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UNIQUENESS AND VALUE-SHARING OF MEROMORPHIC FUNCTIONS

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Abstract. Concerning the uniqueness and sharing values of meromorphic functions, many results about meromorphic functions that share more than or equal to two values have been obtained. In this paper, we shall study meromorphic functions that share only one value, and prove the following result: For $n \ge 11$ and two meromorphic functions $f(z)$ and $g(z)$, if $f^n f'$ and $g^n g'$ share the same nonzero and finite value a with the same multiplicities, then $f \equiv dg$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where d is an $(n+1)$th root of unity, c, c_1 and c_2 being constants. As an application, we solve some non-linear differential equations.

1. Introduction and main result

In this paper, a meromorphic function always means a function which is meromorphic in the whole complex plane. Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, $a \in \overline{\mathbf{C}}$. We say that f and g share the value a CM if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities. We shall use the standard notations of value distribution theory, $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, ... (Hayman [6], Laine [9], Nevanlinna [11] and Yang [13]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to +\infty$, possibly outside of finite measure.

In the 1920's, R. Nevanlinna [11] proved the following result (the Nevanlinna four-value theorem).

Theorem A. *Let f and g be two nonconstant meromorphic functions. If f and g share four distinct values CM, then f is a Möbius transformation of g.*

For instance, $f = e^z$, $g = e^{-z}$ share 0, ± 1, ∞, and $f = 1/g$.

In this paper, we shall show that similar conclusions hold for certain types of differential polynomials when they share only one value.

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Theorem 1. *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbf{C}-\{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, and c_2 are constants and satisfy $(c_1c_2)^{n+1}c^2 = -a^2$.*

Remark 1. The following example shows that $a \neq 0$ is necessary. For $f = e^{e^z}$ and $g = e^z$, we see that $f^n f'$ and $g^n g'$ share 0 CM for any integer n, but f and g do not satisfy the conclusion of Theorem 1.

Remark 2. If f and g are entire, we only need to assume that $n \geq 7$ in Theorem 1. A particular result was obtained by Fang and Hua [4].

In order to prove the above result, we shall first prove the following two theorems.

Theorem 2. *Let f and g be two nonconstant meromorphic functions, $n \geq 6$. If $f^n f' g^n g' = 1$, then $g = c_1 e^{cz}$, $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and $(c_1c_2)^{n+1}c^2 = -1$.*

Theorem 3. *Let f and g be two nonconstant entire functions, $n \geq 1$. If $f^n f' g^n g' = 1$, then $g = c_1 e^{cz}$, $f = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and $(c_1c_2)^{n+1}c^2 = -1$.*

In addition, as applications of our theorems, we present a method of solving some non-linear differential equations in Section 6.

2. Some basic lemmas

By using Chuang's inequality [2] and Borel's monotone theorem we can derive the following lemma.

Lemma 1. *Suppose that f is a nonconstant meromorphic function and $n \geq 0$ is an integer. Then $m(r, f'/f) = S(r, f^{(n)})$.*

The following lemma is presented in Yang [13].

Lemma 2. *Let f and g be two nonconstant meromorphic functions. Then*

$$N\Big(r, \frac{f}{g}\Big) - N\Big(r, \frac{g}{f}\Big) = N(r, f) + N\Big(r, \frac{1}{g}\Big) - N(r, g) - N\Big(r, \frac{1}{f}\Big).$$

Lemma 3. *Let f and g be two nonconstant meromorphic functions. If f and g share 1 CM, one of the following three cases holds:*

(i) $T(r, f) \leq \overline{N}(r, f) + \overline{N}_{(2}(r, f) + \overline{N}(r, g) + \overline{N}_{(2}(r, g) + \overline{N}(r, 1/f)$
$+\overline{N}_{(2}(r, 1/f) + \overline{N}(r, 1/g) + \overline{N}_{(2}(r, 1/g) + S(r, f) + S(r, g),$

the same inequality holding for $T(r, g)$;

(ii) $f \equiv g$;

(iii) $fg \equiv 1$,

where $\overline{N}_{(2}(r, 1/f) = \overline{N}(r, 1/f) - N_{1)}(r, 1/f)$ and $N_{1)}(r, 1/f)$ is the counting function of the simple zeros of f in $\{z : |z| \leq r\}$.

Proof. This result has been obtained by Hua [7], Mues–Reinders [10] and Yi–Yang [14] more or less independently. Here we present a different proof, which is of interest in itself. Set

$$\phi = \frac{f''}{f'} - 2\frac{f'}{f-1} - \frac{g''}{g'} + 2\frac{g'}{g-1}. \tag{1}$$

Since f and g share 1 CM, a simple computation on local expansions shows that $\phi(z_0) = 0$ if z_0 is a simple zero of $f-1$ and $g-1$. Next we consider two cases: $\phi \not\equiv 0$ and $\phi \equiv 0$.

If $\phi(z) \not\equiv 0$, then

$$\begin{aligned} N_{1)}\Big(r, \frac{1}{f-1}\Big) &= N_{1)}\Big(r, \frac{1}{g-1}\Big) \le N\Big(r, \frac{1}{\phi}\Big) \\ &\le T(r,\phi) + O(1) \le N(r,\phi) + S(r,f) + S(r,g), \end{aligned} \tag{2}$$

where $N_{1)}\big(r, 1/(f-1)\big)$ is the counting function of the simple zeros of $f-1$ in $\{z : |z| \le r\}$. Since f and g share 1 CM, any root of $f(z) = 1$ cannot be a pole of $\phi(z)$. In addition, we can easily see from (1) that any simple pole of f and g is not a pole of ϕ. Therefore, by (1), the poles of ϕ only occur at zeros of f' and g' and the multiple poles of f and g. If $f'(z_0) = f(z_0) = 0$, then z_0 is a multiple zero of f. We denote by $N_0(r, 1/f')$ the counting function of those zeros of f' but not that of $f(f-1)$. From (1), (2) and the above observations we deduce that

$$\begin{aligned} N_{1)}\Big(r, \frac{1}{f-1}\Big) \le{}& \overline{N}_{(2}(r,f) + \overline{N}_{(2}(r,g) + \overline{N}_0\Big(r, \frac{1}{f'}\Big) + \overline{N}_0\Big(r, \frac{1}{g'}\Big) \\ &+ \overline{N}_{(2}\Big(r, \frac{1}{f}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{g}\Big) + S(r,f) + S(r,g). \end{aligned} \tag{3}$$

On the other hand, by the second fundamental theorem we have

$$T(r,f) \le \overline{N}(r,f) + \overline{N}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{f-1}\Big) - N_0\Big(r, \frac{1}{f'}\Big) + S(r,f), \tag{4}$$

and by the first fundamental theorem

$$\begin{aligned} N\Big(r, \frac{1}{g'}\Big) - N\Big(r, \frac{1}{g}\Big) + \overline{N}\Big(r, \frac{1}{g}\Big) &= N\Big(r, \frac{g}{g'}\Big) \le T\Big(r, \frac{g'}{g}\Big) + O(1) \\ &= \overline{N}(r,g) + \overline{N}\Big(r, \frac{1}{g}\Big) + S(r,g). \end{aligned}$$

This implies that

$$N\Big(r,\frac{1}{g'}\Big) \le \overline{N}(r,g) + N\Big(r,\frac{1}{g}\Big) + S(r,g).$$

It is easy to see from the definition of $N_0(r,1/g')$ that

$$\overline{N}_0\Big(r,\frac{1}{g'}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{g-1}\Big) + N_{(2}\Big(r,\frac{1}{g}\Big) - \overline{N}_{(2}\Big(r,\frac{1}{g}\Big) \le N\Big(r,\frac{1}{g'}\Big).$$

The above two inequalities yield

$$\overline{N}_0\Big(r,\frac{1}{g'}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{g-1}\Big) \le \overline{N}(r,g) + \overline{N}\Big(r,\frac{1}{g}\Big) + S(r,g). \tag{5}$$

Since f and g share 1 CM, we have

$$\overline{N}\Big(r,\frac{1}{f-1}\Big) = N_{1)}\Big(r,\frac{1}{f-1}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{g-1}\Big). \tag{6}$$

Combining (3)–(6), we obtain (i).

If $\phi(z)\equiv 0$, we deduce from (1) that

$$f \equiv \frac{Ag+B}{Cg+D}, \tag{7}$$

where A, B, C and D are finite complex numbers satisfying $AD-BC\ne 0$. Then, by the first fundamental theorem,

$$T(r,f) = T(r,g) + S(r,f). \tag{8}$$

Next we consider three respective subcases.

Subcase 1. $AC\ne 0$. Then

$$f - \frac{A}{C} = \frac{B-AD/C}{Cg+D}.$$

By the second fundamental theorem, we have

$$\begin{aligned} T(r,f) &\le \overline{N}(r,f) + \overline{N}\Big(r,\frac{1}{f-(A/C)}\Big) + \overline{N}\Big(r,\frac{1}{f}\Big) + S(r,f) \\ &= \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}\Big(r,\frac{1}{f}\Big) + S(r,f). \end{aligned} \tag{9}$$

We get (i).

Subcase 2. $A \neq 0$, $C = 0$. Then $f \equiv (Ag + B)/D$. If $B \neq 0$, by the second fundamental theorem

$$
\begin{aligned}
T(r,f) &\leq \overline{N}(r,f) + \overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{f-B/D}\Big) + S(r,f) \\
&= \overline{N}(r,f) + \overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{g}\Big) + S(r,f).
\end{aligned} \tag{10}
$$

We obtain (i). If $B = 0$, then $f \equiv Ag/D$. If $A/D = 1$, then $f \equiv g$; this is (ii). If $A/D \neq 1$, then by the assumption that f and g share 1 CM, it it easy to see that $f \neq 1$ and $g \neq 1$, which yields $f \neq 1,\ A/D$. By the second fundamental theorem we have

$$T(r,f) \leq \overline{N}(r,f) + S(r,f),$$

and (i) follows.

Subcase 3. $A = 0$, $C \neq 0$. Then $f \equiv B/(Cg + D)$. If $D \neq 0$, by the second fundamental theorem we have

$$
\begin{aligned}
T(r,f) &\leq \overline{N}(r,f) + \overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{f-B/D}\Big) + S(r,f) \\
&= \overline{N}(r,f) + \overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{g}\Big) + S(r,f).
\end{aligned}
$$

Thus we get (i). If $D = 0$, then $f = B/Cg$. If $B/C = 1$, then $fg \equiv 1$ and we obtain (iii). If $B/C \neq 1$, by the assumption that f and g share 1 CM, we have $f \neq 1,\ B/C$. By the second fundamental theorem we get

$$T(r,f) \leq \overline{N}(r,f) + S(r,f).$$

This implies (i). The proof of Lemma 3 is complete.

Remark 3. In some known results such as in Brosch [1], Jank–Terglane [8], Song–Wang [12], etc., it is assumed that the counting functions of the poles and the zeros of f and g are $S(r,f)$ and $S(r,g)$, respectively. Thus all these kinds of theorems can be strengthened according to our result, conclusion (i).

The following lemma, called Borel's unicity theorem, is very useful in the study of value distribution theory and its applications such as factorizations or uniqueness and value sharing problems (cf. Chuang and Yang [3] or Gross [5]).

Lemma 4. *Let g_j $(j = 1,\dots,n)$ be entire and a_j $(j = 1,\dots,n)$ meromorphic functions. If*

$$T(r,a_j) = o\{T(r,e^{g_i - g_k})\} \qquad (1 \leq j \leq n,\ i \neq k,\ i,\, k = 1,\dots,n),$$

the identity $\sum_{j=1}^{n} a_j(z) e^{g_j(z)} \equiv 0$ implies that $a_j(z) \equiv 0$ $(j = 1,\dots,n)$.

Remark 4. In fact, we only need to assume that the growth condition of Lemma 4 holds on a set of values r of infinite linear measure.

3. Proof of Theorem 2

We proceed in the proof step by step as follows.

Step 1. We prove that

$$f \neq 0, \qquad g \neq 0. \tag{11}$$

In fact, suppose that f has a zero z_0 with order m. Then z_0 is a pole of g (with order p, say) by

$$f^n f' g^n g' = 1. \tag{12}$$

Thus, $nm+m-1 = np+p+1$, i.e., $(m-p)(n+1) = 2$. This is impossible since $n \geq 6$ and m, p are integers.

Step 2. We claim that

$$N(r,f) + N(r,g) \leq 2m\Big(r, \frac{1}{fg}\Big) + O(1). \tag{13}$$

By Step 1 and (12) we deduce that

$$(n+1)N(r,g) + \overline{N}(r,g) = N\Big(r, \frac{1}{f'}\Big). \tag{14}$$

From Lemma 2 we have

$$\begin{aligned} N\Big(r, \frac{f}{f'}\Big) - N\Big(r, \frac{f'}{f}\Big) &= N(r,f) + N\Big(r, \frac{1}{f'}\Big) - N(r,f') - N\Big(r, \frac{1}{f}\Big) \\ &= N\Big(r, \frac{1}{f'}\Big) - \overline{N}(r,f). \end{aligned}$$

By the first fundamental theorem, the left side is $m(r, f'/f) - m(r, f/f') + O(1)$, so we have

$$N\Big(r, \frac{1}{f'}\Big) = \overline{N}(r,f) + m\Big(r, \frac{f'}{f}\Big) - m\Big(r, \frac{f}{f'}\Big) + O(1). \tag{15}$$

Now we rewrite (12) in the form $g'/g = (f/f')(1/fg)^{n+1}$. Then

$$m\Big(r, \frac{f}{f'}\Big) \geq m\Big(r, \frac{g'}{g}\Big) - (n+1)m\Big(r, \frac{1}{fg}\Big) - O(1).$$

Combining this, (14), and (15), we get

$$(n+1)N(r,g) + \overline{N}(r,g) \leq \overline{N}(r,f) + m\Big(r, \frac{f'}{f}\Big) - m\Big(r, \frac{g'}{g}\Big) + (n+1)m\Big(r, \frac{1}{fg}\Big) + O(1).$$

By symmetry,

$$(n+1)N(r,f) + \overline{N}(r,f) \leq \overline{N}(r,g) + m\Big(r, \frac{g'}{g}\Big) - m\Big(r, \frac{f'}{f}\Big) + (n+1)m\Big(r, \frac{1}{fg}\Big) + O(1).$$

By adding the above two inequalities we obtain (13).

Step 3. We prove that fg is constant. Let $h = 1/fg$. Then h is entire by Step 1, and (12) can be written as

$$\left(\frac{g'}{g} + \frac{1}{2}\frac{h'}{h}\right)^2 = \frac{1}{4}\left(\frac{h'}{h}\right)^2 - h^{n+1}.$$

Let

$$\alpha = \frac{g'}{g} + \frac{1}{2}\frac{h'}{h}.$$

The above equation becomes

$$\alpha^2 = \frac{1}{4}\left(\frac{h'}{h}\right)^2 - h^{n+1}. \tag{16}$$

If $\alpha \equiv 0$, then $h^{n+1} = \frac{1}{4}(h'/h)^2$. Combining this with Step 1 we obtain $T(r,h) = m(r,h) = S(r,h)$; thus h is a constant. Next we assume that $\alpha \not\equiv 0$. Differentiating (16) yields

$$2\alpha\alpha' = \frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)' - (n+1)h'h^n.$$

From this and (16) it follows that

$$h^{n+1}\left((n+1)\frac{h'}{h} - 2\frac{\alpha'}{\alpha}\right) = \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\right). \tag{17}$$

If $((n+1)h'/h) - 2(\alpha'/\alpha) \equiv 0$, there exists a constant c such that $\alpha^2 = ch^{n+1}$. This and (16) give

$$(c+1)h^{n+1} = \frac{1}{4}\left(\frac{h'}{h}\right)^2.$$

If $c = -1$, then $h' \equiv 0$, and so h is constant. If $c \neq -1$, we have $T(r,h) = S(r,h)$, and h is constant. Next we suppose that

$$(n+1)\frac{h'}{h} - 2\frac{\alpha'}{\alpha} \not\equiv 0.$$

Then, by (17) and the fact that h is entire,

$$\begin{aligned}
(n+1)T(r,h) &= (n+1)m(r,h)\\
&\le m\left(r, h^{n+1}\left((n+1)\frac{h'}{h} - 2\frac{\alpha'}{\alpha}\right)\right)\\
&\quad + m\left(r, \frac{1}{(n+1)h'/h - 2\alpha'/\alpha}\right) + O(1)\\
&\le m\left(r, \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\right)\right) + T\left(r, (n+1)\frac{h'}{h} - 2\frac{\alpha'}{\alpha}\right)\\
&\le \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}\left(r, \frac{1}{\alpha}\right) + S(r,h) + S(r,\alpha).
\end{aligned}$$

Now by (16) and (13) we have

$$T(r,\alpha) \le \tfrac{1}{2}(n+3)T(r,h) + S(r,h),$$

and

$$N(r,f) + N(r,g) \le 2m(r,h) + O(1).$$

Combining the above three inequalities we obtain

$$\tfrac{1}{2}(n-5)T(r,h) \le S(r,h).$$

Thus h must be a constant.

Step 4. We prove our conclusion. By Step 3, h is a constant. Then, by (12),

$$\frac{g'}{g} = c, \qquad c = ih^{(n+1)/2}.$$

Thus

$$g(z) = c_1 e^{cz}, \qquad f = c_2 e^{cz},$$

where c, c_1 and c_2 are constants and satisfy $(c_1c_2)^{n+1}c^2 = -1$ by (12). This completes the proof of the theorem.

4. Proof of Theorem 3

From

$$f^n f' g^n g' = 1$$

and the assumption that f and g are entire we immediately see that f and g have no zeros. Thus there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$f(z) = e^{\alpha(z)}, \qquad g(z) = e^{\beta(z)}.$$

Inserting these in the above equality, we get

$$\alpha'\beta' e^{(n+1)(\alpha+\beta)} \equiv 1.$$

Thus α' and β' have no zeros and we may set

$$\alpha' = e^{\delta(z)}, \qquad \beta' = e^{\gamma(z)}.$$

The above two equalities yield

$$e^{(n+1)(\alpha+\beta)+\delta+\gamma} \equiv 1.$$

Differentiating this gives

$$(n+1)(e^{\delta} + e^{\gamma}) + \delta' + \gamma' \equiv 0.$$

By Lemma 4, $\delta = \gamma + (2m+1)\pi i$ for some integer m. Inserting this in the above equality we deduce that $\delta' \equiv \gamma' \equiv 0$, and so δ and γ are constants, i.e., α' and β' are constants. From this we can easily obtain the desired result.

5. Proof of Theorem 1

Let $F = f^{n+1}/a(n+1)$ and $G = g^{n+1}/a(n+1)$. Then the condition that $f^n f'$ and $g^n g'$ share the value a CM implies that F' and G' share the value 1 CM. Obviously,

$$N(r,F') = (n+1)N(r,f) + \overline{N}(r,f),$$

$$N(r,G') = (n+1)N(r,g) + \overline{N}(r,g), \tag{18}$$

$$\overline{N}(r,F') = \overline{N}_{(2}(r,F') = \overline{N}(r,f) \le \frac{1}{n+2}T(r,F') + O(1), \tag{19}$$

$$\begin{aligned} \overline{N}\Big(r,\frac{1}{F'}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{F'}\Big) &= 2\overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{f'}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{f'}\Big) \\ &\le 2\overline{N}\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{f'}\Big) \\ &\le 2T(r,f) + N\Big(r,\frac{1}{f'}\Big) + O(1). \end{aligned} \tag{20}$$

Since

$$\begin{aligned} nm(r,f) &= m\Big(r,a\frac{F'}{f'}\Big) \le m(r,F') + m\Big(r,\frac{1}{f'}\Big) + O(1) \\ &= m(r,F') + T(r,f') - N\Big(r,\frac{1}{f'}\Big) + O(1) \\ &\le m(r,F') + T(r,f) + \overline{N}(r,f) - N\Big(r,\frac{1}{f'}\Big) + m\Big(r,\frac{f'}{f}\Big) + O(1) \\ &= m(r,F') + T(r,f) + \overline{N}(r,f) - N\Big(r,\frac{1}{f'}\Big) + m\Big(r,\frac{F'}{F}\Big) + O(1), \end{aligned}$$

it follows from this, (18), and Lemma 1 that

$$(n-1)T(r,f) \le T(r,F') - N(r,f) - N\Big(r,\frac{1}{f'}\Big) + S(r,F').$$

This and Lemma 1 imply that

$$\begin{aligned} 2T(r,f) + N\Big(r,\frac{1}{f'}\Big) &= \frac{2}{n-1}\Big\{(n-1)T(r,f) + N\Big(r,\frac{1}{f'}\Big)\Big\} + \frac{n-3}{n-1}N\Big(r,\frac{1}{f'}\Big) \\ &\le \frac{2}{n-1}\{T(r,F') - N(r,f)\} + \frac{n-3}{n-1}\big(T(r,f) + \overline{N}(r,f)\big) \\ &\quad + m\Big(r,\frac{f'}{f}\Big) + O(1) \\ &\le \Big(\frac{2}{n-1} + \frac{n-3}{(n-1)^2}\Big)T(r,F') \\ &\quad + \Big(\frac{n-5}{n-1} - \frac{n-3}{(n-1)^2}\Big)\overline{N}(r,f) + S(r,F'). \end{aligned}$$

Combining this, (19), and (20), we obtain

$$\overline{N}\Big(r,\frac{1}{F'}\Big)+\overline{N}_{(2}\Big(r,\frac{1}{F'}\Big)\le\frac{4n^2-6n-2}{(n-1)^2(n+2)}T(r,F')+S(r,F'). \tag{21}$$

We similarly derive for G' that

$$\overline{N}(r,G')=\overline{N}_{(2}(r,G')=\overline{N}(r,g)\le\frac{1}{n+2}T(r,G')+O(1), \tag{22}$$

$$\overline{N}\Big(r,\frac{1}{G'}\Big)+\overline{N}_{(2}\Big(r,\frac{1}{G'}\Big)\le\frac{4n^2-6n-2}{(n-1)^2(n+2)}T(r,G')+S(r,G'). \tag{23}$$

Without loss of generality, we suppose that there exists a set $I\subset[0,\infty)$ such that $T(r,G')\le T(r,F')$. Next we always let $r\in I$. If we apply Lemma 3 to F' and G', it follows that there are three cases to be considered.

Case (i).

$$\begin{aligned}T(r,F')\le{}&\overline{N}(r,F')+\overline{N}_{(2}(r,F')+\overline{N}(r,G')+\overline{N}_{(2}(r,G')+\overline{N}\Big(r,\frac{1}{F'}\Big)\\&+\overline{N}_{(2}\Big(r,\frac{1}{F'}\Big)+\overline{N}\Big(r,\frac{1}{G'}\Big)+\overline{N}_{(2}\Big(r,\frac{1}{G'}\Big)+S(r,F')+S(r,G').\end{aligned}$$

Setting (19), (21), (22), and (23) into the above inequality and keeping in mind that $T(r,G')\le T(r,F')$, we get

$$\frac{n^3-12n^2+17n+2}{(n-1)^2(n+2)}T(r,F')\le S(r,F'). \tag{24}$$

We denote by $p(n)$ the numerator of the coefficient on the left-hand side above. Then $p'(n)=3n^2-24n+17>0$ for $n\ge 8$. Note that $p(11)=68$; thus $p(n)$ is positive for $n\ge 11$. It follows from (24) that F' must be a rational function. But then, by the above derivations, $S(r,F')=O(1)$. Using (24) again, F' must be a constant, which is impossible.

Case (ii). $F'=G'$. Then we deduce that $f^{n+1}=g^{n+1}+c$ ($c\in\mathbf{C}$). Let $f=hg$, and we have

$$(h^{n+1}-1)g^{n+1}=c. \tag{25}$$

If $h^{n+1}\equiv 1$, then h is an $(n+1)$th unit root and we obtain the desired result. If $h^{n+1}\not\equiv 1$, then by (25),

$$g^{n+1}=\frac{c}{h^{n+1}-1}.$$

Thus h is not constant. We write this in the form

$$g^{n+1}=\frac{c}{(h-u_1)\cdots(h-u_{n+1})},$$

where $u_1,\dots,u_{n+1}$ are different $(n+1)$th roots of unity. Thus h has at least $n+1$ (≥ 14) multiple values. However, from Nevanlinna's second fundamental theorem we know that h has at most 4 multiple values, a contradiction.

Case (iii). $F'G' \equiv 1$, i.e., $a^{-2}f^nf'g^ng' \equiv 1$. Let $\hat{f} = a^{-1/(n+1)}f$ and $\hat{g} = a^{-1/(n+1)}g$. Then $\hat{f}^n\hat{f}'\hat{g}^n\hat{g}' = 1$. The conclusion follows from Theorem 2.

6. Applications

Generally speaking, solving any non-linear differential equation presents a very difficult problem. As applications of our result, we present the derivations of the meromorphic solutions of the following two non-linear differential equations:

$$y'y^n - 1 = e^{\alpha(z)}(e^{\beta(z)} - 1), \qquad n \geq 7, \tag{26}$$

where α and β are entire functions. If e^β can be represented as $g'g^n$ for some entire function g, we see that g has no zeros and poles. Thus we may set $g = e^\gamma$ and obtain $\gamma' = e^{\beta-(n+1)\gamma}$. This implies that γ' has no zeros, and thus there exists an entire function δ such that $\gamma' = e^\delta$, which results in $\beta = (n+1)\gamma + \delta + 2k\pi i$. Therefore

$$\beta' = (n+1)e^\delta + \delta'. \tag{27}$$

Now, by Remark 2 in Section 1, the possible solution for the equation (26) is either $y = dg$ or $y = c_1e^{cz}$ and $g = c_2e^{cz}$, where d is an $(n+1)$th root of unity and c_1, c_2 and c are constants. In the second case γ is linear, and so $\beta' = (n+1)c$. Combining the above discussions, we have the following result.

Theorem 4. *Suppose that for a non-linear differential equation of* (26) *with* $n \geq 7$, *there exists an entire function* δ *such that* (27) *holds. Then every solution of* (26) *is of the form*

$$y = c\exp\left(\int e^\delta\, dz\right)$$

or

$$y = c_1\exp\bigl(-\beta'/(n+1)\bigr) \qquad (\beta' \equiv \text{constant}).$$

Remark 5. In particular, if β is a polynomial, then by (27), δ can only be constant, and hence β is linear.

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