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THE APOLLONIAN METRIC: UNIFORMITY AND QUASICONVEXITY

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Abstract. In this paper we examine implications of a mapping being bilipschitz with respect to the Apollonian metric. The main results are improvements of results by Gehring and Hag which aim at describing Apollonian isometries. We also derive results on quasi-isotropy and quasiconvexity of the Apollonian metric.

1. Introduction

This paper continues the investigation by the author on the Apollonian metric started in [12], which in turn was a continuation of work by A. Beardon [3], A. Rhodes [19], P. Seittenranta [20], F. Gehring and K. Hag [9] and Z. Ibragimov [15]. The same metric has also been considered, from a different point of view, in [1], [4], [5] and [16]. This section contains the statements of the main results, which concern Apollonian bilipschitz mappings. We start by presenting some previous results from the above-mentioned papers. The notation used conforms largely to that of [2] and [24], the reader can consult Section 2 of this paper, if necessary.

We will be considering domains (open connected non-empty sets) G in the Möbius space $\overline{\mathbf{R}^n} := \mathbf{R}^n \cup \{\infty\}$. The Apollonian metric, for $x, y \in G \subsetneq \mathbf{R}^n$, is defined by

$$\alpha_G(x,y) := \sup_{a,b\in\partial G} \log \frac{|a-x|}{|a-y|}\frac{|b-y|}{|b-x|} \tag{1.1}$$

(with the understanding that if $a=\infty$ then we set $|a-x|/|a-y| = 1$ and similarly for b). It is in fact a metric if and only if the complement of G is not contained in a hyperplane or sphere, as was noted in [3, Theorem 1.1].

In the paper [3] Alan Beardon speculated that the isometries of the Apollonian metric are only the Möbius mappings, at least for many domains. He proved that conformal mappings of plane domains whose boundary is a compact subset of the extended negative real axis which are Apollonian isometries are indeed Möbius mappings, [3, Theorem 1.3]. The next step in the investigation of Apollonian isometries was taken in [9], where the following theorem was proved:

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Theorem 1.2 ([9, Theorem 3.11]). *Let $G \subseteq \mathbf{R}^2$ be a quasidisk and $f\colon G \to \mathbf{R}^2$ be an Apollonian bilipschitz mapping. The following conditions are equivalent*:

(1) *$f(G)$ is a quasidisk.*

(2) *f is quasiconformal in G.*

Moreover, if either of the two conditions holds then $f = g|_G$, where $g\colon \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$ is quasiconformal.

Remark 1.3. Gehring and Hag actually considered Apollonian isometries instead of Apollonian bilipschitz mappings. Their proof carries over directly to the bilipschitz case, however.

If we replace quasidisks, quasiconformal mappings and bilipschitz mappings by disks, conformal mappings and isometries in the previous theorem we get the following result, from the same paper.

Theorem 1.4 ([9, Theorem 3.16]). *Let $G \subseteq \mathbf{R}^2$ be a disk and let $f\colon G \to \mathbf{R}^2$ be an Apollonian isometry. The following conditions are equivalent*:

(1) *$f(G)$ is a disk.*

(2) *f is a Möbius mapping of G.*

Moreover, if either of the two conditions holds then $f = g|_G$, where $g\colon \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$ is a Möbius mapping.

As a last result from the paper of Gehring and Hag we quote the following theorem, which is a stronger version of the previous one:

Theorem 1.5 ([9, Theorem 3.29]). *If $G \subseteq \mathbf{R}^2$ is a disk and $f\colon G \to \mathbf{R}^2$ is an Apollonian isometry then*

(1) *$f(G)$ is a disk and*

(2) *$f = g|_G$, where $g\colon \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$ is a Möbius mapping.*

Note that this result solves Beardon's problem for the disk.[1] In the paper [12] the first step was taken in generalizing these results to $\mathbf{R}^n$. Specifically, it was proven that the implication (1) $\Rightarrow$ (2) of Theorem 1.2 holds in $\mathbf{R}^n$ as well.

Theorem 1.6 ([12, Corollary 1.7]). *Let $G \subseteq \mathbf{R}^n$ be a quasiball and $f\colon G \to \mathbf{R}^n$ be an Apollonian bilipschitz mapping. If $f(G)$ is a quasiball then $f = g|_G$ where $g\colon \overline{\mathbf{R}^n} \to \overline{\mathbf{R}^n}$ is quasiconformal.*

In this paper we complement these results by three new ones, two in space and one in $\mathbf{R}^2$. Our first result is the strong version of Theorem 1.2 and is valid only in the plane.

[1] After the completion of this paper the author was informed that Zair Ibragimov has considered this problem in his thesis, [15]. He concentrates on domains that are complements of so-called constant width sets and thus his results are complementary to those derived here.

Theorem 1.7. *If $G \subseteq \mathbf{R}^2$ is a quasidisk and $f\colon G \to \mathbf{R}^2$ is an Apollonian bilipschitz mapping then*
(1) *$f(G)$ is a quasidisk and*
(2) *$f = g|_G$, where $g\colon \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$ is quasiconformal.*

The next result is an extension of Theorem 1.2, which is, however, not stated in terms of quasiballs but in terms of A-uniform domains. A-uniform domains are introduced in Definition 6.5 of this paper and are defined as those domains that satisfy the relation $k_G \le K\alpha_G$ for some fixed $K \ge 1$, where k_G denotes the quasihyperbolic metric from [11]. We show that in general quasiballs are A-uniform domains (Corollary 6.9) and that in the plane these two concepts define the same class of simply connected domains (Corollary 6.10). Whether these classes of domains coincide in space is an open problem. It follows, then, that the next result implies Theorem 1.2, although it is not the most natural generalization of that result.

Theorem 1.8. *Let $G \subseteq \mathbf{R}^n$ be A-uniform and let $f\colon G \to \mathbf{R}^n$ be an Apollonian bilipschitz mapping. The following conditions are equivalent*:
(1) *$f(G)$ is A-uniform.*
(2) *f is quasiconformal in G.*

Notice that we are not able to prove the last statement of Theorem 1.2 (that f would be a restriction of a quasiconformal mapping from $\overline{\mathbf{R}^n}$ onto $\overline{\mathbf{R}^n}$) for the case $n \ge 3$.

Our last result along this line of investigation is a generalization of [9, Theorem 3.29] to $\mathbf{R}^n$, which is also proved quite similarly, although the geometry becomes a bit more complicated in space.

Theorem 1.9. *If $G \subseteq \mathbf{R}^n$ is a ball and $f\colon G \to \mathbf{R}^n$ is an Apollonian isometry then*
(1) *$f(G)$ is a ball and*
(2) *$f = g|_G$, where $g\colon \overline{\mathbf{R}^n} \to \overline{\mathbf{R}^n}$ is a Möbius mapping.*

We present a schema of the results in Table 1, where the results from this paper are in boldface. We consider the results as varying in three dichotomic dimensions. One dimension is whether they are valid in the plane or in space, a second is whether we consider quasiballs, quasiconformal mappings and Apollonian bilipschitz mappings or balls, conformal mappings and Apollonian isometries and a third is whether the result is weak (i.e. implies the equivalence of the conditions) or strong (i.e. implies the conditions). Notice that the results for quasiballs are lacking.

As a final result the following theorem summarizes several characterizations of planar quasidisks in terms of the Apollonian metric. These add to the legion of equivalent conditions given e.g. in [8].

	Disk	Quasidisk	Ball	Quasiball
Weak	Th. 1.4	Th. 1.2	**Th. 1.8**	(Ths. 1.6, **1.9**)
Strong	Th. 1.5	**Th. 1.7**	**Th. 1.8**	

Table 1. A schematic representation of the main results.

Theorem 1.10. *Let G be a simply connected planar domain. The following statements are equivalent*:

(1) *The domain G is a quasidisk.*

(2) *There exists a constant K such that $h_G \le K\alpha_G$, where h_G denotes the hyperbolic metric* [9, Theorem 3.1].

(3) *The domain G is A-uniform, i.e. there exists a constant K such that $k_G \le K\alpha_G$* (Corollary 6.10).

(4) *The metric α_G is quasiconvex, i.e. there exists a constant K such that for every $x, y \in G$ there exists a path γ connecting x and y in G with $l_{\alpha_G}(\gamma) \le K\alpha_G(x,y)$* (Corollary 7.4).

Remark 1.11. Notice that of the conditions in the previous theorem the fourth one involves only the Apollonian metric.

The structure of the rest of this paper is as follows. In the next section we review the terminology and notation that is in common use; this section can be skipped or merely perused by the reader acquainted with the field. In Section 3 we present notation and terminology that is not as well known, a large part of which is specific to the Apollonian metric. In Section 4 we introduce the concept of quasi-isotropy, consider its connection to the comparison property and derive some preliminary results that are used in later sections. In Section 5 we introduce the inner metric approach and consider two methods of deriving estimates for the inner metric of the Apollonian metric. In Section 6 we introduce A-uniform domains, which allow us to use the results on quasi-isotropy and inner metrics to prove Theorem 1.8. In Section 7 we prove Theorem 1.7 combining the inner metric approach with an estimate of the hyperbolic metric and a lemma from [12]. In Section 8 we prove Theorem 1.9 using lemmata from [9]. In an appendix we prove some simple results relating to Ferrand's and Seittenranta's metrics.

2. Common notation and terminology

As mentioned in the introduction, the notation used conforms largely to that in [2] and [24]. We denote by $\{e_1, e_2, \dots, e_n\}$ the standard basis of $\mathbf{R}^n$ and by n the dimension of the Euclidean space under consideration and assume that $n \ge 2$. We denote by x_i the i^{th} coordinate of $x \in \mathbf{R}^n$. The following notation will be used for balls, spheres and the upper half-space ($x \in \mathbf{R}^n$ and $0 < r < \infty$):

$$\begin{aligned} B^n(x,r) &:= \{y \in \mathbf{R}^n : |x-y| < r\}, \\ S^{n-1}(x,r) &:= \partial B^n(x,r), \\ B^n &:= B^n(0,1), \\ S^{n-1} &:= S^{n-1}(0,1), \\ H^n &:= \{y \in \mathbf{R}^n : y_n > 0\}. \end{aligned}$$

We use the notation $\overline{\mathbf{R}^n} := \mathbf{R}^n \cup \{\infty\}$ for the one point compactification of $\mathbf{R}^n$. We define the spherical (chordal) metric q in $\overline{\mathbf{R}^n}$ by means of the canonical projection onto the Riemann sphere, hence

$$q(x,y) := \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \qquad q(x,\infty) := \frac{1}{\sqrt{1+|x|^2}}.$$

We consider $\overline{\mathbf{R}^n}$ as the metric space $(\overline{\mathbf{R}^n}, q)$, hence its balls are the (open) balls of $\mathbf{R}^n$, complements of closed balls and half-spaces. If $G \subseteq \overline{\mathbf{R}^n}$ we will denote by ∂G, G^c and $\bar{G}$ its boundary, complement and closure, respectively, all with respect to $\overline{\mathbf{R}^n}$. In contrast to topological operations, we will always consider metric operations with respect to the ordinary Euclidean metric, unless specified otherwise.

Let (G,d) and (G',d') be metric spaces. The mapping $f\colon G \to G'$ is said to be K-bilipschitz if

$$d(x,y)/K \le d'\bigl(f(x), f(y)\bigr) \le K d(x,y)$$

for all $x, y \in G$. If no metric spaces are specified then K-bilipschitz is understood to mean K-bilipschitz when considered a mapping from $(G, |\cdot|)$ to $(G', |\cdot|)$. A mapping is bilipschitz if it is K-bilipschitz for some $1 \le K < \infty$. The expression "f is bilipschitz with respect to the Apollonian metric in G" means that f is bilipschitz when considered as a mapping from (G, α_G) to $\bigl(f(G), \alpha_{f(G)}\bigr)$ and similarly for other domain dependent metrics.

Let $G \subseteq \overline{\mathbf{R}^n}$ be a domain and $f\colon G \to \overline{\mathbf{R}^n}$ be an embedding. The linear dilatation of f at $x \in G \setminus \{\infty, f^{-1}(\infty)\}$ is defined by

$$H(f,x) := \limsup_{r \to 0} \frac{\sup\{|f(x)-f(y)| : |x-y| = r\}}{\inf\{|f(x)-f(z)| : |x-z| = r\}}.$$

The linear dilatation constant of f, $H(f)$, is the essential supremum of $H(f,x)$ over $x \in G \setminus \{\infty, f^{-1}(\infty)\}$. A mapping is said to be quasiconformal if $\sup H(f,x) < \infty$, where the supremum is again over $G \setminus \{\infty, f^{-1}(\infty)\}$. For the connection between different constants of quasiconformality, see [6]. Clearly every K-bilipschitz mapping is quasiconformal with linear dilatation constant less than or equal to K^2. For basic theory of quasiconformal mappings see e.g. [17] for $n = 2$ and [21] for $n \geq 2$. We say that a domain $G \subseteq \mathbf{R}^n$ is a quasiball if there exists a quasiconformal mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $G = f(B^n)$. A quasiball in $\mathbf{R}^2$ is called a quasidisk.

The cross-ratio $|a, b, c, d|$ is defined by

$$|a, b, c, d| := \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)} \left(= \frac{|a - c||b - d|}{|a - b||c - d|} \right)$$

for $a \neq b$, $c \neq d$ and $a, b, c, d \in \overline{\mathbf{R}^n}$, where the second equality is valid if $a, b, c, d \in \mathbf{R}^n$. A homeomorphism $f\colon \overline{\mathbf{R}^n} \to \overline{\mathbf{R}^n}$ is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|$$

for every quadruple $a, b, c, d \in \overline{\mathbf{R}^n}$ with $a \neq b$ and $c \neq d$. For more information on Möbius mappings see e.g. [2]. Using the cross-ratio we can express the Apollonian metric as

$$\alpha_G(x, y) = \log \sup_{a,b \in \partial G} |a, y, x, b|,$$

for $x, y \in G \subsetneq \mathbf{R}^n$. Indeed, one can define the Apollonian metric for domains in $\overline{\mathbf{R}^n}$ instead of domains of $\mathbf{R}^n$. However, since we are ultimately interested in the metric "modulo" Möbius mappings, the normalization $\infty \notin G$ is no real restriction. Moreover, the j_G metric is only defined in proper subdomains of $\mathbf{R}^n$, hence not assuming $\infty \notin G$ would imply that we would have to start and end every proof by using an auxiliary Möbius mapping, or use a Möbius invariant version of j_G such as the metric $j_{G,a}$ from [14]. We note that the inversion in S^{n-1}, defined by $x \mapsto x/|x|^2$ for $x \in \mathbf{R}^n \setminus \{0\}$, $0 \mapsto \infty$ and $\infty \mapsto 0$, is a Möbius mapping.

Some miscellaneous notation and terminology:

– For $x \in G \subsetneq \mathbf{R}^n$ we denote $\delta(x) := d(x, \partial G) := \min\{|x - z| : z \in \partial G\}$.
– We denote by xy the line through x and y and by $[x, y]$ the closed segment between x and y.
– We use the expressions "satisfies condition X", "has property X" and "is an X domain" interchangeably.

We end this introduction by citing the so-called Bernoulli inequalities which hold for $s \geq 0$:

$$\log(1 + as) \leq a \log(1 + s) \quad \text{for } 1 \leq a,$$
$$\log(1 + as) \geq a \log(1 + s) \quad \text{for } 0 \leq a \leq 1.$$

These inequalities follow from the fact that $a \mapsto \log(1 + as)/a$ is decreasing on $(0, \infty)$ for constant $s > 0$.

3. Specific notation and terminology

In this section we present notation and results which are more or less specific to the Apollonian metric. Let us start by defining some auxiliary metrics, which have also been used in other contexts.

Assume throughout this paragraph that $x, y \in G \subsetneq \mathbf{R}^n$. The j_G metric from [23] is defined by

$$j_G(x,y) := \log\left(1 + \frac{|x-y|}{\min\{\delta(x), \delta(y)\}}\right).$$

Note that this metric is sometimes defined, following [10], by

$$\hat{\jmath}_G(x,y) = \frac{1}{2}\log\left[\left(1+\frac{|x-y|}{\delta(x)}\right)\left(1+\frac{|x-y|}{\delta(y)}\right)\right],$$

a fact which makes no greater difference in the present context, since $\frac{1}{2}j_G \le \hat{\jmath}_G \le j_G$ for every $G \subsetneq \mathbf{R}^n$, and our approximations will not be this exact, anyway. The quasihyperbolic metric from [11] is defined by

$$k_G(x,y) := \inf_{\gamma} \int \frac{|dz|}{\delta(z)},$$

where the infimum is taken over all rectifiable curves joining x and y in G.

Definition 3.1. We say that a domain $G \subsetneq \mathbf{R}^n$ has the *comparison property* if there exists a constant K such that $j_G/K \le \alpha_G \le 2j_G$.

Note that the upper bound $\alpha_G \le 2j_G$ is valid in every domain $G \subsetneq \mathbf{R}^n$ by [3, Theorem 3.2]. The comparison property was a key concept in [12], allowing us to prove Theorem 1.6 among other results. Comparison domains will be important to us also in this investigation. Indeed, the proof of Theorem 1.8 consists essentially of showing that the Apollonian quasiconvexity property implies the comparison property for simply connected planar domains (Proposition 7.3).

We end this section by presenting the Apollonian balls approach. This approach has previously been used in [3], [5] and [20, Theorem 4.1], although this presentation is from Section 5.1 of [12].

Let us define

$$q_x := \sup_{b\in\partial G}\frac{|b-y|}{|b-x|}, \qquad q_y := \sup_{a\in\partial G}\frac{|a-x|}{|a-y|}.$$

Then, by definition, $\alpha_G(x,y) = \log(q_x q_y)$. Moreover the balls

$$(3.2) \qquad \begin{aligned} B_x &:= \{z \in \mathbf{R}^n : |z-x|/|z-y| < 1/q_x\} \qquad \text{and} \\ B_y &:= \{z \in \mathbf{R}^n : |z-y|/|z-x| < 1/q_y\} \end{aligned}$$

lie completely in G. We collect some immediate results regarding these balls.

(1) $B_x \subseteq G$ and $\overline{B_x} \cap \partial G \neq \emptyset$, similarly for B_y.
(2) If B_1 and B_2 are balls that satisfy the conditions of item (1) then there exists $x \in B_1$ and $y \in B_2$ such that $B_x = B_1$ and $B_y = B_2$.
(3) If i_x and i_y denote the inversions in the spheres ∂B_x and ∂B_y then $y = i_x(x) = i_y(x)$.
(4) Since $\infty \notin G$ we have $q_x, q_y \geq 1$. If moreover $\infty \notin \overline{G}$, then $q_x, q_y > 1$.
(5) Let x_0 denote the center of B_x and r_x its radius. We have

$$|x - x_0| = \frac{|x-y|}{q_x^2 - 1} = \frac{r_x}{q_x}.$$

(6) The ball B_x in (3.2) is decreasing (in the partial order defined by set inclusion) in q_x for fixed x and y.

4. Quasi-isotropy

In this section we introduce the concept of quasi-isotropy of a metric and consider some basic implications of quasi-isotropy for the Apollonian metric.

Definition 4.1. We say that a metric space (G, d) with open $G \subseteq \mathbf{R}^n$ is K*-quasi-isotropic* if

$$\limsup_{r\to 0} \frac{\sup\{d(x,z) : |x-z| = r\}}{\inf\{d(x,y) : |x-y| = r\}} \leq K$$

for every $x \in G$. A metric which is 1-quasi-isotropic is said to be *isotropic*, whereas a metric that is not K-quasi-isotropic for any K is said to be *anisotropic*.

Since the only metric that we will be considering which is not isotropic is the Apollonian metric (see Example 4.4), we say that a domain $G \subsetneq \mathbf{R}^n$ is quasi-isotropic if (G, α_G) is, similarly for isotropic and anisotopic.

Since the metric j_G is isotropic, it follows that every domain which has the comparison property is also quasi-isotropic, as can be seen from the following inequalities

$$\limsup_{r\to 0} \frac{\sup\{\alpha_G(x,z) : |x-z| = r\}}{\inf\{\alpha_G(x,y) : |x-y| = r\}} \leq \limsup_{r\to 0} \frac{\sup\{2j_G(x,z) : |x-z| = r\}}{\inf\{j_G(x,y)/L : |x-y| = r\}} = 2L,$$

where L is the constant from the definition of the comparison condition.

The following lemma provides an alternative characterization for the quasi-isotropic domains, except that the constant may be off by a factor of 2.

Lemma 4.2. *If the domain $G \subsetneq \mathbf{R}^n$ is L-quasi-isotropic then*

$$\frac{1}{L} \le \inf_{x\in G} \liminf_{z\to x} \frac{\alpha_G(x,z)}{j_G(x,z)}. \tag{4.3}$$

Conversely, if (4.3) *holds then G is $2L$-quasi-isotropic.*

Proof. Let us start by noting that

$$\alpha_G(x,y) = \sup_{a,b\in\partial G} \log \frac{|a-x|}{|a-y|}\frac{|b-y|}{|b-x|} = \sup_{a,b\in G^c} \log \frac{|a-x|}{|a-y|}\frac{|b-y|}{|b-x|}$$

so that it does not matter whether we take the supremum over ∂G or over G^c.

Assume that G is L-quasi-isotropic and fix $x \in G$. Let $w \in \partial G$ be such that $|x-w| = \delta(x)$ and set $r := (w-x)/|x-w|$. For $0 < \varepsilon < |x-w|$

$$\alpha_G(x, x+\varepsilon r)/j_G(x, x+\varepsilon r) \ge 1,$$

as is directly seen by choosing $a = w$ and $b = \infty$ in definition of the Apollonian metric, (1.1), for a lower bound. This implies that

$$\begin{aligned}\liminf_{z\to x} \frac{\alpha_G(x,z)}{j_G(x,z)} &= \liminf_{|x-z|=\varepsilon\to 0} \frac{\alpha_G(x,z)}{\alpha_G(x,x+\varepsilon r)} \frac{j_G(x,x+\varepsilon r)}{j_G(x,z)} \frac{\alpha_G(x,x+\varepsilon r)}{j_G(x,x+\varepsilon r)} \\ &\ge \liminf_{|x-z|=\varepsilon\to 0} \frac{\alpha_G(x,z)}{\alpha_G(x,x+\varepsilon r)} \frac{j_G(x,x+\varepsilon r)}{j_G(x,z)}.\end{aligned}$$

It is easy to see that

$$\frac{j_G(x,x+\varepsilon r)}{j_G(x,z)} \ge \frac{\log\big(1+\varepsilon/\big(\delta(x)+\varepsilon\big)\big)}{\log\big(1+\varepsilon/\big(\delta(x)-\varepsilon\big)\big)},$$

for $|x-z| = \varepsilon < \delta(x)$. Since $t \mapsto \log(1+t)/t$ is decreasing we find that

$$\frac{\log(1+u)}{\log(1+v)} \ge \frac{u}{v}$$

(for positive u and v) if and only if $u \le v$. Therefore we get

$$\frac{j_G(x,x+\varepsilon r)}{j_G(x,z)} \ge \frac{\log\big(1+\varepsilon/\big(\delta(x)+\varepsilon\big)\big)}{\log\big(1+\varepsilon/\big(\delta(x)-\varepsilon\big)\big)} \ge \frac{\delta(x)-\varepsilon}{\delta(x)+\varepsilon}.$$

Combining the previous two estimates gives

$$\liminf_{z\to x} \frac{\alpha_G(x,z)}{j_G(x,z)} \ge \liminf_{|x-z|=\varepsilon\to 0} \frac{\alpha_G(x,z)}{\alpha_G(x,x+\varepsilon r)} \frac{\delta(x)-\varepsilon}{\delta(x)+\varepsilon} \ge \frac{1}{L}.$$

Assume conversely that (4.3) holds. Then

$$\frac{1}{L} \le \liminf_{z\to x} \frac{\alpha_G(x,z)}{j_G(x,z)} = \liminf_{|x-y|=|x-z|\to 0} \frac{\alpha_G(x,z)}{j_G(x,y)} \le \liminf_{|x-y|=|x-z|\to 0} \frac{\alpha_G(x,z)}{\alpha_G(x,y)/2},$$

where we again used that j_G is isotropic, as in the previous paragraph. Hence G is $2L$-quasi-isotropic. □

We are now ready to present an example of a domain which is not quasi-isotropic.

Example 4.4. The domain $G := H^n \setminus [0, e_n]$ is not quasi-isotropic. For let $t > 0$ and consider the points $x_t := (1+t)e_n$ and $y_t = x_t + re_1$ (r is specified later). We define

$$p_1 := \sup_{a\in\partial H^n} \frac{|x_t - a|}{|y_t - a|}, \quad p_2 := \sup_{b\in\partial H^n} \frac{|y_t - b|}{|x_t - b|}, \quad \text{and} \quad p_3 := \sup_{c\in[0,e_n]} \frac{|y_t - c|}{|x_t - c|}.$$

By what amounts to dividing the supremum in the definition of α_G into parts we get

$$\alpha_G(x_t, y_t) = \log \max\{p_1 p_2, p_3 p_2\}.$$

Since $p_1, p_2, p_3 \ge 1$ and $\alpha_{H^n}(x_t, y_t) = \log(p_1 p_2)$ we get

$$\alpha_G(x_t, y_t) = \log \max\{p_1 p_2, p_3 p_2\} \le \log(p_1 p_2 p_3) = \alpha_{H^n}(x_t, y_t) + \log p_3.$$

Since $\alpha_{B^n} = h_{B^n}$ (see for instance [3]) it follows from [2, p. 35] that

$$\alpha_{H^n}(x_t, y_t) = \operatorname{arcosh}\big(1 + r^2/\big(2(1+t)^2\big)\big).$$

A simple calcualtion shows that $p_3 = \sqrt{t^2 + r^2}\,/t = \sqrt{1 + (r/t)^2}$. Therefore we have shown that

$$\alpha_G(x_t, y_t) \le \operatorname{arcosh}\big(1 + r^2/\big(2(1+t)^2\big)\big) + \log\sqrt{1 + (r/t)^2}\,.$$

On the other hand we see that $j_G(x_t, y_t) = \log(1 + r/t)$. Let us choose $r = t^2$. Then we find that

$$\frac{\alpha_G(x_t, y_t)}{j_G(x_t, y_t)} \le \frac{\operatorname{arcosh}\big(1 + t^4/\big(2(1+t)^2\big)\big) + \log(1+t^2)/2}{\log(1+t)}.$$

But this means that

$$\frac{\alpha_G(x_t, y_t)}{j_G(x_t, y_t)} \to 0$$

as $t \to 0$. Hence, by Lemma 4.2, G is not quasi-isotropic.

Using the previous lemma we can also show that the quasi-isotropy property implies a local version of the comparison property.

Lemma 4.5. *Let G be L-quasi-isotropic. For every compact subset K of G and every $\varepsilon > 0$ there exists a constant $\delta > 0$ such that $\alpha_G(x, y) \ge j_G(x, y)/(L+\varepsilon)$ for every $x, y \in K$ with $|x - y| < \delta$.*

Proof. By Lemma 4.2 we know that

$$\liminf_{y\to x} \alpha_G(x,y)/j_G(x,y) \ge 1/L.$$

Next we note that it follows easily from the definitions of α_G and j_G that $t \mapsto \alpha_G(x,x+te)/j_G(x,x+te)$ is continuous for $|t| < \frac{1}{2}\delta(x)$, where e is a fixed unit vector. It follows that for every $x \in K$ there exists a $t_0(x,re) > 0$ such that $\alpha_G(x,x+te)/j_G(x,x+te) \ge 1/(L+\varepsilon)$ for $|t| < t_0(x,e)$, moreover, by continuity of α_G/j_G, the function t_0 may be chosen to be continuous in x and e. Since x is in the compact set K and e is in the compact set S^{n-1}, we see that the claim of the lemma holds for $\delta = \min_{x,e} t_0(x,e) > 0$. ◻

Lemma 4.6. *Let $G \subsetneq \mathbf{R}^n$ be an L-quasi-isotropy domain and $x \in G$. For $K \ge 1$ we have*

$$\limsup_{z\to x}\biggl\{\sup\biggl\{\frac{\alpha_G(x,y)}{\alpha_G(x,z)} : \frac{|x-z|}{K} \le |x-y| \le K|x-z|\biggr\}\biggr\} \le 2KL.$$

Proof. For $y,z \in B^n\bigl(x,\delta(x)/K\bigr)$ such that $|x-z|/K \le |x-y| \le K|x-z|$ let $w = w_{y,z}$ be the point on the ray from x through y with $|x-w| = |x-z|$. Since $\log(1+u)/\log(1+v) \le \max\{1,u/v\}$ for $u,v>0$ (proved as in the proof of Lemma 4.2) we find that

$$\frac{j_G(x,y)}{j_G(x,w)} \le \max\biggl\{1, \biggl(\frac{\delta(x)+K|x-z|}{\delta(x)-K|x-z|}\biggr)\frac{|x-y|}{|x-w|}\biggr\} \le K\frac{\delta(x)+K|x-z|}{\delta(x)-K|x-z|}.$$

Hence we get

$$\begin{aligned}\limsup_{z\to x}\biggl\{\sup\frac{\alpha_G(x,y)}{\alpha_G(x,z)}\biggr\} &\le 2\limsup_{z\to x}\biggl\{\sup\frac{j_G(x,y)}{\alpha_G(x,z)}\biggr\}\\ &\le 2\limsup_{z\to x}\biggl\{K\frac{\delta(x)+K|x-z|}{\delta(x)-K|x-z|}\sup\frac{j_G(x,w)}{\alpha_G(x,z)}\biggr\}\\ &\le 2KL,\end{aligned}$$

where the suprema are over the same set of points as the supremum in the statement of the lemma. The last inequality follows from Lemma 4.2. ◻

Proposition 4.7. *Let $G \subsetneq \mathbf{R}^n$ be a K-quasi-isotropy domain and $f\colon G \to f(G) \subseteq \mathbf{R}^n$ be a quasiconformal mapping which is also M-Apollonian bilipschitz. Then $f(G)$ is $2KLM^2$-quasi-isotropic, where $L := \sup_{x\in f(G)} H(f^{-1},x)$.*

Proof. Fix a point $x' =: f(x)$ in $G' := f(G)$ and $\varepsilon > 0$. Let $U \subseteq G$ be a neighborhood of x such that

$$1/(K+\varepsilon) \le \alpha_G(x,y)/\alpha_G(x,z) \le K+\varepsilon,$$

whenever $|x-y| = |x-z|$ and $y,z \in U$ (such a U exists since G is K-quasi-isotropic). Let $V' \subseteq G'$ be a neighborhood of x' such that

$$1/(L+\varepsilon) \le |x-y|/|x-z| \le L+\varepsilon,$$

for $|x'-y'| = |x'-z'|$, $y := f^{-1}(y')$, $z := f^{-1}(z')$ and $y', z' \in V'$ (such a V' exists since $H(f^{-1},x) \le L$).

For $y', z' \in f(U) \cap V'$ with $|z'-x'| = |y'-x'|$ we have

$$\frac{\alpha_{f(G)}(x',y')}{\alpha_{f(G)}(x',z')} \le M^2 \frac{\alpha_G(x,y)}{\alpha_G(x,z)} \le 2M^2(K+\varepsilon)(L+\varepsilon),$$

where the first inequality follows since f is Apollonian bilipschitz and the second one follows from Lemma 4.6 in view of what was shown of the points x, y, z in the previous paragraph. The claim follows as $\varepsilon \to 0$. □

The previous proposition says that quasi-isotropy is preserved under quasiconformal mappings that are Apollonian bilipschitz. This should be contrasted with Corollary 5.15 of [12], which says that the comparison property is preserved under Euclidean bilipschitz mappings with constants near 1. Notice that quasi-isotropy and quasiconformality are local properties, whereas the comparison and bilipschitz properties are global. Hence one might hope that the condition that f be Apollonian bilipschitz could be removed from the previous proposition. This turns out not to be the case, however.

5. Inner metrics

In this section we define the inner metric of a metric and consider the inner metrics of the metrics from Section 3 and of the Apollonian metric.

By a path we mean a continuous function $\gamma\colon [0,l] \to \mathbf{R}^n$, $l > 0$. We assume throughout this section that the l in the previous sentence equals 1 for every path considered. Let $G \subseteq \mathbf{R}^n$ and d be a metric in G. The length in (G,d) of the path $\gamma \subseteq G$ (as usual, we sometimes identify the path with its image in $\mathbf{R}^n$) is defined by

$$l_d(\gamma) := \sup \sum_{i=0}^{k-1} d\big(\gamma(t_i), \gamma(t_{i+1})\big),$$

where the supremum is taken over all sequences of sequences $\{t_i\}$ satisfying $0 = t_0 < t_1 < \cdots < t_k = 1$. If the supremum is finite, then γ is said to be d-*rectifiable.* We denote by $l(\gamma)$ the Euclidean length of a path and call an $|\cdot|$-rectifiable path rectifiable. For brevity we call a sequence $\{t_i\}_0^k$ satisfying $0 = t_0 < t_1 < \ldots < t_k = 1$ a *length sequence* throughout this section.

Definition 5.1. Let d be a metric in the domain $G \subseteq \mathbf{R}^n$. The *inner metric* of d, denoted by $\tilde{d}$, is defined by $\tilde{d}(x,y) := \inf_\gamma l_d(\gamma)$, where the infimum is taken over all paths connecting x and y in G.

Remark 5.2. It is clear that that $d \le \tilde{d}$, by repeated use of the triangle inequality. Note that it is possible that $\tilde{d}(x,y) = \infty$, in which case the inner metric is not a metric, but this will not happen for the metric spaces that we consider.

The following result is well known.

Lemma 5.3. *For $G \subsetneq \mathbf{R}^n$ we have $\tilde{j}_G = k_G$.*

Proof. Follows for instance using [22, Theorem 3.7(1) and (3)]. □

It turns out to be quite difficult to describe the inner metric of the Apollonian metric, which is perhaps not so surprising, given that this metric is not, in contrast to the j_G metric, isotropic. We therefore only derive some estimates of $\tilde{\alpha}_G$ in this paper, without deriving an explicit formula of it. Using Lemma 4.2 we obtain the following result for the inner metrics of α_G and j_G.

Corollary 5.4. *If the domain $G \subsetneq \mathbf{R}^n$ is L-quasi-isotropic then $k_G/L \le \tilde{\alpha}_G \le 2k_G$.*

Proof. Consider first the second inequality. Fix $x, y \in G$ and let $\gamma \subseteq G$ be a path connecting them. Then for every length sequence we have

$$\alpha_G\big(\gamma(t_i), \gamma(t_{i+1})\big) \le 2 j_G\big(\gamma(t_i), \gamma(t_{i+1})\big),$$

and so

$$\sum_{i=0}^{k-1} \alpha_G\big(\gamma(t_i), \gamma(t_{i+1})\big) \le 2 \sum_{i=0}^{k-1} j_G\big(\gamma(t_i), \gamma(t_{i+1})\big) \le 2 l_{j_G}(\gamma).$$

It follows that

$$l_{\alpha_G}(\gamma) = \sup \sum_{i=0}^{k-1} \alpha_G\big(\gamma(t_i), \gamma(t_{i+1})\big) \le 2 l_{j_G}(\gamma),$$

where the supremum is taken over all length sequences. This implies that the same inequality holds also for the infima, and so the inequality $\tilde{\alpha}_G \le 2k_G$ follows.

By Lemma 4.5 we know that for $\varepsilon > 0$ there exists $r_0 > 0$ such that

$$\alpha_G(x,y) \ge j_G(x,y)/(L+\varepsilon),$$

for $x, y \in \gamma$ with $|x-y| < r_0$. We may then argue as in the first part of the proof to show that

$$l_{\alpha_G}(\gamma) \ge l_{j_G}(\gamma)/(L+\varepsilon),$$

since the restriction to length sequences satisfying $|\gamma(t_i) - \gamma(t_{i+1})| < r_0$ is not important, as we may assume that $|\gamma(t_i) - \gamma(t_{i+1})| \to 0$, anyway. Since ε was arbitrary it follows that $l_{\alpha_G}(\gamma) \geq l_{j_G}(\gamma)/L$, and since γ was arbitrary, it follows that

$$\tilde{\alpha}_G(x,y) = \inf_\gamma l_{\alpha_G}(\gamma) \geq \inf_\gamma l_{j_G}(\gamma)/L = k_G(x,y)/L. \square$$

Definition 5.5. The *directed density* of the metric d at the point $x \in G$ in direction $r \in \mathbf{R}^n \setminus \{0\}$ is defined by

$$\bar{d}(x;r) = \lim_{t \to 0+} d(x, x+tr)/(t|r|),$$

if the limit exists.

If $\bar{d}(x;r)$ is independent of r in every point of G then (G,d) is isotropic and we may denote $\bar{d}(x) := \bar{d}(x;e_1)$ and call this function the *density* of d at x. For the density of the j_G metric we have the following expression.

Lemma 5.6. *For $x \in G \subsetneq \mathbf{R}^n$ we have $\bar{j}_G(x) = 1/\delta(x)$.*

Proof. Follows directly from the definition of the density. $\square$

We next present a geometric method of calculating the density of the Apollonian metric.

Definition 5.7. Let $G \subsetneq \mathbf{R}^n$ and $x \in G$ and $e \in S^{n-1}$. Let $r_\pm \in (0,\infty]$ be such that $B^n(x+se,|s|) \subseteq G$ if and only if $-r_- \leq s \leq r_+$ (excluding equality for $r_+ = \infty$ or $r_- = \infty$). We define the *Apollonian spheres through x in direction e* by $S_+ := S^{n-1}(x + r_+e, r_+)$ and $S_- := S^{n-1}(x - r_-e, r_-)$ for finite r_+ or r_- and the limiting hyper-plane otherwise.

The next lemma shows that the Apollonian spheres from the previous definition correspond to the Apollonian balls B_x and B_y as $y \to x$ from direction e. The reason for now considering spheres instead of balls is simply to allow for the expression "S_+ through x" which corresponds to "B_x about x" for the non-limiting case.

Lemma 5.8. *Let $x \in G \subsetneq \mathbf{R}^n$, $r \in S^{n-1}$ and $r_\pm$ be the radii of the Apollonian spheres $S_\pm$ at x in direction r. Then*

$$\bar{\alpha}_G(x;r) = \frac{1}{2r_+} + \frac{1}{2r_-}.$$

Proof. Let us denote

$$f(t,a,b) := \frac{1}{t} \log\left(\frac{|x-a|}{|x+tr-a|} \frac{|x+tr-b|}{|x-b|} \right).$$

Then by definition we have

$$\bar{\alpha}_G(x;r) = \lim_{t\to 0+} \sup_{a,b\in\partial G} f(t,a,b),$$

provided the limit exists. We start by showing the limit indeed exists and that

$$\lim_{t\to 0+} \sup_{a,b\in\partial G} f(t,a,b) = \sup_{a,b\in\partial G} \lim_{t\to 0+} f(t,a,b). \tag{5.9}$$

Let us denote $g(a,b) := \lim_{t\to 0+} f(t,a,b)$ and show that this limit exists. Using the formula $|x+tr-a|^2 = |x-a|^2 + t^2 - 2|x-a|t\cos\theta$ and the corresponding one for $|x+tr-b|$ we find that

$$\lim_{t\to 0+} \frac{1}{t} \log \frac{|x-a|}{|x+tr-a|} \frac{|x+tr-b|}{|x-b|} = \frac{\cos\theta}{|x-a|} + \frac{\cos\phi}{|x-b|},$$

where θ is the angle between r and $x-a$ and ϕ is the angle between $-r$ and $x-b$.

Hence $(a,b) \mapsto g(a,b)$ is continuous and we see that there exist points a_0 and b_0 in ∂G such that $\sup_{a,b\in\partial G} g(a,b) = g(a_0,b_0)$. It is easy to see that $\lim_{t\to 0+} f(t,a_0,b_0) = g(a_0,b_0)$ and so it follows that

$$\liminf_{t\to 0+} \sup_{a,b\in\partial G} f(t,a,b) \geq \sup_{a,b\in\partial G} \lim_{t\to 0+} f(t,a,b).$$

To prove the opposite inequality fix $\varepsilon > 0$. Since $(a,b) \mapsto f(t,a,b)$ is continuous for $t \leq \frac{1}{2}\delta(x)$ we see that

$$h(t) := \max_{a,b\in\partial G} |g(a,b) - f(t,a,b)|$$

exists for all such t. Since $t \mapsto f(t,a,b)$ is continuous, h is continuous as well, and since $h \to 0$ as $t \to 0+$ we can find $t_0 > 0$ such that $h(t) < \varepsilon$ for every positive $t < t_0$. Then

$$\sup_{a,b\in\partial G} f(t,a,b) \leq \sup_{a,b\in\partial G} g(a,b) + \varepsilon$$

for the same range of t and it follows that

$$\limsup_{t\to 0+} \sup_{a,b\in\partial G} f(t,a,b) \leq \sup_{a,b\in\partial G} \lim_{t\to 0+} f(t,a,b) + \varepsilon.$$

Since ε was arbitrary we find that

$$\limsup_{t\to 0+} \sup_{a,b\in\partial G} f(t,a,b) \leq \sup_{a,b\in\partial G} \lim_{t\to 0+} f(t,a,b) \leq \liminf_{t\to 0+} \sup_{a,b\in\partial G} f(t,a,b),$$

so that (5.9) is proved.

We have thus shown that

$$\bar{\alpha}_G(x;r) = \sup_{a,b\in\partial G} \frac{\cos\theta}{|x-a|} + \frac{\cos\phi}{|x-b|}$$

with θ and ϕ as before. Let r'_+ and r'_- denote the radii of the spheres through x with center on the line $\{x+tr : t\in \mathbf{R}\}$ which also pass through a and b, respectively. By elementary trigonometry one derives $\cos\theta/|x-a| = 1/(2r'_+)$ and $\cos\phi/|x-b| = 1/(2r'_-)$ so that $g(a,b) = 1/(2r'_+)+1/(2r'_-)$. We therefore see that the supremum is achieved by choosing a and b such that $r'_+ = r_+$ and $r'_- = r_-$. □

Remark 5.10. The content of the previous lemma seems to correspond to that of Lemma 5.1.4 of [5], where the directed density is called a Lagrangian structure.

Using the directed densities we can restate Lemma 4.2 in a form which is more practical to check.

Corollary 5.11. *Let $G\subsetneq \mathbf{R}^n$. If G is L-quasi-isotropic then $\bar{\alpha}_G(x;r)\delta(x)\ge 1/L$ for every $x\in G$ and $r\in S^{n-1}$. If conversely $\bar{\alpha}_G(x;r)\delta(x)\ge 1/L$ for every $x\in G$ and $r\in S^{n-1}$ then G is $2L$-quasi-isotropic.*

Proof. This follows from Lemmata 4.2 and 5.6, since

$$\begin{aligned}\liminf_{z\to x}\frac{\alpha_G(x,z)}{j_G(x,z)} &= \liminf_{z\to x}\frac{\alpha_G(x,z)}{|x-z|}\,\frac{|x-z|}{j_G(x,z)}\\ &= \liminf_{z\to x}\frac{\alpha_G(x,z)}{|x-z|}\,\lim_{z\to x}\frac{|x-z|}{j_G(x,z)}\\ &= \inf_{r\in S^{n-1}}\frac{\bar{\alpha}_G(x;r)}{\bar{j}_G(x)},\end{aligned}$$

where the second equality follows since

$$z\mapsto \frac{\alpha_G(x,z)}{|x-z|} \qquad\text{and}\qquad z\mapsto\frac{|x-z|}{j_G(x,z)}$$

are continuous in $G\setminus\{x\}$ with both limit inferior and superior unequal to 0 and ∞. □

Lemma 5.12. *Let $G\subsetneq\mathbf{R}^n$ be such that $\bar{\alpha}_G(x;r)\ge h(x)$ for every $x\in G$ and $r\in S^{n-1}$ and some continuous $h\colon G\to\mathbf{R}$. Then for $x,y\in G$ we have*

$$\tilde{\alpha}_G(x,y)\ge \inf_\gamma \int_\gamma h(z)\,|dz|,$$

where the infimum is taken over all rectifiable paths γ connecting x and y in G.

Proof. Let γ be a path connecting x and y in G. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\alpha_G(z,w)/|z-w| \ge \bar{\alpha}_G(z;z-w)/(1+\varepsilon) \ge h(z)/(1+\varepsilon),$$

for $z,w \in \gamma$ with $|z-w| < \delta$. It follows by considering the supremum over length sequences with $|\gamma(t_{i+1}) - \gamma(t_i)| < \delta$ for all i that

$$\begin{aligned}(1+\varepsilon)l_{\alpha_G}(\gamma) &= (1+\varepsilon)\sup \sum \alpha_G\bigl(\gamma(t_{i+1}),\gamma(t_i)\bigr)\\ &\ge \sum \bar{\alpha}_G\bigl(\gamma(t_i);\gamma(t_{i+1})-\gamma(t_i)\bigr)|\gamma(t_{i+1})-\gamma(t_i)|\\ &\ge \sum h\bigl(\gamma(t_i)\bigr)|\gamma(t_{i+1})-\gamma(t_i)|.\end{aligned}$$

The right-hand side is just the Riemann sum for the integral in the lemma, and so we have shown that

$$\inf_\gamma l_{\alpha_G}(\gamma) \ge \inf_\gamma \int_\gamma h(z)\,|dz|,$$

where both infima are over rectifiable paths connecting x and y in G. By definition the first infimum equals $\tilde{\alpha}_G(x,y)$ which completes the proof. □

6. Quasiconvexity and A-uniform domains

In this section we introduce the concept of A-uniform domains, which allows us to integrate the results from the previous sections and to prove two of the main theorems. The following definition is from [22, Section 2].

Definition 6.1. A metric space (X,d) is said to be *K-quasiconvex* if for every $x,y \in X$ there exists a path $\gamma \subseteq X$ joining x and y in X such that $l_d(\gamma) \le Kd(x,y)$. A domain $G \subseteq \mathbf{R}^n$ is said to be quasiconvex if the metric space $(G,|\cdot|)$ is quasiconvex.

We note first that an inner metric is K-quasiconvex for every $K>1$ and that it may or may not be 1-quasiconvex. Hence if d is K-quasiconvex then $\tilde{d} \le Kd$ and if $\tilde{d} \le Kd$ then d is K'-quasiconvex for every $K' > K$.

Definition 6.2. A domain $G \subsetneq \mathbf{R}^n$ is said to be *uniform* with constant K if for every $x,y \in G$ there exists a rectifiable path γ, parameterized by arc-length, connecting x and y in G, such that

(1) $l(\gamma) \le K|x-y|$ and
(2) $K\delta\bigl(\gamma(t)\bigr) \ge \min\{t, l(\gamma)-t\}$.

Notice that the first condition implies that G is quasiconvex. Uniform domains were introduced in [18, 2.12], but Definition 6.2 is an equivalent form from [10, (1.1)]. From the latter paper we also need the following result, which says that a domain G is uniform if and only if j_G is quasiconvex.

Lemma 6.3 ([10, Corollary 1]). *The domain $G \subsetneq \mathbf{R}^n$ is uniform if and only if there exists a constant K such that $k_G \le K j_G$.*

Note that in [10] the second condition is in the form $k_G \le c j_G + d$. However, the two forms are equivalent since (for instance) $2j_G(x,y) \ge k_G(x,y)$ in every domain $G \subsetneq \mathbf{R}^n$ for points x and y with $j_G(x,y) < \log\bigl(\frac{3}{2}\bigr)$ by [23, (2.34)]. From the previous lemma it also follows that every quasiball is uniform. In $\mathbf{R}^2$ we have the following stronger result:

Lemma 6.4 ([18, Theorem 2.24]). *Let G be a simply connected planar domain. Then G is uniform if and only if it is a quasidisk.*

Definition 6.5. A domain $G \subsetneq \mathbf{R}^n$ is *A-uniform* with constant K if $k_G \le K\alpha_G$. A domain $G \subsetneq \mathbf{R}^n$ is said to be A-uniform if it is A-uniform with some constant $K < \infty$.

Since $\alpha_G \le 2j_G$ in every domain $G \subsetneq \mathbf{R}^n$, it is clear that A-uniformity implies uniformity. The following proposition makes the relationship clearer.

Proposition 6.6. *Let $G \subsetneq \mathbf{R}^n$ be a domain. The following conditions are equivalent*:

(1) *G is A-uniform*;
(2) *G is uniform and has the comparison property*;
(3) *G is quasi-isotropic and α_G is quasiconvex.*

Proof. Suppose first that G is A-uniform with constant K. Then

$$j_G \le k_G \le K\alpha_G \le 2Kj_G.$$

From this it is directly seen that $j_G \le K\alpha_G$ and $k_G \le 2Kj_G$, the comparison property and uniformity, respectively; hence (1) $\Rightarrow$ (2).

Suppose next that (2) holds. It is clear that the comparison property implies that G is quasi-isotropic. By Corollary 5.4, uniformity with constant K and the comparison property with constant L we conclude that $\tilde{\alpha}_G \le 2k_G \le 2Kj_G \le 2KL\alpha_G$, so that α_G is also quasiconvex. We have thus proved that (2) $\Rightarrow$ (3).

Suppose finally that α_G is quasiconvex and G is quasi-isotropic, with constants K and L, respectively. Then, using Corollary 5.4 for the first inequality, we find that $k_G \le L\tilde{\alpha}_G \le KL\alpha_G$ and hence G is A-uniform, which proves the implication (3) $\Rightarrow$ (1). □

Using the previous proposition we see that our old acquaintance $H^n \setminus [0, e_n]$ is not A-uniform (recall that we showed in Example 4.4 that this domain is not quasi-isotropic). Nevertheless $H^n \setminus [0, e_n]$ is uniform provided that $n \ge 3$, as can be seen directly from the definition. We thus see that the class of A-uniform domains is a proper subset of the class of uniform domains for $n \ge 3$.

We are now ready to prove the first main result, Theorem 1.8, the generalization to space of the quasidisk theorem. The proof consists of a bunch of references to the previous auxiliary results, but the argument is basically that quasiconvexity is preserved under bilipschitz mappings, a quite trivial fact. Note that [9, Theorem 3.29] was based on a similar argument, which will be repeated in the proof of Theorem 1.9.

Proof of Theorem 1.8. Assume first that f is quasiconformal. Since G is A-uniform, α_G is quasiconvex by Proposition 6.6. Since f is an Apollonian bilipschitz mapping, $\alpha_{G'}$ is also quasiconvex, where $G' := f(G)$. It then follows from Proposition 4.7 that G' is quasi-isotropic. Hence, using Proposition 6.6 again, we see that G' is A-uniform.

Assume conversely that G' is A-uniform. Then both G and G' have the comparison property, by Proposition 6.6, and so it follows as in the proof of [12, Theorem 1.4] that f is quasiconformal in G. □

We next give a geometric characterization of A-uniform domains. The following definition is taken from [12].

Definition 6.7. We say that a domain $G \subsetneq \mathbf{R}^n$ satisfies an *interior double ball condition* with constant L (abbreviated L-IDB condition) if there exists a boundary point $z \in \partial G \setminus \{\infty\}$ and a real number $r > 0$ such that $B^n(z, 2r) \cap G$ contains two disjoint balls with radii r/L.

In [12, Theorem 5.13] it was shown that a domain does not have the L-IDB property for every $L > 1$ if and only if it has the comparison property. This, combined with Proposition 6.6, implies the following result, which gives a geometric characterization of domains that are A-uniform.

Corollary 6.8. *The domain G is A-uniform if and only if it is uniform and for some $L > 1$ it does not have the L-IDB property.*

We end this section by considering the relationship between A-uniform domains and quasiballs.

Corollary 6.9. *Every quasiball is A-uniform.*

Proof. As was noted after Lemma 6.3, every quasiball is uniform. It was shown in [1, Corollary 1.3] that quasiballs have the comparison property, hence the claim follows from Proposition 6.6. □

Corollary 6.10. *A simply connected planar domain is a quasidisk if and only if it is A-uniform.*

Proof. The sufficiency follows from the previous corollary. If G is A-uniform, then it follows from Proposition 6.6 that it is uniform, hence a quasidisk, by Lemma 6.4. □

It remains an open question whether there exists a domain topologically equivalent to B^n which is A-uniform but not a quasiball. If we do not require that a domain be a topologically ball the claim is obviously false, consider for instance $B^n(0,1) \setminus B^n\big(0,\frac{1}{2}\big)$.

7. Results in the plane

In this section we derive some results that are only valid in the plane, in particular, we prove Theorem 1.7. We start with a more general lemma, which is also valid in $\mathbf{R}^n$. We make use of the hyperbolic metric in the half-space, h_{H^n}. For properties of this metric the reader is referred to [2], [24, Section 2], or any introductory text on the hyperbolic metric.

Lemma 7.1. *Let $G \subseteq \mathbf{R}^n$ be a domain such that $G \cap B^n = H^n \cap B^n$. Then for every $0 < s < 1$ and every path $\gamma \subset G$ connecting se_n with S^{n-1} we have*

$$l_{\alpha_G}(\gamma) \geq \tfrac{1}{2}(\operatorname{arsinh} s^{-1} - \operatorname{arsinh} 1).$$

Proof. Let us define

$$C := \big\{x \in H^n \cap B^n : x_n + |x - x_n e_n| < 1\big\},$$

where x_n denotes the n^{th} coordinate of x. We will show that $\bar{\alpha}_G(x;r) \geq \frac{1}{2}\bar{h}_{H^n}(x)$ for every $x \in C$ and every $r \in S^{n-1}$. From this it follows as in Lemma 5.12 that $l_{\alpha_G}(\gamma) \geq \frac{1}{2} l_{h_{H^n}}(\gamma)$ for all paths $\gamma \subseteq C$.

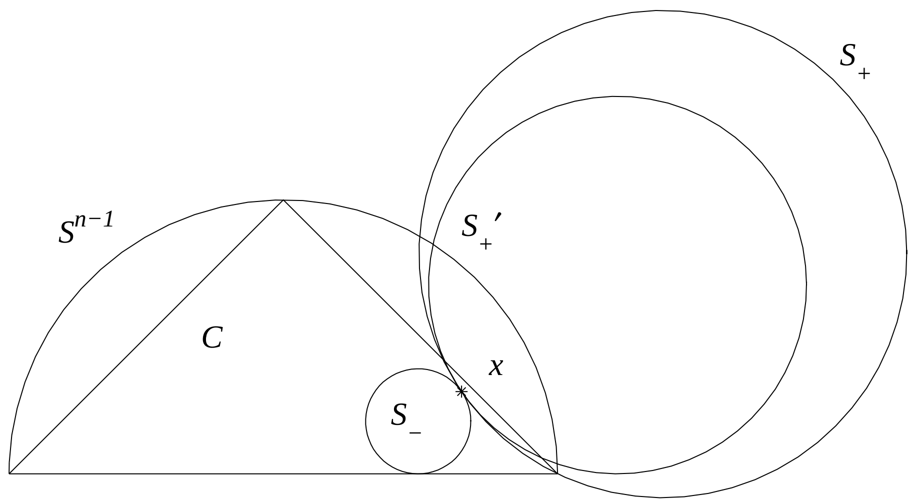

Figure 1. The density at $x = 0.65e_1 + 0.3e_2$.

Let then $x \in C$. In order to get a lower bound of $\bar{\alpha}_G(x;r)$ we need to get an upper bound for the radius of at least one of the Apollonian spheres through x, by Lemma 5.8. Consider the direction $r \in S^{n-1} \cap H^n$. Since $\partial G \supset B^n \cap \partial H^n$, we may assume that $\partial G = (B^n \cap \partial H^n) \cup \{\infty\}$ since we are deriving a lower bound for $\bar{\alpha}_G$, which, like α_G, is decreasing in ∂G. It is then clear that the radius r_- is the same as it would be in H^n, and less than r_+ (see Figure 1). Hence we conclude that

$$\bar{\alpha}_G(x;r) = \frac{1}{2r_+} + \frac{1}{2r_-} \geq \frac{1}{4r_-} + \frac{1}{4r_-} \geq \frac{1}{4r_-} + \frac{1}{4r'_+} = \frac{\bar{h}_{H^n}(x)}{2},$$

where r'_+ is the radius of the Apollonian sphere through x in the direction r in the domain H^n. The second inequality follows since $r_- \leq r'_+$ and the last equality follows since $\alpha_{H^n} = h_{H^n}$, hence we may use the Apollonian spheres to obtain $\bar{h}_{H^n}(x)$, as well.

Let us then evaluate $\inf_\gamma l_{h_{H^n}}(\gamma)$, where the infimum is taken over all paths γ connecting se_n with C. Since the hyperbolic metric is 1-quasiconvex, it suffices to calculate the distance $h_{H^n}(se_n, C)$. Let $B_h(se_n, R)$ denote the hyperbolic ball about se_n with radius R. It is known that the hyperbolic balls about se_n are Euclidean balls, more specifically, $B_h(se_n, R) = B^n\big(s\cosh(R)e_n, s\sinh R\big)$, by [24, (2.11)]. Let R_0 be such that $B_h(se_n, R_0)$ is tangent to C. Then we have

$$h_{H^n}(se_n, C) = \max_{B_h(se_n, R) \subseteq C} R = R_0.$$

By elementary trigonometry we get the formula $\sqrt{2}\, s\sinh R_0 = 1 - s\cosh R_0$ for the radius R_0. From this equation we derive $\sqrt{2}\,\big(e^{R_0} - e^{-R_0}\big) + e^{R_0} + e^{-R_0} = 2/s$ and so we find that $e^{R_0} = \big(1/s + \sqrt{1 + 1/s^2}\,\big)/\big(1 + \sqrt{2}\,\big)$ from which it follows that

$$h_{H^n}(se_n, C) = R_0 = \log\big(1/s + \sqrt{1 + 1/s^2}\,\big) - \log\big(1 + \sqrt{2}\,\big) = \operatorname{arsinh} s^{-1} - \operatorname{arsinh} 1.$$

Since every path connecting se_n with S^{n-1} has a subpath connecting se_n with C we are finished. □

The next lemma, which is valid only in the plane, is a variant of Lemma 5.4, [12]. Note that the quite complicated looking conditions say basically that B is split into two large parts by ∂G.

Lemma 7.2. *Let $G \subsetneq \mathbf{R}^2$ be a simply connected domain and assume that there exist points $x, y \in G$ such that $N\alpha_G(x,y) < j_G(x,y)$ for some $N > 40$. Then there exists a disk $B = B^2(b,r)$ and a unit vector $e \in S^1$ such that*

(1) *for all $z \in G^c \cap B$ we have $\langle z - b, e\rangle \leq 4N^{-1/2} r$ and*
(2) *the points $b \pm 0.9re$ belong to different path components of $B \cap G$.*

(Here $\langle\,\cdot\,,\cdot\,\rangle$ denotes the usual inner product.)

Proof. It follows from Lemma 5.4 in [12] that there exists a point $w\in\partial G$, a unit vector $e\in S^{n-1}$ and $R>0$ such that for every $z\in G^c\cap B^n(w,R)$ we have $\langle z-w,e\rangle < 2R/\sqrt{N}$. For simplicity we assume without loss of generality that $w=0$ and $e=e_1$. Denote $r:=\frac{1}{2}R$ and consider the balls $B_+ := B^n(re_2,r)$ and $B_- := B^n(-re_2,r)$. Both balls satisfy condition (1) of the lemma. Let us denote $a_\pm := \pm re_2 + 0.9re_1$ and $b_\pm := \pm re_2 - 0.9re_1$. Suppose that neither ball satisfies condition (2) so that there exist paths $\gamma_+\subset G\cap B_+$ connecting a_+ and b_+ and $\gamma_-\subset G\cap B_-$ connecting a_- and b_-. Then the path formed by concatenating γ_+, $[b_+,b_-]$, γ_+ and $[a_-,a_+]$, which lies in G, is closed and loops around the boundary point w. But this contradicts the assumption that G is simply connected; hence either B_+ or B_- satisfies condition (2), and so is the ball whose existence we wanted to prove. □

Proposition 7.3. *If $G\subsetneq \mathbf{R}^2$ is simply connected and α_G is K-quasiconvex then G has the comparison property with constant* $1250\exp\{3.5K\}$.

Proof. Suppose that G does not have the comparison property with constant $N\geq 8000$ (the constant in the proposition is at least $1250e^{3.5}>8000$). This means that there exist $x,y\in G$ such that $N\alpha_G(x,y)<j_G(x,y)$. Let B be a disk which satisfies conditions (1) and (2) of Lemma 7.2. We assume without loss of generality that $B=B^2(0,1)$ and that $e=e_2$.

Consider the points $\pm te_2$, where $t:=2N^{-1/4}$. Since the disks

$$B_+ := B^2\big(\tfrac{1}{2}(1+t^2)e_2, \tfrac{1}{2}(1-t^2)\big)$$

and

$$B_- := B^2\big(-\tfrac{1}{2}(1+t^2)e_2, \tfrac{1}{2}(1-t^2)\big)$$

lie in G and since $\pm te_2$ are each others inverses in $\partial B_\pm$ (see Figure 2), it follows that the Apollonian disks about $\pm te_2$ are at least as large as these disks, hence

$$\alpha_G(te_2,-te_2)\leq 2\log\frac{t+t^2}{t-t^2} = 2\log\frac{1+t}{1-t} < 0.86.$$

It is clear that every path connecting te_2 and $-te_2$ passes through S^1, since the points are in different path components of $B\cap G$. Hence it follows that $\tilde\alpha_G(te_2,-te_2)\geq 2l$, where l is the minimum Apollonian length of a path connecting te_2 with S^1 in B. Let γ be any such path. If we move the boundary ∂G further away from every point in γ then we get a lower bound for its Apollonian length, since this makes the Apollonian spheres larger. This means that we can consider the domain G' with $B\cap\partial G' = B\cap\partial H'$, where

$$H' = \{x\in\mathbf{R}^2 : x+t^2e_2\in H^2\}$$

when deriving a lower bound for the part of γ in the upper component of $B\cap G$. (The boundary of G' is the heavy line in Figure 3.) But then G' is a domain

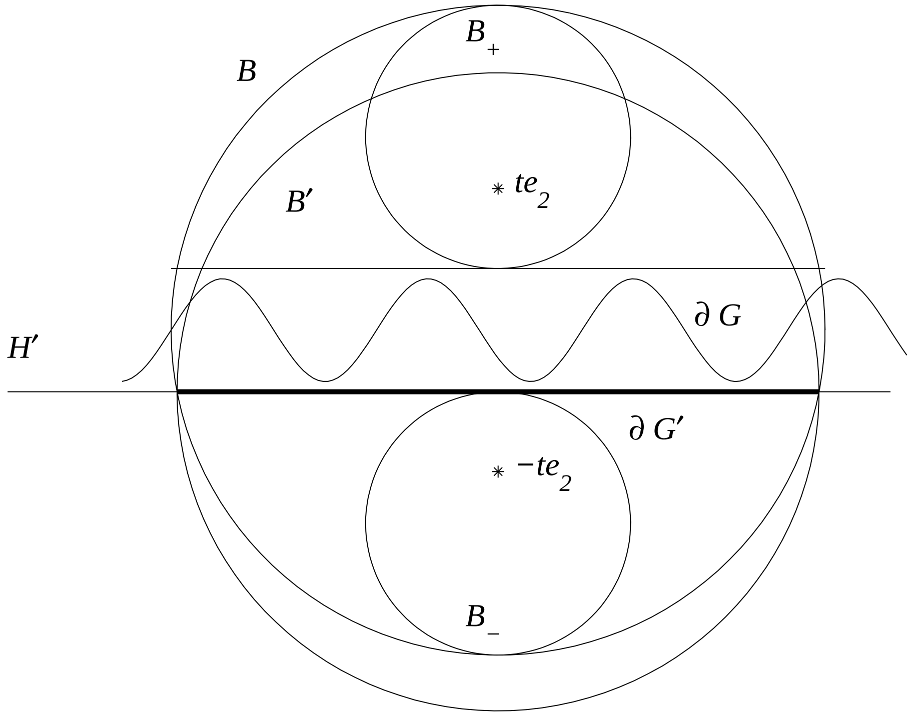

Figure 2. An example domain with a sine curve as boundary.

with $G' \cap B' = H' \cap B'$, where $B' := B^2\big(-t^2 e_2, \sqrt{1-t^2}\,\big)$. After a translation and scaling the domain G' satisfies the conditions of Lemma 7.1, with

$$s = (t+t^2)/(1-t^2) = t\sqrt{(1+t)/(1-t)} \le 1.24t.$$

Hence

$$2l_{\alpha_G}(\gamma) \ge 2l_{\alpha_{G'}}(\gamma) \ge \operatorname{arsinh} s^{-1} - \operatorname{arsinh} 1 \ge \operatorname{arsinh}(0.8t^{-1}) - \operatorname{arsinh} 1.$$

Then since G is K-quasiconvex in the Apollonian metric it follows that

$$\operatorname{arsinh}(0.8t^{-1}) - \operatorname{arsinh} 1 \le 2l \le \tilde{\alpha}_G(te_2, -te_2) \le K\alpha_G(te_2, -te_2) < 0.86K.$$

This implies that $\operatorname{arsinh}(0.8t^{-1}) \le 0.86K + \operatorname{arsinh} 1$ and so, by the addition formula for the hyperbolic sine, $\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$,

$$0.8t^{-1} \le \sqrt{2}\,\sinh\{0.86K\} + \cosh\{0.86K\} \le \big(1+\sqrt{2}\,\big)\exp\{0.86K\}.$$

From this it follows that $\frac{1}{2}N^{1/4} = t^{-1} \le 3.02\exp\{0.86K\}$ and so we find that

$$N \le 1327\exp\{3.44K\} \le 1250\exp\{3.5K\},$$

as claimed. □

Corollary 7.4. *A simply connected planar domain $G \subsetneq \mathbf{R}^2$ is a quasidisk if and only if α_G is quasiconvex.*

Proof. From Corollary 6.10 we see that a quasidisk is A-uniform, hence α_G is quasiconvex, by Proposition 6.6.

If α_G is quasiconvex then G has the comparison property by Proposition 7.3. The comparison property trivially implies quasi-isotropy and hence G is A-uniform by Proposition 6.6. It then follows from Corollary 6.10 that G is a quasidisk. □

We can now prove the strong version of the Theorem 1.2.

Proof of Theorem 1.7. Since G is a quasidisk it is A-uniform by Corollary 6.10 and hence α_G is quasiconvex by Proposition 6.6, say with constant K. Since f is Apollonian bilipschitz, say with constant L, it follows that

$$\tilde{\alpha}_{f(G)}\big(f(x), f(y)\big) \le L\tilde{\alpha}_G(x,y) \le KL\alpha_G(x,y) \le KL^2\alpha_{f(G)}\big(f(x), f(y)\big),$$

and so $\alpha_{f(G)}$ is quasiconvex, as well. It is easy to see that f is a homeomorphism so that $f(G)$ is simply connected. Hence, by Proposition 7.3, $f(G)$ has the comparison property, and so is trivially quasi-isotropic. It then follows from Proposition 6.6 that $f(G)$ is A-uniform and from Corollary 6.10 that it is a quasidisk. Since G and $f(G)$ are both quasidisks, the claim follows from [9, Theorem 3.11] or, equivalently, from [12, Corollary 1.7]. □

Remark 7.5. The previous theorem has the shortcoming of not being asymptotically sharp, i.e. even if we assume that the bilipschitz constant equals one and that G is a ball the theorem still only allows us to affirm that $f(G)$ is a quasidisk and f is quasiconformal.

8. Convexity of the Apollonian metric in space

In this section we show how to extend to $\mathbf{R}^n$ the proof by Gehring and Hag that a domain is Apollonian convex (= 1-quasiconvex) if and only if it is a disk or a half-plane ([9, Lemmata 3.18, 3.22, 3.23 and Theorem 3.26]).

Let us first note that Lemmata 3.18 and 3.22 [9] hold as such in space also, the proofs being essentially identical (change Apollonian disk to Apollonian ball etc.) The latter lemma, translated to $\mathbf{R}^n$, reads as follows:

Lemma 8.1 ([9, Lemma 3.22]). *Let $G \subsetneq \mathbf{R}^n$, $z \in G$ and $w \in \partial G$ and suppose that there exists an Apollonian geodesic connecting z with any point $z' \in G$. Then for every $r > 0$ there exists a ball $B \subseteq G$ such that $d(z, B) < r$ and $d(w, B) < r$.*

We skip Lemma 3.23 of [9] and prove the following result directly. This proposition corresponds to [9, Theorem 3.26].

Proposition 8.2. *If $G \subsetneq \mathbf{R}^n$ is such that $\alpha_G \equiv \tilde\alpha_G$, then G is a ball or a half-space.*

Proof. The cases where G^c is a subset of hyperplane (i.e. when α_G is not a metric) and when $\overline{G} \neq \overline{\mathbf{R}^n}$ are handled exactly as in the proof of [9, Theorem 3.26]. It remains to consider the case when $\overline{G} = \overline{\mathbf{R}^n}$, in which the argument of Gehring and Hag needs some nontrivial modifications.

Since ∂G is not contained in a hyperplane it follows that there exists $n+1$ finite boundary points, $w, z_1, \dots, z_n$, which are the vertices of a simplex S. Let R be the ray starting at w and going through the barycenter $m := (z_1 + \cdots + z_n)/n$ of the opposite side of the simplex. We assume without loss of generality that $w = 0$ and R is the positive real axis. Let us denote $\hat e_i := z_i/|z_i|$. Since S is a simplex the set $\{\hat e_1, \dots, \hat e_n\}$ is a basis of $\mathbf{R}^n$.

Let $s > 2d(S)$ and $w_s := s e_1 \in R$. Let us define

$$\varepsilon(s) := \min\bigl\{1/(2s^2), 2s^2, |z_1|/2, \dots, |z_n|/2, |z_1|^2/(8s^2), \dots, |z_n|^2/(8s^2)\bigr\}.$$

By Lemma 8.1 there exists a ball $B_s \subseteq G$ which intersects $B^n\bigl(0, \varepsilon(s)\bigr)$ and $B^n\bigl(w_s, \varepsilon(s)\bigr)$, since we can find a point of G arbitrarily close to w_s. For every $s > 2d(S)$ let us fix one such ball and denote by b_s its center and by R_s its diameter.

For $1 \le i \le n$ consider the point z_i and the half-space $H_i := \bigl\{x \in \mathbf{R}^n : \langle x, \hat e_i\rangle < \frac{1}{2}|z_i|\bigr\}$, the boundary of which is the midpoint normal plane of the segment $[w, z_i]$. Now if $|b_s|$ and $|b_s - z_i|$ would equal R_s then it would be clear that $b_s \in \overline{H_i}$. Since we in fact have only $|b_s| \le R_s + \varepsilon(s)$ (since $B^n(b_s, R_s)$ intersects $B^n\bigl(0, \varepsilon(s)\bigr)$) and $|b_s - z_i| > R_s$ (since $B_s \subseteq G$ and $z_i \in \partial G$) we consider instead of H_i the half-space $H_i' := \{x \in \mathbf{R}^n : \langle x, \hat e_i\rangle < |z_i|\}$ which contains H_i. It turns out that even these larger half-spaces do not always contain b_s, however, we will show that b_s is in H_i' or $|b_s|$ is large.

Let us consider the intersection of the boundary of H_i' with the midpoint normal planes of segments $[z_i, w']$ where $w' \in B^n\bigl(w, \varepsilon(s)\bigr)$. Notice that for some such w' the point b_s lies above the mid-point normal plane of the segment $[w', z_i]$. Let us estimate this distance, as indicated in Figure 3. We consider the hyperplanes normal to rays from z_i through $w' \in B^n\bigl(w, \varepsilon(s)\bigr)$ at a distance $u := \frac{1}{2}\bigl(|z_i| - \varepsilon(s)\bigr)$ from z_i. We easily see that such a plane intersects $\partial H_i'$ at distance at least t from the origin, where t is as in the figure with w' extremal. Considering the congruent triangles we find that

$$t = \frac{u\sqrt{|z_i|^2 - \varepsilon(s)^2}}{\varepsilon(s)} \ge \frac{\bigl(|z_i| - \varepsilon(s)\bigr)^2}{2\varepsilon(s)} \ge \frac{\frac{1}{2}(|z_i|)^2}{2\varepsilon(s)} \ge s^2,$$

where the second inequality follows since $\varepsilon(s) \le \frac{1}{2}|z_i|$ and the third one since $\varepsilon(s) \le |z_i|^2/(8s^2)$. We thus see that either $b_s \in H_i'$ or $|b_s| > s^2$. Let A be the set of balls B_s with $|b_s| > s^2$ and let B be the set of all other balls B_s.

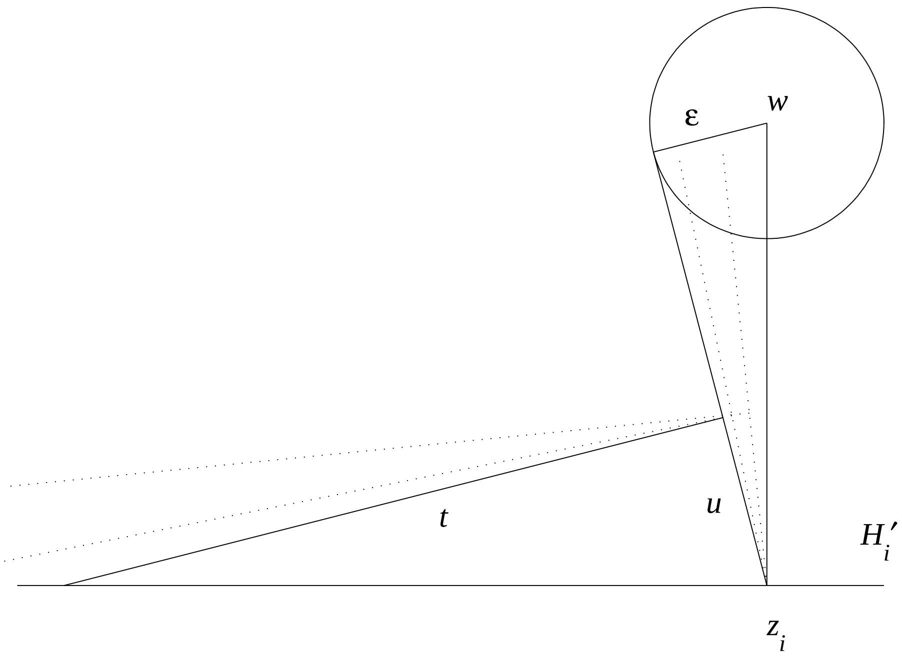

Figure 3. The center of B_s is far away.

For $B_s \in B$ we have $b_s \in \bigcap_{i=1}^{n} H_i'$, which means that

$$\frac{n\langle b_s, m\rangle}{M} = \frac{\langle b_s, z_1 + \cdots + z_n\rangle}{\max\{|z_1|^2, \ldots, |z_n|^2\}} \le \frac{\langle b_s, z_1\rangle}{|z_1|^2} + \cdots + \frac{\langle b_s, z_n\rangle}{|z_n|^2}$$
$$= \frac{\langle b_s, \hat{e}_1\rangle}{|z_1|} + \cdots + \frac{\langle b_s, \hat{e}_n\rangle}{|z_n|} < n,$$

where we have set $M := \max\{|z_1|^2, \ldots, |z_n|^2\}$. Since $m = |m|e_1$ it follows that $\langle b_s, e_1\rangle < M/|m|$. Consider then $s > 2M/|m| + 1 + \varepsilon(s)$. As shown in Figure 4, we find that $(M/|m|)^2 + h^2 \ge R_s^2$ and $(s - M/|m|)^2 + h^2 \le \big(R_s + \varepsilon(s)\big)^2$. From this we conclude that $1 + \varepsilon(s)^2 + 2M/|m| \le 2R_s\varepsilon(s) + \varepsilon(s)^2$, which means that $R_s \ge \big(2\varepsilon(s)\big)^{-1} \ge s^2$, since $\varepsilon(s) \le 1/(2s^2)$ by assumption. This means that $B_s \in A$ for every $s > 2M/|m| + 1$.

We have shown that for large s the balls B_s have radius greater than $s^2 - \varepsilon(s) \approx s^2$. This means that as $s \to \infty$ the ball B_s is very close to a half-space the boundary of which contains the e_1-axis, at least in the ball $B^n\big(w, \operatorname{diam}(S)\big)$, which contains all of the z_i. Since the $\hat{e}_i$ span the space $\mathbf{R}^n$, it is clear that every half-space the boundary of which contains the e_1-axis contains one of the z_i. Since there is only a finite number of z_i's, it is clear that for sufficiently large s

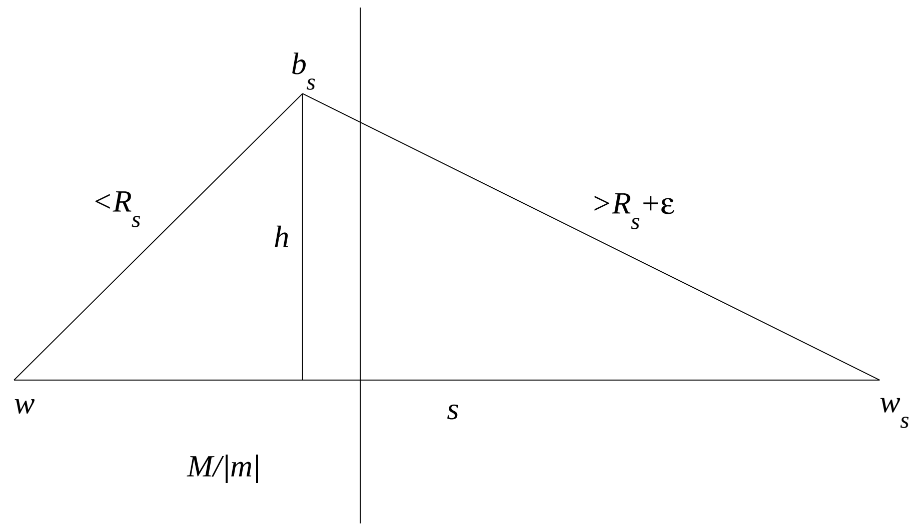

Figure 4. B_s is in A for large s.

the ball B_s becomes so close to a half-space that it contains some z_i, contrary to the assumption that the $B_s \subseteq G$. This contradiction shows that the assumption that $\overline{G} = \overline{\mathbf{R}^n}$ is false, which completes the proof. □

Proof of Theorem 1.9. Since G is a ball, it follows that $\alpha_G = h_G$ is convex. Since f is an isometry $\alpha_{f(G)}$ is also convex. By Proposition 8.2, $f(G)$ is then a ball, and so we have $\alpha_{f(G)} = h_{f(G)}$. This means that f is a hyperbolic isometry from a ball to a ball and so it is clear that $f = g|_G$, where $g\colon \overline{\mathbf{R}^n} \to \overline{\mathbf{R}^n}$ is a Möbius mapping. □

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Appendix

In this appendix we consider Seittenranta's and Ferrand's metrics, which can be thought of as the Möbius invariant variants of the j_G and the k_G metric, respectively. The results in this section are needed in an upcoming paper, [13].

Assume throughout this appendix that $x, y \in G \subsetneq \mathbf{R}^n$. Seittenranta's metric, δ_G, from [20] is defined by

$$\delta_G(x,y) := \sup_{a,b\in\partial G} \log\{1 + |a,x,b,y|\}.$$

As with the Apollonian metric, the requirement that $\infty \notin G$ is a nonessential simplification. The Ferrand metric from [7] is defined by

$$\sigma_G(x,y) := \inf_\gamma \int \sup_{a,b\in\partial G} \frac{|a-b|}{|a-z||b-z|}\,|dz|,$$

where the infimum is over all rectifiable curves joining x and y in G. We have $j_G \le \delta_G \le 2j_G$ and $k_G \le \sigma_G \le 2k_G$, by [20, Theorem 3.4] and [7, Section 6], respectively.

Ferrand's metric is the inner metric of δ_G:

Lemma A.1. *For $G \subsetneq \mathbf{R}^n$ we have $\tilde\delta_G = \sigma_G$.*

Proof. A lower bound for $\delta_G(\gamma)$ is proved as the proof of [20, Theorem 3.12] and the upper bound can be established similarly. □

It is easy to see that Seittenranta's metric is isotropic. The following lemma gives two expressions of its density.

Lemma A.2. *Let $x \in G \subsetneq \mathbf{R}^n$ and i_x denote the inversion in the sphere $S^{n-1}(x,1)$. Then*

$$\bar\delta_G(x) = \sup_{a,b\in\partial G} \frac{|a-b|}{|a-x||b-x|} = \operatorname{diam} i_x(\partial G).$$

Proof. The first equality is clear. To prove the second one, we note that if i_x denotes the inversion in the sphere $S^{n-1}(x,1)$ then $|i_x(z) - i_x(w)| = |z-w|/(|z-x||w-x|)$ (see [24, (1.5)]). Hence

$$\sup_{a,b\in\partial G} \frac{|a-b|}{|a-x||b-x|} = \sup_{a,b\in\partial G} |i_x(a) - i_x(b)| = \sup_{a,b\in\partial i_x(G)} |a-b| = \operatorname{diam} i_x(\partial G). \qquad \square$$

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