}$. Then there exists a subset $S_k \subset N_k$, where S_k contains q_k elements, such that $\sum_{j\in S_k} j$ is equal to either T_k or $T_k + 1$, whichever is an even number. Then we can write $P_k(f_n)$ in the form

$$\prod_{2j\in S_k}\frac{|n^j e^{nz}|^2}{1+|n^j e^{nz}|^2}\times\prod_{2j\in(N_k-S_k)}\frac{1}{1+|n^j e^{nz}|^2}\times\frac{a_k}{1+|e^{nz}|^2},$$

where $a_k = |e^{nz}|$ if T_k is even and k is odd, $a_k = 1/n$ if k is even and T_k is odd, $a_k = 1$ if both T_k and k are even, and $a_k = |e^{nz}|/n$ if both T_k and k are odd. Thus, in the product, each of the k factors is less than 1, so it follows that $P_k(f_n)$ is less than 1 whenever $k \ge 4$, $k \ne 5$. (The set S_k does not exist for $k = 5$.) To cover the case $k = 5$, we write $P_5(f_n)$ in the form

$$\prod_{j=3}^{4}\frac{|n^j e^{nz}|^2}{1+|n^j e^{nz}|^2}\times\prod_{\substack{j=0\\ j\ne 1}}^{2}\frac{1}{1+|n^j e^{nz}|^2}\times\frac{|ne^{nz}|}{1+|ne^{nz}|^2}.$$

Each of the factors indicated is less than 1, so we have shown that $P_5(f_n)$ is less than 1 for each n. This shows the counter-example. We note that for this counter-example, the functions f_n have no zeros. For a counter-example where the functions have zeros, let $\mathscr{G} = \{nz^k\}$. The reasoning for this counter-example is similar to that for the family $\mathscr{F} = \{e^{nz}\}$.

We note that there are no results similar to the results of Theorem 5 for normal functions, as the example in [5, Theorem 3] shows.

The technique used to prove the case $k = 3$ in Theorem 5 yields a generalization involving three consecutive factors from the product $P_k(f)(z)$ for $k \ge 3$, as follows.

Theorem 6. *Let α be a complex number, let $\nu \in \{2,3\}$, let k be a positive integer with $k \geq \nu$, and let p be a non-negative integer such that $0 \leq p \leq k-\nu$. Suppose that $\mathscr{F}$ is a family of functions meromorphic in a domain D such that $\mathscr{F} \subset \mathscr{M}_k(D,\alpha)$. Further, suppose that for each compact subset K of D there exists a constant $M_k(K)$ such that*

$$\prod_{j=p}^{p+\nu-1} (f^{(j)})^{\#}(z) \leq M_k(K)$$

for each $f \in \mathscr{F}$ and $z \in K$. Then $\mathscr{F}$ is a normal family on D.

We remark that the number of factors of the product under consideration is exactly ν.

Proof. Assume that $\mathscr{F}$ is not a normal family on D. Let λ be a real number such that $0 < \lambda < 1$. According to Theorem CG, there exist a sequence of functions $\{f_n\}$ in $\mathscr{F}$, a sequence of points $\{z_n\}$ in D with $z_n \to z_0 \in D$, and a sequence of positive numbers $\{\varrho_n\}$ with $\varrho_n \to 0$, such that the sequence $\{g_n(t) = (\varrho_n)^{-p-\lambda} f_n(z_n + \varrho_n t)\}$ converges uniformly on each compact subset of $\mathbf{C}$ to a non-constant meromorphic function g. Using this sequence $\{g_n(t)\}$ in place of the ones used in the proof of Theorem 5, the argument used in the proof of Theorem 5 yield the desired conclusions in the two cases $\nu = 2$ and $\nu = 3$.

We can also prove a result of a related but somewhat different character.

Theorem 7. *Let $\mathscr{F}$ be a family of functions meromorphic on D, and let k be a positive integer greater than 1. Suppose both*

(1) *$\mathscr{F} \subset \mathscr{M}_k(D,0)$, and*
(2) *for each compact subset K of D, there exist a number $\delta > 0$ and a positive number M such that $|f^{(k-1)}(z) f^{(k)}(z)| \leq M$ whenever $\sum_{j=0}^{k-1} |f^{(j)}(z)| \leq \delta$, $z \in K$, and $f \in \mathscr{F}$.*

Then $\mathscr{F}$ is a normal family on D.

Proof. Let $0 < \beta < \frac{1}{2}$. If $\mathscr{F}$ is not a normal family in a neighborhood of the point z_0, then, by Theorem CG, there exist sequences $\{f_n\}$ of functions in $\mathscr{F}$, $\{z_n\}$ of points in D, and $\{\varrho_n\}$ of positive numbers such that $\varrho_n \to 0$, $z_n \to z_0$, and the sequence $\{g_n(t) = (\varrho_n)^{1-k-\beta} f(z_n + \varrho_n t)\}$ converges uniformly on each compact subset of the complex plane to a non-constant function $g(t)$ meromorphic in the complex plane. By (1), $g^{(k)}$ cannot be identically zero. Also, we may assume that $\varrho_n < 1$ for each n, and that, given a number $s > 0$ there exists a compact set K_0 of D such that $z_n + \varrho_n t \in K_0$ whenever $|t| \leq s$. Then, we have

$$g_n^{(j)}(t) = (\varrho_n)^{1-k-\beta+j} f_n^{(j)}(z_n + \varrho_n t), \qquad 0 \leq j \leq k-1$$

and hence $|g_n^{(j)}(t)| \geq |f_n^{(j)}(z_n + \varrho_n t)|$, $0 \leq j \leq k-1$. Let $\delta > 0$ and M be the constants given by (2) corresponding to the compact set K_0. Then, for $|t| \leq s$, $\sum_{j=0}^{k-1} |g_n^{(j)}(t)| \leq \delta$ implies that $\sum_{j=0}^{k-1} |f_n^{(j)}(z_n + \varrho_n t)| \leq \delta$, and thus,

$$|f_n^{(k-1)}(z_n + \varrho_n t) f_n^{(k)}(z_n + \varrho_n t)| \leq M.$$

Thus, if $|t| \leq s$ and $\sum_{j=0}^{k-1} |g_n^{(j)}(t)| \leq \delta$, then

$$|g_n^{(k-1)}(t) g_n^{(k)}(t)| = (\varrho_n)^{1-2\beta} |f_n^{(k-1)}(z_n + \varrho_n t) f^{(k)}(z_n + \varrho_n t)| \leq (\varrho_n)^{1-2\beta} M \to 0$$

as $n \to \infty$. Hence it follows that g must satisfy on K_0 a "strong version of (2)", that is, $\sum_{j=0}^{k-1} |g^{(j)}(t)| \leq \frac{1}{2}\delta$ implies that $|g^{(k-1)}(t) g^{(k)}(t)| = 0$. (This is actually independent of the choice of s, since the s considered above was arbitrary. Also, K_0 can be any compact subset of D having z_0 as an interior point.) Thus, if the hypothesis of this "strong version of (2)" ever occurs on an open set—as would happen, for example, if g ever assumed the value zero (because of condition (1))—then we would have that $g^{(k)} \equiv 0$ on an open set, which contradicts the implication built into condition (1) that the kth derivative of g cannot be identically zero (since $g \in \mathscr{M}_k(D, 0)$ and g is not identically zero). Thus, to avoid this inconsistency, we must have that $\sum_{j=0}^{k-1} |g^{(j)}(t)| \geq \frac{1}{2}\delta$ for all $t \in \mathbf{C}$. Then g never assumes the value zero, and, for $t \in \mathbf{C}$,

$$\sum_{j=0}^{k-1} \left| \frac{g^{(j)}(t)}{g(t)} \right| \geq \frac{\delta}{2|g(t)|}$$

which means that

$$\frac{1}{|g(t)|} \leq \frac{2}{\delta} \left(1 + \sum_{j=1}^{k-1} \left| \frac{g^{(j)}(t)}{g(t)} \right| \right)$$

and hence $m(r, 1/g) \leq \sum_{j=1}^{k-1} m(r, g^{(j)}/g) + O(1) = o\big(T(r, g)\big)$ where this last equality, an extension of Nevalinna's "Lemma on the logarithmic derivative", is valid outside a set of r with finite measure (see for example, [4, Theorem 3.11, pp. 67–68]). Thus, outside this set, we have

$$T(r, g) = O(1) + T\Big(r, \frac{1}{g}\Big) = O(1) + m\Big(r, \frac{1}{g}\Big) = o\big(T(r, g)\big)$$

and this last is not possible, since g is a non-constant function. This completes the proof.

We remark that the boundedness of the product $(f^{(k)})^{\#}(z)(f^{(k-1)})^{\#}(z)$ implies (2).

Finally, we remark that we cannot replace the product of the two factors $f^{(k-1)}(z)f^{(k)}(z)$ by the product of the three factors $f^{(k-2)}(z)f^{(k-1)}(z)f^{(k)}(z)$ in (2) of Theorem 7. For example, if we let $f_n(z) = nz^3$, then

$$|f_n(z)| + |f_n'(z)| + |f_n''(z)| = n|z|^3 + 3n|z|^2 + 6n|z|.$$

If this sum is less than some $\delta > 0$, then $6n|z|$ is also less than δ, and this means that

$$|f_n'(z)f_n''(z)f_n^{(3)}(z)| = 108n^3|z|^3 \le \tfrac{1}{2}(6n|z|)^3 \le \tfrac{1}{2}\delta^3,$$

which implies that both (1) and the version of (2) using three factors are valid, but the family $\{f_n(z)\}$ is not a normal family on the unit disk D.

References

[1] Aulaskari, R., and P. Lappan: Some integral conditions involving the spherical derivative. - Conference on Complex Analysis, Joensuu 1987, Lecture Notes in Math. 1351, 1988, 28–37.

[2] Aulaskari, R., and P. Lappan: An integral criterion for normal functions. - Proc. Amer. Math. Soc. 103, 1988, 438–440.

[3] Chen Huaihui and Gu Yong-Xing: An improvement of Marty's criterion and its application. - Sci. China Ser. A 36, 1993, 674–681.

[4] Gross, F.: Factorization of meromorphic functions. - Mathematics Research Center, Naval Research Laboratory, Washington, D. C., 1972.

[5] Lappan, P.: The spherical derivative and normal functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 3, 1977, 301–310.

[6] Lehto, O., and K.I. Virtanen: Boundary behaviour and normal meromorphic functions. - Acta Math. 97, 1957, 47–65.

[7] Lohwater, A.J., and Ch. Pommerenke: On normal meromorphic functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 550, 1973, 12 pp.

[8] Marty, F.: Recherches sur la repartition des valeurs d'une fonction meromorphe. - Ann. Fac. Sci. Toulouse Math. (3) 23, 1931, 183–261.

[9] Pang Xue-cheung: Bloch's principle and normal criterion. - Sci. China Ser. A 32, 1989, 782–791.

[10] Yamashita, S.: On normal meromorphic functions. - Math Z. 141, 1975, 139–145.

[11] Zalcman, L.: A heuristic principle in complex function theory. - Amer. Math. Monthly 82, 1975, 813–817.

Received 29 June 1994