Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 34, 2009, 503–522

MAXIMAL FUNCTIONS IN VARIABLE EXPONENT SPACES: LIMITING CASES OF THE EXPONENT

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Abstract. In this paper we study the Hardy–Littlewood maximal operator in variable exponent spaces when the exponent is not assumed to be bounded away from 1 and ∞. Within the framework of Orlicz–Musielak spaces, we characterize the function space X with the property that $Mf \in L^{p(\cdot)}$ if and only if $f \in X$, under the assumptions that p is log-Hölder continuous and $1 \leqslant p^- \leqslant p^+ \leqslant \infty$.

1. Introduction

Variable exponent spaces have been the subject of quite a lot of interest recently, as surveyed in [13, 39]. These investigations have dealt both with the spaces themselves, e.g. [7, 12, 17, 29, 31], with related differential equations [1, 3, 15, 16, 32, 41, 42], and with applications [4, 38]. A critical step in the development of the theory was establishing the boundedness of the Hardy–Littlewood maximal operator. To describe this results and ours, we use the by now standard notation of variable exponent spaces. The reader may consult Section 2 if necessary.

The boundedness of the maximal operator was originally proved by Diening [9] assuming that

(1) the exponent is locally log-Hölder continuous;
(2) the exponent is constant outside a compact set; and
(3) the exponent is bounded and bounded away from 1, i.e. $1 < p^- \leqslant p^+ < \infty$.

Pick and Růžička [37] complemented this result by showing that the local log-Hölder continuity in (1) is the optimal continuity modulus for this assertion.

Diening's second assumption, that p be constant outside a compact set, is quite unnatural and is a result of the method of proof. Cruz-Uribe, Fiorenza and Neugebauer [8, Theorem 1.5] showed that it can be replaced by a weaker decay condition

2000 Mathematics Subject Classification: Primary 42B25, 46E30.

Key words: Variable exponent Lebesgue spaces, generalized Lebesgue spaces, Orlicz–Musielak spaces, maximal functions, non-uniformly convex norms, non-doubling modulars.

*Indebted to the Landesstiftung Baden-Württemberg for facilitating the analysis entailed in this paper.

†Supported in part by the Academy of Finland and INTAS.

‡Supported in part by the Academy of Finland, INTAS and Emil Aaltonen Foundation.

at infinity. Their condition is naturally termed global log-Hölder continuity (cf. Remark 2.4). These authors also comment on the necessity of the decay condition in [8, Theorem 1.6]. Nekvinda [36] independently proved the same result under a slightly weaker decay assumption at infinity, replacing the continuity modulus by a Dini-type condition at infinity. It remains an open question whether Nekvinda's Dini-type condition is sufficient also for finite points.

With the first two assumptions being optimal, it is natural to look at the third condition also. The upper bound in (3), $p^+ < \infty$, seems strange, since it is immediately clear that $M\colon L^\infty \hookrightarrow L^\infty$. In Section 3 we show that this condition was an artefact of previous proof methods and that $M\colon L^{p(\cdot)} \hookrightarrow L^{p(\cdot)}$ when p is globally log-Hölder continuous and $1 < p^- \leqslant p^+ \leqslant \infty$. Since $M\colon L^1 \not\hookrightarrow L^1$, the lower bound condition $p^- > 1$, on the other hand, looks quite reasonable. Indeed, Cruz-Uribe, Fiorenza and Neugebauer [8, Theorem 1.7] showed that if p is lower semicontinuous and $M\colon L^{p(\cdot)} \hookrightarrow L^{p(\cdot)}$, then $p^- > 1$. In Section 6 we improve this result and show that $M\colon L^{p(\cdot)} \hookrightarrow L^{p(\cdot)}$ implies $p^- > 1$ even without the semicontinuity assumption (Theorem 6.3).

Despite these quite definite looking assertions about the lower bound, other approaches were devised. Hästö [26], Futamura and Mizuta [19] and Mizuta, Ohno and Shimomura [33] studied embeddings from $L^{p(\cdot)}$ to L^1 and Cruz-Uribe, Fiorenza and Neugebauer [8], Aguilar Cañestro and Ortega Salvador [2] and Harjulehto and Hästö [23] studied weak-type inequalities including the case $p^- = 1$. In this paper we take a different route. We fix the target space to be $L^{p(\cdot)}$ and give a concrete description of the space $M^{-1}[L^{p(\cdot)}]$, i.e. we describe the space X for which $Mf \in L^{p(\cdot)}$ if and only if $f \in X$. In view of the previous paragraph $X \subsetneq L^{p(\cdot)}$.

The relevant classical results [40, Section 1] for such a characterization are that

$$Mf \in L^p \text{ if and only if } f \in L^p \ (p > 1) \quad \text{and} \quad Mf \in L^1 \text{ if and only if } f \in L \log L.$$

The latter result is, of course, restricted to bounded domains: if $Mf \in L^1(\mathbf{R}^n)$, then $f \equiv 0$. On an intuitive level, then, we see that we need some kind of modified scale of spaces $\widetilde{L}^{p(\cdot)}$ where $p = 1$ corresponds to $L \log L$, not L^1, if we want to characterize functions f for which $Mf \in L^{p(\cdot)}$. It is possible to construct such a space within the framework of Orlicz–Musielak spaces.

Recall that the Orlicz–Musielak space with modular $\Phi(x,t)$ is defined by the Luxemburg type-norm

$$\|f\|_{L^\Phi(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \Phi\Big(x, \frac{f(x)}{\lambda}\Big)\, dx \leqslant 1 \right\}$$

[34]. (The function Φ must satisfy certain conditions, which we will not detail here.) We need a function which behaves like a logarithm when $p = 1$ and fades away when $p > 1$. Since the embedding constant of $M\colon L^p \hookrightarrow L^p$ is p', the function

$$\min\big\{p', \log(e + |t|)\big\}, \quad p' := p/(p-1),$$

would be a natural choice. Unfortunately, it does not yield a convex modular. The following variant fixes this problem:

$$\psi_p(t) = \begin{cases} \log(e + |t|), & \text{for } |t| < e^{p'} - e, \\ 2p' - \frac{e^{p'}}{e+|t|} p', & \text{for } |t| \geqslant e^{p'} - e. \end{cases}$$

Note that $t \mapsto t^p \psi_p(t)$ is convex on $[0, \infty)$ and that

$$\tfrac{1}{2}\psi_p(t) \leqslant \min\big\{p', \log(e + |t|)\big\} \leqslant \psi_p(t),$$

so ψ_p is equivalent up to a contant to the natural choice of modular.

The norm $\|f\|_{L^{p(\cdot)}\psi_{p(\cdot)}[L]}$ is then given by the modular

$$\Phi(x, t) = |t|^{p(x)} \psi_{p(x)}(t).$$

Note that $\|f\|_{L^{p(\cdot)}\psi_{p(\cdot)}[L]} \approx \|f\|_{L^{p(\cdot)}}$ if $p^- > 1$, but the constant of proportionality blows up as $p^- \to 1$. We are now ready for the main theorem:

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let $1/p \colon \Omega \to \mathbf{R}$ be* log*-Hölder continuous with $1 \leqslant p^- \leqslant p^+ \leqslant \infty$. Then*

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \lesssim \|f\|_{L^{p(\cdot)}\psi_{p(\cdot)}[L](\Omega)}.$$

The embedding is sharp in the sense that

$$Mf \in L^{p(\cdot)}(B) \quad \text{if and only if} \quad f \in L^{p(\cdot)}\psi_{p(\cdot)}[L](B)$$

for balls B (and other sufficiently nice domains).

In the special case when $p^- > 1$ we get a result in all of $\mathbf{R}^n$, as expected:

Theorem 1.2. *Let $p \in \mathscr{P}^{\log}(\mathbf{R}^n)$ with $1 < p^- \leqslant p^+ \leqslant \infty$. Then M is bounded from $L^{p(\cdot)}(\mathbf{R}^n)$ to $L^{p(\cdot)}(\mathbf{R}^n)$, more specifically,*

$$\|Mf\|_{p(\cdot)} \leqslant A(p^-)' \|f\|_{p(\cdot)}.$$

Here $A > 0$ depends only on the dimension n and the constant of log*-Hölder continuity of $\frac{1}{p}$.*

After this paper was finished, Diening, Cruz-Uribe and Fiorenza [5] found a new proof for Theorem 1.2.

Cruz-Uribe and Fiorenza [6] have also recently investigated the behavior of the maximal operator in variable exponent spaces when $p \to 1$. In fact, improving their result was the inital motivation for this study. Their main result is the following:

Theorem 1.5 of [6]**.** *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let $p \colon \Omega \to \mathbf{R}$ be* log*-Hölder continuous with $1 \leqslant p^- \leqslant p^+ < \infty$. For every $\delta > 0$ there exist non-negative, continuous exponents q and ε with $q|_{\{p=1\}} \equiv 1$,* spt *$q \subset \{p < 1 + \delta\}$ and* spt *$\varepsilon \subset \{1 < p < 1 + \delta\}$ so that*

$$M \colon L^{p(\cdot)+\varepsilon(\cdot)} \log L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega).$$

Since ε equals 0 in the set $\{p = 1\}$ in the previous theorem, the result allows us to regain the classical, constant exponent embeddings as special cases. However, the embedding is clearly not optimal in the variable exponent case. Surely, the exponent ε is a strange manifestation of the technique of proof which does not correspond to anything in the constant exponent case. As a consequence of our embedding theorem, we get the following improvement of their result:

Theorem 1.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let $1/p \colon \Omega \to \mathbf{R}$ be* log*-Hölder continuous with $1 \leqslant p^- \leqslant p^+ \leqslant \infty$. For every $\delta > 0$ there exist a non-negative, continuous exponent q with* spt *$q \subset \{p < 1 + \delta\}$ and $q|_{\{p=1\}} \equiv 1$ so that*

$$M \colon L^{p(\cdot)} \log L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega).$$

Cruz-Uribe and Fiorenza [6, Example 1.9] also gave an example which shows that the embedding of Theorem 1.3 is strict in the sense that there exists a function f with $Mf \in L^{p(\cdot)}(\Omega)$ but $f \notin L^{p(\cdot)} \log L^{q(\cdot)}(\Omega)$. We give a slight improvement of this result in Proposition 5.1 by relaxing the continuity asssumptions on the exponents.

Let us conclude the introduction by mentioning some other directions of study of the maximal operator in variable exponent spaces which are not yet covered by our results. Lerner [31] constructed examples of variable exponents which are not log-Hölder continuous but for which the maximal operator is nevertheless bounded. Diening [10] gave an abstract characterization of the boundedness of the maximal operator, which allows us, for instance, to conclude that the maximal operator is bounded for the exponent p if and only if it is bounded for the conjugate exponent p'. The weighted case was studied by Kokilashvili, N. Samko and S. Samko [27, 28, 29]. In further generality still, the maximal operator was considered in the metric measure space case by Harjulehto, Hästö and Pere [25] and Futamura, Mizuta and Shimomura [20, 21]. In all these cases it remains unclear what happens when $p \to 1$ or $p \to \infty$.

Another question related to the methods of this paper is modifying the Lebesgue space scale to allow for optimal results. Other recent modifications in the variable exponent case, again not covered by our results, are due to Harjulehto, Hästö and Latvala [24], Harjulehto and Hästö [23] and Diening, Hästö and Roudenko [14].

The structure of the rest of this paper is as follows. In Section 2 we recall the definitions and conventions used throughout the paper. In Section 3 we tackle the case of unbounded exponents which are bounded away from 1. In Section 4 we consider the lower bound on p^- and prove Theorem 1.1. In Section 5 we show how this theorem implies Theorem 1.3 and prove that a theorem of this type cannot be sharp (Proposition 5.1). Finally, in Section 6 we show without any a priori assumptions on the exponent that the maximal operator is never bounded from $L^{p(\cdot)}(\Omega)$ to itself when $p^- = 1$. The proofs in Sections 3 and 6 are based on the Habilitation's Thesis of Lars Diening [11].

2. Notation and conventions

By C we denote a generic constant, i.e. a constant whose value may change from appearance to appearance. We write $f \lesssim g$ if there exists a constant C so that $f \leqslant Cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. We always assume that $\Omega \subset \mathbf{R}^n$ is an open set. For a function $f\colon \Omega \to \mathbf{R}$ we denote the set $\{x \in \Omega :\ a < f(x) < b\}$ simply by $\{a < f < b\}$, etc. For $f \in L^1_{\mathrm{loc}}(\Omega)$ and $A \subset \mathbf{R}^n$ with positive finite measure we write

$$f_A = \fint_A f(y)\,dy := |A|^{-1} \int_{A\cap\Omega} f(y)\,dy.$$

We use M to denote the centered Hardy–Littlewood maximal operator, $Mf(x) = \sup_{r>0} |f|_{B(x,r)}$.

Let $p\colon \Omega \to [1,\infty]$ be a measurable function, which we call a variable exponent. For $A \subset \mathbf{R}^n$ we write $p^+_A = \operatorname{esssup}_{x\in A\cap\Omega} p(x)$ and $p^-_A = \operatorname{essinf}_{x\in A\cap\Omega} p(x)$, and abbreviate $p^+ = p^+_\Omega$ and $p^- = p^-_\Omega$. We define

$$\varphi_p(x,t) := \begin{cases} t^{p(x)} & \text{for } 0 < p(x) < \infty, \\ 0 & \text{for } p(x) = \infty, t \in (0,1], \\ \infty & \text{for } p(x) = \infty, t \in (1,\infty). \end{cases}$$

The reason to define $\varphi_\infty(1) = 0$ is to get a left-continuous function, as in the general theory of Orlicz–Musielak spaces. Note that $\varphi_\infty(t) \leqslant \lim_{q\to\infty} \varphi_q(t) \leqslant \varphi_\infty(2t)$ for all $t \geqslant 0$. Let $q, r, s \in [1,\infty]$ with $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$. Then Young's inequality reads

$$\varphi_q(x, ab) \leqslant \varphi_r(x, a) + \varphi_s(x, b)$$

for all $a, b \geqslant 0$.

Let $\varphi_p^{-1}(x,\cdot)$ be the left-continuous inverse of $\varphi_p(x,\cdot)$, i.e.

$$\varphi_p^{-1}(x, s) := \inf\{t \geqslant 0\colon \varphi_p(x,t) \geqslant s\}.$$

This implies $\varphi_p^{-1}(x,t) = t^{\frac{1}{p(x)}}$ for $p(x) < \infty$ and $\varphi_p^{-1}(x,t) = \chi_{(0,\infty]}(t)$ for $p(x) = \infty$. Except where special emphasis is needed, we will simply abbreviate $\varphi_p(x,t)$ and $\varphi_p^{-1}(x,t)$ by $t^{p(x)}$ and $t^{1/p(x)}$. It is to be understood that this means $\infty\chi_{(1,\infty)}(t)$ and $\chi_{(0,\infty)}$, respectively, at points where $p(x) = \infty$.

The variable exponent *modular* is defined for measurable functions by

$$\varrho_{p(\cdot)}(f) = \int_\Omega \varphi_p(x, |f(x)|)\, dx.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of measurable functions $f\colon \Omega \to \mathbf{R}$ with $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg norm on this space by the formula

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{\lambda > 0\colon \varrho_{p(\cdot)}(f/\lambda) \leqslant 1\big\}.$$

Here Ω could of course be replaced by some subset as in $\|f\|_{L^{p(\cdot)}(A)}$; the abbreviation $\|f\|_{p(\cdot)}$ is used for the norm $\|f\|_{L^{p(\cdot)}(\Omega)}$ over all of Ω. Notice that an immediate consequence of the definition is that $\|f\|_{p(\cdot)} \leqslant 1$ if and only if $\varrho_{p(\cdot)}(f) \leqslant 1$.

Remark 2.1. If $p^+ = \infty$, then our definition is not exactly the same as the one given by Kováčik and Rákosník [30] who used the modular $\varphi_{p(\cdot)}(f\chi_{\{p<\infty\}}) + \|f\chi_{\{p=\infty\}}\|_\infty$. However, both definitions give the same space up to equivalence of norms, and our definition is easier to use in the case $p^+ = \infty$.

The following inequality remains valid also when we include $p^+ = \infty$, although for technical reasons we must assume either $\varrho_{p(\cdot)}(f) > 0$ or $p^+ < \infty$:

Lemma 2.2. *For any measurable exponent* $p\colon \Omega \to [1,\infty]$ *with* $p^- < \infty$ *we have*

$$\min\Big\{\varrho_{p(\cdot)}(f)^{\frac{1}{p^-}}, \varrho_{p(\cdot)}(f)^{\frac{1}{p^+}}\Big\} \leqslant \|f\|_{p(\cdot)} \leqslant \max\Big\{\varrho_{p(\cdot)}(f)^{\frac{1}{p^-}}, \varrho_{p(\cdot)}(f)^{\frac{1}{p^+}}\Big\}$$

when $\varrho_{p(\cdot)}(f) > 0$ *or* $p^+ < \infty$.

Proof. Suppose that $p^+ < \infty$. If $\varrho_{p(\cdot)}(f) \leqslant 1$, then we need to prove that

$$\varrho_{p(\cdot)}(f)^{\frac{1}{p^-}} \leqslant \|f\|_{p(\cdot)} \leqslant \varrho_{p(\cdot)}(f)^{\frac{1}{p^+}}.$$

By homogeneity, the latter inequality is equivalent to $\|f/\varrho_{p(\cdot)}(f)^{\frac{1}{p^+}}\|_{p(\cdot)} \leqslant 1$, which is equivalent to the modular being less than or equal to one:

$$\int_\Omega \left(\frac{f(x)}{\varrho_{p(\cdot)}(f)^{\frac{1}{p^+}}}\right)^{p(x)} dx \leqslant 1.$$

But since $\varrho_{p(\cdot)}(f)^{-\frac{p(x)}{p^+}} \leqslant \varrho_{p(\cdot)}(f)^{-1}$, this is clear. The other inequality and the case $\varrho_{p(\cdot)}(f) \geqslant 1$ are similar.

Consider then $p^+ = \infty$. In this case the upper inequality becomes $\|f\|_{p(\cdot)} \leqslant \max\{\varrho_{p(\cdot)}(f)^{1/p^-}, 1\}$. If $\varrho_{p(\cdot)}(f) \leqslant 1$, then $\|f\|_{p(\cdot)} \leqslant 1$, so the inequality holds. If $\varrho_{p(\cdot)}(f) > 1$, then we need to show that

$$\int_\Omega \Big(\frac{f(x)}{\varrho_{p(\cdot)}(f)^{1/p^-}}\Big)^{p(x)}\,dx \leqslant 1.$$

Since $\varrho_{p(\cdot)}(f)^{-1} < 1$, we conclude that

$$\varrho_{p(\cdot)}(f)^{\frac{-p(x)}{p^-}} \leqslant \begin{cases} 0, & \text{if } p(x) = \infty, \\ \varrho_{p(\cdot)}(f)^{-1}, & \text{if } p(x) < \infty. \end{cases}$$

Hence

$$\int_\Omega \Big(\frac{f(x)}{\varrho_{p(\cdot)}(f)^{1/p^-}}\Big)^{p(x)}\,dx \leqslant \int_\Omega f(x)^{p(x)}\varrho_{p(\cdot)}(f)^{-1}\,dx = 1.$$

The proof of the lower inequality is analogous. □

The following has emerged as a central condition in the theory of variable exponent spaces.

Definition 2.3. Let $\alpha \in C(\Omega)$. We say that α is *locally* log-*Hölder continuous* if there exists $c_{\log} > 0$ so that

$$|\alpha(x) - \alpha(y)| \leqslant \frac{c_{\log}}{\log(e + 1/|x-y|)}$$

for all $x, y \in \Omega$.

We say that α is (*globally*) log-*Hölder continuous* if it is locally log-Hölder continuous and there exists $\alpha_\infty \in \mathbf{R}$ so that the decay condition

$$|\alpha(x) - \alpha_\infty| \leqslant \frac{c_{\log}}{\log(e + |x|)}$$

holds for all $x \in \Omega$.

Remark 2.4. If Ω is a bounded set then the notions of local and global log-Hölder continuity coincide. If Ω is unbounded, then α is globally log-Hölder continuous if and only if it can be extended to ∞ in such a way that

$$|\alpha(x) - \alpha(y)| \leqslant \frac{c_{\log}}{\log(e + 1/q(x,y))}$$

for all $x, y \in \Omega \cup \{\infty\}$, where q denotes the spherical-chordal metric,

$$q(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \quad \text{and} \quad q(x,\infty) = \frac{1}{\sqrt{1+|x|^2}},$$

for $x, y \in \mathbf{R}^n$. This motivates calling the condition global log-Hölder continuity.

The following lemma is an easy consequence of the definition.

Lemma 2.5. *Let α be locally* log-*Hölder continuous. Then*

$$|B|^{-|\alpha(x)-\alpha(y)|} \leqslant (2e\Omega_n^{-1})^{n\,c_{\log}}$$

for all balls $B \subset \mathbf{R}^n$ containing x and y, where Ω_n denotes the volume of the unit ball.

The notation $\mathscr{P}^{\log}(\Omega)$ is used for the set of functions p from Ω to $(0,\infty]$ for which $1/p$ is globally log-Hölder continuous.

3. The maximal operator in spaces with unbounded exponents

In this section we develop the theory of maximal operators when the exponent is not bounded. Therefore the conventions regarding t^∞ and $t^{1/\infty}$ introduced in the previous section are particularly relevant.

Lemma 3.1. *Let $p \in \mathscr{P}^{\log}(\mathbf{R}^n)$ with $0 < p^- \leqslant p^+ \leqslant \infty$. Then there exists $\beta \in (0,1)$, which only depends on the* log-*Hölder continuity constant of $\frac{1}{p}$, such that*

$$\big(\beta\,(\lambda\,|B|^{-1})^{1/p_B^-}\big)^{p(x)} \leqslant \lambda\,|B|^{-1}$$

for all $\lambda \in [0,1]$, every ball $B \subset \mathbf{R}^n$ and $x \in B$.

Proof. If $p_B^- = \infty$, then $p(x) = \infty$ and $\varphi_\infty\big(\beta\,\varphi_\infty^{-1}(\lambda\,|B|^{-1})\big) = \varphi_\infty(\beta) = 0 \leqslant \lambda\,|B|^{-1}$. Assume now that $p_B^- < \infty$ and $p(x) < \infty$. Due to Lemma 2.5 there exists $\beta \in (0,1)$ such that

$$\beta\,|B|^{1/p(x)-1/p_B^-} \leqslant 1.$$

Multiply this by $|B|^{-1/p(x)}$ and raise the result to the power of $p(x)$ to prove the claim for $\lambda = 1$. If $0 \leqslant \lambda < 1$, then

$$\Big(\beta\,\big(\lambda\,|B|^{-1}\big)^{1/p_B^-}\Big)^{p(x)} = \lambda^{p(x)/p_B^-}\Big(\beta\,\big(|B|^{-1}\big)^{1/p_B^-}\Big)^{p(x)} \leqslant \lambda^{p(x)/p_B^-}\,|B|^{-1} \leqslant \lambda\,|B|^{-1}$$

by convexity and the case $\lambda = 1$.

It remains to consider the case $p(x) = \infty$ and $p_B^- < \infty$. Since $p_B^- < \infty$ we can choose points $x_i \in B$ such that each $p(x_i)$ is finite and $p(x_i) \to \infty$. Then $\varphi_p(x,t) \leqslant \lim_{i\to\infty} \varphi_p(x_i,t)$ for all $t \geqslant 0$ by continuity. Hence, this case is reduced to the previous case. $\square$

If A and B are normed spaces contained in a vector space, then the norm $\|\cdot\|_{A+B}$ is defined by

$$\|f\|_{A+B} = \inf_{a,b} \|a\|_A + \|b\|_B,$$

where the infimum is taken over elements $a \in A$ and $b \in B$ such that $a + b = f$.

Lemma 3.2. *Let $p \in \mathscr{P}^{\log}(\mathbf{R}^n)$ with $1 \leqslant p^- \leqslant p^+ \leqslant \infty$. Define $r \in \mathscr{P}^{\log}(\mathbf{R}^n \times \mathbf{R}^n)$ by*

$$\frac{1}{r(x,y)} = \max\left\{0, \frac{1}{p(x)} - \frac{1}{p(y)}\right\}.$$

Then for $\gamma \in (0,1)$ there exists $\beta \in (0,1)$ such that

$$\left(\beta \fint_B |f(y)|\,dy\right)^{p(x)} \leqslant \fint_B |f(y)|^{p(y)}\,dy + \fint_B \gamma^{r(x,y)}\,dy$$

for every ball $B \subset \mathbf{R}^n$ containg x, and function $f \in L^{p(\cdot)}(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$ satisfying $\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)+L^\infty(\mathbf{R}^n)} \leqslant 1$. Here, β depends on p only via the constant of local log-*Hölder continuity of $\frac{1}{p}$.*

Proof. We assume without loss of generality that f is non-negative. Since we have $\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)+L^\infty(\mathbf{R}^n)} \leqslant 1$, we can write $f = h + k$ where $\|h\|_\infty + \|k\|_{p(\cdot)} \leqslant 1$. Then $f_p = \min\{f, |k|\}$ and $f_\infty = f - f_p$ are non-negative and $\|f_\infty\|_\infty + \|f_p\|_{p(\cdot)} \leqslant 1$. By convexity

$$\Big(\frac{\beta}{2}\fint_B f(y)\,dy\Big)^{p(x)} \leqslant \frac{1}{2}\Big(\beta\fint_B f_p(y)\,dy\Big)^{p(x)} + \frac{1}{2}\Big(\beta\fint_B f_\infty(y)\,dy\Big)^{p(x)}.$$

Thus it suffices to prove the claim for $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ with $\|f\|_{p(\cdot)} \leqslant 1$ or $\|f\|_\infty \leqslant 1$.

Let $B \subset \mathbf{R}^n$ be a ball and $x \in B$. Without loss of generality let $f(y) = 0$ for $y \in \mathbf{R}^n \setminus B$. If $p_B^- = \infty$, then $p(y) = \infty$ for all $y \in B$ and the claim is just Jensen's inequality for the convex function φ_∞, even without the last term. So assume in the following $p_B^- < \infty$.

Let $\beta > 0$ be as in Lemma 3.1. We can assume that $\beta \leqslant \gamma$. We split f into three parts:

$$f_1 := f\,\chi_{\{f>1\}}, \quad f_2 := f\,\chi_{\{f\leqslant 1,\, p(\cdot)\leqslant p(x)\}}, \quad \text{and} \quad f_3 := f\,\chi_{\{f\leqslant 1,\, p(\cdot)>p(x)\}}.$$

Then $f = f_1 + f_2 + f_3$ and $f_j \leqslant f$. By convexity of $\varphi_p(x,\cdot)$ we conclude that

$$\varphi_p\Big(x, \frac{\beta}{3}\fint_B f(y)\,dy\Big) \leqslant \frac{1}{3}\sum_{j=1}^{3}\varphi_p\Big(x, \beta\fint_B f_j(y)\,dy\Big) =: \frac{1}{3}\big((I) + (II) + (III)\big).$$

Therefore it suffices to consider the functions f_1, f_2, and f_3 independently.

We start with f_1. Hölder's inequality with exponent p_B^- implies that

$$(I) = \varphi_p\Big(x, \beta\fint_B f_1(y)\,dy\Big) \leqslant \varphi_p\Bigg(x, \beta\Big(\fint_B f_1(y)^{p_B^-}\,dy\Big)^{1/p_B^-}\Bigg).$$

Since $f_1(y) > 1$ or $f_1(y) = 0$ and $p_B^- \leqslant p(y)$, we have by $f_1(y)^{p_B^-} \leqslant \varphi_p(y, f_1(y))$ so that

$$(I) \leqslant \varphi_p\Bigg(x, \beta\Big(\fint_B \varphi_p(y, f_1(y))\,dy\Big)^{1/p_B^-}\Bigg).$$

If $\|f\|_\infty \leqslant 1$, then $f_1 = 0$ and $(I) = 0$. So we assume that $\|f\|_{p(\cdot)} \leqslant 1$. Then we apply Lemma 3.1 with $\lambda = \int_B \varphi_p(y, f_1(y))\,dy \leqslant 1$ and conclude that

$$(I) \leqslant \fint_B \varphi_p(y, f_1(y))\,dy.$$

Since $\beta\, f_2(y) \leqslant f_2(y) \leqslant 1$ and $p(y) \leqslant p(x)$ in the support of f_2, we conclude that $\varphi_p\big(x, \beta\, f_2(y)\big) \leqslant \varphi_p\big(y, f_2(y)\big)$. To estimate (II) we apply Jensen's inequality followed by this inequality and derive

$$(II) = \varphi_p\Big(x, \fint_B \beta\, f_2(y)\,dy\Big) \leqslant \fint_B \varphi_p\big(x, \beta\, f_2(y)\big)\,dy \leqslant \fint_B \varphi_p\big(y, f_2(y)\big)\,dy.$$

Finally, for (III), Jensen's inequality, Young's inequality and $\beta \leqslant \gamma$ give

$$\begin{aligned}(III) &\leqslant \fint_B \varphi_p\big(x, \beta\, f_3(y)\big)\,dy \leqslant \fint_B \varphi_p\Big(y, \beta\,\frac{f_3(y)}{\gamma}\Big) + \gamma^{r(x,y)}\,dy \\ &\leqslant \fint_B \varphi_p(y, f_3(y))\,dy + \fint_B \gamma^{r(x,y)}\,dy.\end{aligned}$$

Collecting the estimates for (I), (II) and (III) proves the lemma. □

Lemma 3.3. *Let $p \in \mathscr{P}^{\log}(\mathbf{R}^n)$ with $1 \leqslant p^- \leqslant p^+ \leqslant \infty$. Then for any $k \geqslant 0$ there exists $\beta \in (0,1)$ such that*

$$\Big(\beta \fint_B |f(y)|\, dy\Big)^{p(x)} \leqslant \fint_B |f(y)|^{p(y)}\, dy + h_B(x),$$

for every ball $B \subset \mathbf{R}^n$ containg x, and function $f \in L^{p(\cdot)}(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$ satisfying $\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)+L^\infty(\mathbf{R}^n)} \leqslant 1$, where

$$h_B(x) := \min\{|B|^k, 1\} \Big((e+|x|)^{-k} + \fint_B (e+|y|)^{-k}\, dy\Big).$$

Here, β depends on p only via the constant of log*-Hölder continuity of $\frac{1}{p}$.*

Proof. Let $\gamma := \exp(-K)$ for some $K > 0$ and let β and r be as in Lemma 3.2. We have $\varphi_{r(x,y)}(\gamma) = \varphi_{r(x,y)/2}(\gamma) \cdot \varphi_{r(x,y)/2}(\gamma)$. We will show that $\varphi_{r(x,y)/2}(\gamma) \leqslant \min\{|B|^k, 1\}$ and $\varphi_{r(x,y)/2}(\gamma) \leqslant (e+|x|)^{-k} + (e+|y|)^{-k}$ for a suitable choice of K. The claim follows easily from this and Lemma 3.2. If $r(x,y) = \infty$, then $\varphi_{r(x,y)}(\gamma) = \varphi_\infty(\gamma) = 0$ and there is nothing to prove. So we can assume that $r(x,y) < \infty$.

The local log-Hölder continuity of $\frac{1}{p}$ implies that

$$\left|\frac{1}{r(x,y)}\right| \leqslant \left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leqslant \frac{A}{\log(e+1/|B|)},$$

with A a constant multiple of the constant of local log-Hölder continuity of $\frac{1}{p}$. Hence we get

$$\gamma^{\frac{r(x,y)}{2}} \leqslant \exp\Big(\frac{K \log|B|}{2A}\Big) = |B|^{\frac{K}{2A}} \leqslant |B|^k$$

for $K \geqslant 2kA$ and $|B| \leqslant 1$. If $|B| > 1$, then we use $\gamma^{r(x,y)/2} \leqslant 1$ which follows from $\gamma < 1$. Overall, we get $\gamma^{r(x,y)/2} \leqslant \min\{|B|^k, 1\}$ for $K \geqslant 2kA$.

Define s by $\frac{1}{s(x)} = |\frac{1}{p(x)} - \frac{1}{p_\infty}|$. Then $\frac{1}{r(x,y)} \leqslant \frac{1}{s(x)} + \frac{1}{s(y)} \leqslant 2\max\{\frac{1}{s(x)}, \frac{1}{s(y)}\}$. Hence, $r(x,y) \geqslant \frac{1}{2}\min\{s(x), s(y)\}$ and

$$\gamma^{\frac{r(x,y)}{2}} \leqslant \gamma^{\frac{1}{4}\min\{s(x),s(y)\}} \leqslant \gamma^{\frac{s(x)}{4}} + \gamma^{\frac{s(y)}{4}}.$$

Due to the decay condition on $\frac{1}{p}$ at infinity, $\frac{1}{s(x)} \leqslant \frac{A_2}{\log(e+|x|)}$ and $\frac{1}{s(y)} \leqslant \frac{A_2}{\log(e+|y|)}$ where $A_2 > 0$ is the decay constant of $\frac{1}{p}$. This implies that

$$\gamma^{\frac{s(x)}{4}} \leqslant \exp\Big(-\frac{K\log(e+|x|)}{4A_2}\Big) = (e+|x|)^{-\frac{K}{4A_2}} \leqslant (e+|x|)^{-k},$$

and similarly $\gamma^{\frac{s(y)}{4}} \leqslant (e+|y|)^{-k}$ for $K \geqslant 4kA_2$. Thus

$$\gamma^{\frac{r(x,y)}{2}} \leqslant \gamma^{\frac{s(x)}{4}} + \gamma^{\frac{s(y)}{4}} \leqslant (e+|x|)^{-k} + (e+|y|)^{-k}. \qquad \square$$

Recall that the space weak-L^1 is defined to consist of those functions f for which

$$\sup_{\lambda>0} \lambda \big|\{f > \lambda\}\big| < \infty.$$

One easily sees that the supremum is less than or equal to $\|f\|_{L^1}$ (sometimes called Chebyshev's inequality), and so $L^1 \subset$ weak-L^1, as the name implies.

Corollary 3.4. *Let $p \in \mathscr{P}^{\log}(\mathbf{R}^n)$ with $1 \leqslant p^- \leqslant p^+ \leqslant \infty$. Then there exists $h \in \text{weak-}L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ and $\beta \in (0,1)$ such that*

$$\big(\beta\, Mf(x)\big)^{p(x)} \leqslant M\big(|f(\cdot)|^{p(\cdot)}\big)(x) + h(x) \tag{3.5}$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$ with $\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)+L^\infty(\mathbf{R}^n)} \leqslant 1$. Here, β depends on p only via the constant of local log-*Hölder continuity of $\frac{1}{p}$.*

Proof. Fix $k > n$ and define

$$h(x) = (e+|x|)^{-k} + M\big((e+|\cdot|)^{-k}\big)(x).$$

We easily see that the function $x \mapsto (e+|x|)^{-k}$ belongs to $L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. Since $M\colon L^1(\mathbf{R}^n) \hookrightarrow \text{weak-}L^1(\mathbf{R}^n)$ and $M\colon L^\infty(\mathbf{R}^n) \hookrightarrow L^\infty(\mathbf{R}^n)$, we have $h \in \text{weak-}L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. Now the claim follows from Lemma 3.3, since $h_B \leqslant h$. □

With these results we are ready to prove the first part of our main result, that $p^+ = \infty$ is not an obstacle to the boundedness of the maximal operator.

Proof of Theorem 1.2. If $p^- = \infty$, then $p \equiv \infty$ and the result is classical. So assume in the following that $p^- < \infty$. Define $q := \frac{p}{p^-}$ and note that $q \in \mathscr{P}^{\log}(\mathbf{R}^n)$ with $1 \leqslant q^- \leqslant q^+ \leqslant \infty$. Since $q \leqslant p \leqslant \infty$, we have $L^{p(\cdot)} \hookrightarrow L^{q(\cdot)} + L^\infty$, with embedding constant $K \geqslant 1$ independent of p and q. Since the claim is homogeneous, it suffices to show that

$$\|Mf\|_{p(\cdot)} \leqslant A(p^-)'$$

for all functions f with $\|f\|_{p(\cdot)} \leqslant 1/K$. By the norm-modular inequality, Lemma 2.2, the former follows from

$$\varrho_{p(\cdot)}(\beta\, Mf) \leqslant A^{p^-}\alpha^{p^-},$$

where we denoted $\alpha := (p^-)'$. (Lemma 2.2 is not applicable if $\varrho_{p(\cdot)}(\beta\, Mf) = 0$, but in this case $f \equiv 0$, so the claim is clear.)

Fix now f with $\|f\|_{p(\cdot)} \leqslant 1/K$. Note that, by the choice of K, $\|f\|_{L^{q(\cdot)}+L^\infty} \leqslant 1$. Let β and h be as in Corrollary 3.4, such that (3.5) holds with p replaced by q. Then

$$\begin{aligned}\varphi_p\big(x, \beta\, Mf(x)\big) &= \varphi_q\big(x, \beta\, Mf(x)\big)^{p^-} \\ &\leqslant \Big(M\big[\varphi_q(\cdot, |f(\cdot)|)\big](x) + h(x)\Big)^{p^-} \\ &\leqslant 2^{p^- -1} M\big[\varphi_q(\cdot, |f(\cdot)|)\big](x)^{p^-} + 2^{p^- -1}|h(x)|^{p^-}.\end{aligned}$$

Since $p^- > 1$ and $h \in \text{weak-}L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, we have $h \in L^{p^-}(\mathbf{R}^n)$. Hence,

$$\begin{aligned}\int_{\mathbf{R}^n} \varphi_p\big(x, \beta\, Mf(x)\big)\, dx &\leqslant 2^{p^-} \int_{\mathbf{R}^n} M\big[\varphi_q(\cdot, |f(\cdot)|)\big](x)^{p^-}\, dx + 2^{p^-}\|h\|_{p^-}^{p^-} \\ &\leqslant C\alpha^{p^-} \int_{\mathbf{R}^n} \varphi_q(x, |f(x)|)^{p^-}\, dx + 4^{p^-}\|M(e+|\cdot|)^{-k}\|_{p^-}^{p^-} \\ &\leqslant C^{p^-}\alpha^{p^-}\Big(\varrho_{p(\cdot)}(f) + \|(e+|\cdot|)^{-k}\|_{p^-}^{p^-}\Big) \leqslant C^{p^-}\alpha^{p^-},\end{aligned}$$

where we used that $M\colon L^{p^-}(\mathbf{R}^n) \hookrightarrow L^{p^-}(\mathbf{R}^n)$ with constant $C_n\alpha$, where C_n depends only on the dimension n, in the second and third inequalities. □

4. The preimage of $L^{p(\cdot)}$ under the maximal operator

In this section we consider the other ingredient of Theorem 1.1, namely the lower bound $p^- \geqslant 1$, and eventually prove the theorem itself.

The following Lemma is an easy modification of Lemma 3.3 which is valid for bounded exponents in bounded domains. The proof is omitted.

Lemma 4.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let p be* log-*Hölder continuous with $1 \leqslant p^- \leqslant p^+ < \infty$. Let $f \in L^{p(\cdot)}(\Omega)$ be such that $\varrho_{p(\cdot)}(f) \leqslant K$. Then there exists a constant C depending only on p, n and K so that*

$$Mf(x)^{p(x)} \leqslant C\, M\big[|f|^{p(\cdot)} + \chi_{\{0<f<1\}}\big](x)$$

for every $x \in \Omega$.

The trick with the following theorem is to use a reverse triangle inequality to recombine terms that were originally split using the triangle inequality. This is possible since our exponent tends to 1 in the critical parts of the domain.

Proposition 4.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and let p be* log-*Hölder continuous with $1 \leqslant p^- \leqslant p^+ \leqslant 2$. Then there exists a constant C depending only on p, Ω and the dimension n so that*

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \leqslant C\|f\|_{L^{p(\cdot)}\psi_{p(\cdot)}[L](\Omega)}$$

for every $f \in L^{p(\cdot)}\psi_{p(\cdot)}[L](\Omega)$.

Proof. By the homogenity of the claim, it suffices to consider such non-negative functions f that

$$\int_\Omega f(x)^{p(x)}\psi_{p(x)}(f(x))\,dx \leqslant 1.$$

Then we must show that $\|Mf\|_{L^{p(\cdot)}} \leqslant C$, which is equivalent to $\varrho_{L^{p(\cdot)}}(Mf) \leqslant C$.

We split f into small and large parts as follows:

$$f_s = f\chi_{\{f\leqslant e^{p'(\cdot)}-e\}} \quad \text{and} \quad f_l = f\chi_{\{f> e^{p'(\cdot)}-e\}}.$$

Note that $\varrho_{p(\cdot)}(Mf) \lesssim \varrho_{p(\cdot)}(Mf_s) + \varrho_{p(\cdot)}(Mf_l)$. By Lemma 4.1 we have

$$Mf_s(x)^{p(x)} \lesssim M\big[f_s^{p(\cdot)}\big](x) + 1$$

for $x \in \Omega$. Then the embedding $L\log L \hookrightarrow L^1$ implies that

$$\begin{aligned}\int_\Omega Mf_s(x)^{p(x)}\,dx &\lesssim \int_\Omega f_s(y)^{p(y)}\log\Big(e + f_s(y)^{p(y)}\Big)\,dy + |\Omega| \\ &\lesssim \int_\Omega f(y)^{p(y)}\psi_{p(y)}(f(y))\,dy + |\Omega| \leqslant C.\end{aligned}$$

This takes care of f_s

Next we treat f_l. Let us define $r_i = 1 + 1/i$ for $i \geqslant 1$ and $\Omega_i = \big\{r_i < p \leqslant r_{i-1}\big\}$ so that $\cup\Omega_i = \{p > 1\}$. The sequences $(1)_{i=0}^\infty$ and $(r_i)_{i=0}^\infty$ satisfy the criterion of Nekvinda [35, Theorem 4.1] so we conclude that $l^1 \cong l^{(r_i)}$. We fix $K > 0$ so that

$$\sum_{i\geqslant 1} x_i \leqslant K \quad \text{whenever} \quad \sum_{i\geqslant 1} x_i^{r_i} \leqslant L \tag{4.3}$$

where L is will be specified later.

Define $f_i = f_l\,\chi_{\Omega_i}$ and $p_i = \max\{r_i, p\}$ for $i \geqslant 2$. Since $f_l = 0$ in $\{p = 1\}$, we see that $\sum f_i = f_l$. By the subadditivity of the maximal operator, the triangle inequality and the embedding $L^{p_i(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ [30, Theorem 2.8], we conclude that

$$\|Mf_l\|_{p(\cdot)} \leqslant \sum_{i\geqslant 2} \|Mf_i\|_{p(\cdot)} \leqslant (1+|\Omega|) \sum_{i\geqslant 2} \|Mf_i\|_{p_i(\cdot)}.$$

Next, Theorem 1.2 and the norm-modular inequality, Lemma 2.2, imply that

$$\|Mf_l\|_{p(\cdot)} \lesssim \sum_{i\geqslant 2} (r_i-1)^{-1} \|f_i\|_{L^{p_i(\cdot)}(\Omega_i)} \lesssim \sum_{i\geqslant 2} (r_i-1)^{-1} \varrho_{L^{p_i(\cdot)}(\Omega_i)}(f_i)^{1/r_{i-1}}.$$

By (4.3) the right hand side is bounded by K provided we show that

$$\sum_{i\geqslant 2} (r_i-1)^{-r_{i-1}} \varrho_{L^{p_i(\cdot)}(\Omega_i)}(f_i) \leqslant L. \tag{4.4}$$

But for this we just need to estimate as follows:

$$\begin{aligned}\sum_{i} (r_i-1)^{-r_{i-1}} \varrho_{L^{p_i(\cdot)}(\Omega_i)}(f_i) &\lesssim \sum_{i\geqslant 2} \int_{\Omega_i} (r_i-1)^{-1} f_i(x)^{p_i(x)} dx \\ &\lesssim \sum_{i\geqslant 2} \int_{\Omega_i} (p(x)-1)^{-1} f(x)^{p(x)} dx \\ &\leqslant \varrho_{L^{p(\cdot)}\psi_{p(\cdot)}[L](\Omega)}(f) \leqslant 1.\end{aligned}$$

We collect the implicit absolute constants from these estimates and define L by them. Thus (4.4) holds, and so (4.3) concludes the proof. □

We next show that the space we constructed is optimal for the boundedness of the maximal operator. First we have a constant exponent version of the result we are pursuing.

Lemma 4.5. *Let $p \in (1, 2]$ and fix an open bounded set $\Omega \subset \mathbf{R}^n$. Assume that $f \in L^1(\Omega)$ is a non-negative function with $\int_\Omega f\,dx = 1$ which does not take values in the range $\big(0, e^{Kp'}\big)$ for some $K \geqslant 1$. Then*

$$p' \int_\Omega |f(x)|^p\,dx \lesssim \int_{\Omega'} Mf(x)^p\,dx,$$

where $\Omega' := \big\{x \in \mathbf{R}^n : d(x,\Omega) < \Omega_n^{-1/n} e^{-(K-1)p'/n}\big\}$ is a dilatation of Ω.

Proof. Since the support of f lies in Ω and f has unit mass, we see that $Mf(x) < a$ if $d(x,\Omega) > \big(\Omega_n\,a\big)^{-1/n}$. We set $a = e^{(K-1)p'}$. Then $Mf < \lambda$ in the set $\mathbf{R}^n \setminus \Omega'$ for $\lambda \geqslant a$. Using this for the equality, and [40, Chapter 1, 5.2(b)] for the inequality, we conclude that

$$\big|\big\{x \in \Omega' : Mf(x) > \lambda\big\}\big| = \big|\big\{x \in \mathbf{R}^n : Mf(x) > \lambda\big\}\big| \geqslant \frac{1}{2^n\lambda} \int_{\{f>\lambda\}} f(x)\,dx$$

for every $f \in L^1(\Omega)$ and $\lambda \geqslant a$. We start with the usual kind of estimate, but split our integral in two parts:

$$\begin{aligned} p' \int_\Omega f(x)^p\,dx &= \int_0^\infty p\lambda^{p-2} \int_{\{f>\lambda\}} f(x)\,dx\,d\lambda \\ &= \int_0^a p\lambda^{p-2} \int_\Omega f(x)\,dx\,d\lambda + \int_a^\infty p\lambda^{p-2} \int_{\{f>\lambda\}} f(x)\,dx\,d\lambda \\ &\leqslant p'a^{p-1} \int_\Omega f(x)\,dx + 2^n \int_a^\infty p\lambda^{p-1} \big|\{x \in \Omega' : Mf(x) > \lambda\}\big|\,d\lambda \\ &\leqslant p'e^{(K-1)p} \int_\Omega f(x)\,dx + 2^n \int_{\Omega'} Mf(x)^p\,dx. \end{aligned}$$

Since $f^{p-1} \notin (0, e^{Kp})$, we see that

$$p'e^{(K-1)p} \int_\Omega f(x)\,dx \leqslant p'e^{-p} \int_\Omega f(x)^p\,dx.$$

Thus we can absorb the first term of the right hand side in the previous estimate into the left hand side, and so we conclude that

$$p'(1-e^{-p}) \int_\Omega f(x)^p\,dx \leqslant 2^n \int_{\Omega'} Mf(x)^p\,dx. \qquad \square$$

Theorem 4.6. *Let $p \in \mathscr{P}^{\log}(\Omega)$ with $1 \leqslant p^- \leqslant p^+ \leqslant \infty$ in a ball $\Omega \subset \mathbf{R}^n$. If $Mf \in L^{p(\cdot)}(\Omega)$, then $f \in L^{p(\cdot)}\psi_{p(\cdot)}[L](\Omega)$.*

Proof. We extend p by reflection in $\partial\Omega$ to the dilated ball $\frac{3}{2}\Omega$, and then to all of $\mathbf{R}^n$ without changing the log-Hölder constant by [12, Proposition 3.6]. The extension can be done so that p is bounded away from 1 in the complement of 2Ω. Let $f \in L^1(\Omega)$ be a function with $\int_\Omega Mf(x)^{p(x)}\,dx < \infty$. As in the proof of [22, Theorem 2.7] we conclude by reflection in $\partial\Omega$ that Mf is in fact bounded on $\frac{3}{2}\Omega$. Since f is supported in Ω, we have $Mf(x) \lesssim |x|^{-n}$ in $\mathbf{R}^n \setminus \frac{3}{2}\Omega$. Thus we conclude, that $Mf \in L^{p(\cdot)}(\mathbf{R}^n)$.

We may assume that f is non-negative since otherwise we may estimate function $|f|$. The claim obviously holds in the set $\{p \geqslant 2\}$ since $Mf \geqslant f$ a.e. Thus we assume in the rest of the proof that $p < 2$. Since we may scale the function f by a constant without affecting the claim, we assume that $\varrho_{L^{p(\cdot)}(\mathbf{R}^n)}(Mf) \leqslant 1$ and $\|f\|_1 \leqslant 1$.

We split f into small and large parts for $K > 1$ as follows:

$$f_s = f\chi_{\{f \leqslant e^{Kp'(\cdot)} - e\}} \quad \text{and} \quad f_l = f\chi_{\{f > e^{Kp'(\cdot)} - e\}}.$$

Then we find that

$$\int_\Omega f(x)^{p(x)} \psi_{p(x)}(f(x))\,dx \leqslant K \int_\Omega f_s(x)^{p(x)} \log(e + f_s(x))\,dx + 2 \int_\Omega p'(x)\, f_l(x)^{p(x)}\,dx.$$

Thus it suffices to bound the terms on the right hand side. Since $f_s(x)^{p(x)-1} \leqslant e^{Kp(x)} \leqslant e^{2K}$ we see that

$$\int_\Omega f_s(x)^{p(x)} \log(e + f_s(x))\,dx \leqslant e^{2K} \int_\Omega f_s(x) \log(e + f_s(x))\,dx.$$

Since $Mf \in L^1(\Omega)$ it follows by [40, Chapter 1, 5.2(b)] that the first member of the right hand side is finite.

So it remains to estimate

$$\int_\Omega p'(x)\, f_l(x)^{p(x)}\, dx.$$

We divide Ω into the sets $\Omega_i = \{r_{i+1} < p < r_{i-1}\}$, $r_i = 1 + 2^{-i}$, and write $f_i = f_l \chi_{\Omega_i}$. Let Ω_i' be the dilation of Ω_i, as in Lemma 4.5, corresponding to exponent r_i. Since $r_i' = 2^i + 1$, we get

$$d(\partial\Omega_i', \partial\Omega_i) \leqslant C_n e^{-(K-1)(2^i+1)/n}.$$

On the other hand,

$$d(\partial\Omega_{i-j}, \partial\Omega_i) \geqslant \exp\left(-c_{\log}\frac{r_{i-j+1}r_{i-1}}{r_{i-j+1} - r_{i-1}}\right) \geqslant e^{-c_{\log}2^{i+3}/|2^j-4|}$$

for $j > 2$ by the log-Hölder continuity of the exponent. A similar estimate holds for $j < -1$. Now we may choose K depending on C_n, $c_{\log}$ and n so that $\Omega_i' \subset \bigcup_{j\in J}\Omega_{i+j}$ for every $i \in \mathbf{N}$, where $J \subset \mathbf{Z}$ is finite. We can then use Lemma 4.5 in the sets Ω_i as follows

$$\int_\Omega p'(x)\, f_l(x)^{p(x)}\, dx \lesssim \sum_i \int_{\Omega_i} r_i' \left(f_i^{p(x)/r_i}\right)^{r_i} dx \lesssim \sum_i \int_{\Omega_i'} M\big[f_i^{p(\cdot)/r_i}\big](x)^{r_i}\, dx.$$

By Lemma 4.1, $M(f_i^{p(\cdot)/r_i})^{r_i/p(x)} \lesssim Mf_i(x) + 1 \leqslant Mf(x) + 1$ for $x \in \Omega_i'$. Using this and the finite overlap of Ω_i' gives

$$\begin{aligned}\int_\Omega p'(x)\, f_l(x)^{p(x)}\, dx &\lesssim \sum_i \int_{\Omega_i'} (Mf(x)+1)^{p(x)}\, dx \lesssim \sum_i \int_{\Omega_i'} 2^{p(x)-1}(Mf(x)^{p(x)}+1)\, dx \\ &\lesssim |J| \sum_i \int_{\Omega_i} 2^{p(x)-1}(Mf(x)^{p(x)}+1)\, dx \\ &\leqslant C \int_{\mathbf{R}^n} Mf(x)^{p(x)}\, dx + C\,|\Omega| \leqslant C.\end{aligned}$$

Thus also the integral over the large part of the function is bounded, which concludes the proof. □

We are now ready to prove the main theorem.

Proof of Theorem 1.1. Theorem 4.6 states that if $Mf \in L^{p(\cdot)}(\Omega)$ for a ball Ω, then $f \in L^{p(\cdot)}\psi_{p(\cdot)}[L](\Omega)$. We proceed to show how the reverse implication can be pieced together from the previous results.

By the homogenity of the claim, it suffices to consider such non-negative functions f that

$$\int_\Omega f(x)^{p(x)}\psi_{p(x)}(f(x))\, dx \leqslant 1.$$

Then we must show that $\|Mf\|_{L^{p(\cdot)}} \leqslant C$, which is equivalent to $\varrho_{L^{p(\cdot)}}(\beta\, Mf) \leqslant C$, $\beta \in (0,1)$. Denote $\Omega_a^- = \{p < a\}$, $\Omega_a^+ = \{p \geqslant a\}$ and

$$d_0 = \min\{d(\Omega_{4/3}^-, \Omega_{5/3}^+), d(\Omega_{5/3}^-, \Omega_2^+)\}.$$

Note that Ω_a^- and Ω_a^+ are bounded since Ω is bounded. The uniform continuity of the exponent implies that $d_0 > 0$. Denote by D the set of points $x \in \Omega$ for which $Mf(x) = \fint_{B(x,r)} f(y)\, dy$ with $r \geqslant d_0$. We note that

$$\varrho_{L^{p(\cdot)}(\Omega)}(\beta\, Mf) \leqslant \varrho_{L^{p(\cdot)}(\Omega_{5/3}^-\setminus D)}(Mf) + \varrho_{L^{p(\cdot)}(\Omega_{5/3}^+\setminus D)}(\beta\, Mf) + \varrho_{L^{p(\cdot)}(D)}(\beta\, Mf).$$

For $x \in D$ we have $Mf(x) \lesssim d_0^{-n}\|f\|_1$ and so $\varrho_{L^{p(\cdot)}(D)}(\beta Mf) \leqslant C$. For the other two terms only points where $p < 2$ or $p \geqslant 4/3$ affect the maximal function. Thus Theorem 1.2 implies that $\varrho_{L^{p(\cdot)}(\Omega^+_{5/3}\setminus D)}(\beta Mf) \leqslant C$ and Proposition 4.2 implies that $\varrho_{L^{p(\cdot)}(\Omega^-_{5/3}\setminus D)}(Mf) \leqslant C$, which completes the proof. $\square$

5. Improving the results of Cruz-Uribe and Fiorenza

As a corollary of our embedding theorem, we get the improvement of the result by Cruz-Uribe and Fiorenza [6] stated in the introduction. The proof is as follows.

Proof of Theorem 1.3. We fix $K > 0$ and define

$$q(x) = \max\Big\{0, 1 - K\big(\log p'(x)\big)^{-1}\Big\}.$$

We prove the theorem by showing that

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \lesssim \|f\|_{L^{p(\cdot)}\psi_{p(\cdot)}[L](\Omega)} \lesssim \|f\|_{L^{p(\cdot)}\log L^{q(\cdot)}(\Omega)}.$$

The first inequality follows from Theorem 1.1, so it suffices to show the second one. This in turn follows from the point-wise inequality

$$t^p \min\{p', \log(e+t)\} \lesssim t^p \log^q(e+t)$$

between modulars, which we proceed to prove.

Further we restrict our attention to so small p that $q = 1 - K\big(\log p'\big)^{-1} \geqslant 0$. Then the previous inequality follows from

$$\min\{p', z\} \leqslant e^K z^{1-K/\log p'},$$

where we denoted $z = \log(e+t) \geqslant 1$. Suppose first that $p' \leqslant z$. Then

$$e^K z^{1-K/\log p'} \geqslant e^K p'^{1-K/\log p'} = p',$$

since the exponent of z is non-negative. If, on the other hand, $p' \geqslant z$, then $e^K z^{-K/\log p'} \geqslant e^K p'^{-K/\log p'} = 1$, so the inequality follows in both cases. $\square$

In [6, Example 1.9], Cruz-Uribe and Fiorenza showed that it is not possible to get an optimal result using the scale $L^{p(\cdot)}\log L^{q(\cdot)}$, i.e. that an analogue of Theorem 4.6 is not possible in the $L^{p(\cdot)}\log L^{q(\cdot)}$-scale. This is not a surprising result, but required a good two pages of proof and certain continuity assumptions of the exponents.

Our next result improves their example by assuming the boundedness of the maximal operator, instead of their log-Hölder continuity assumption, removing the continuity assumption

$$|q(x) - q(y)| \leqslant \frac{C}{\log\log(e^e + 1/|x-y|)}$$

on the exponent q, and halving the length of the proof.

Proposition 5.1. *Let $\Omega \subset \mathbf{R}^n$ be an open set. Let p be a continuous bounded variable exponent such that $M\colon L^{p_\varepsilon(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is bounded for every $\varepsilon > 0$, where $p_\varepsilon = \max\{p, 1+\varepsilon\}$. Suppose $q\colon \Omega \to [0,1]$ is a measurable function that has a Lebesgue point in the set $\{p > 1, q > 0\}$. Then there exists a function f with $Mf \in L^{p(\cdot)}(\Omega)$ but $f \notin L^{p(\cdot)}\log L^{q(\cdot)}(\Omega)$.*

Proof. We assume without loss of generality that the Lebesgue point of q is 0 and denote $q(0)/3$ by δ.

Let $B \subset \Omega$ be a ball centered at 0 with radius $r \in (0, 1/2)$ so that $p_B^- = 1+\varepsilon > 1$. Define the function f by $f(y) = |y|^{-\frac{n}{p(y)}} \log(1/|y|)^{-\frac{1+\delta}{p(y)}} \chi_B(y)$. Then

$$\varrho_{p_\varepsilon(\cdot)}(f) = \int_B |y|^{-n} \log(1/|y|)^{-(1+\delta)}\, dy = \int_0^r y^{-1} \log(1/y)^{-(1+\delta)}\, dy < \infty.$$

Therefore $f \in L^{p_\varepsilon(\cdot)}(\Omega)$ and hence $Mf \in L^{p(\cdot)}(\Omega)$.

Let $k = \big(\frac{2(1-\delta)}{\delta}\big)^{1/n}$ and let $i_0 \gg 1$ be an index so large that $q_{B(0,k^{-i})} \geqslant 2\delta$ for every $i \geqslant i_0$. Denote $A := \{q > \delta\}$. Then

$$\begin{aligned}\varrho_{L^{p(\cdot)} \log L^{q(\cdot)}(\Omega)}(f) &= \int_\Omega f(y)^{p(y)} \log(f(y))^{q(y)}\, dy \\ &\geqslant \int_{B(0,k^{-i_0}) \cap A} |y|^{-n} \log(1/|y|)^{q(y)-(1+\delta)}\, dy \\ &\geqslant \sum_{i \geqslant i_0} \int_{(B(0,k^{-i}) \setminus B(0,k^{-i-1})) \cap A} |y|^{-n} \log(1/|y|)^{-1}\, dy.\end{aligned}$$

Since $q \leqslant 1$ and $q_{B(0,k^{-i})} \geqslant 2\delta$, we find that the set $B(0,k^{-i}) \setminus A$ has measure at most $\frac{1-2\delta}{1-\delta}|B(0,k^{-i})|$. Therefore the measure of the set $(B(0,k^{-i}) \setminus B(0,k^{-i-1})) \cap A$ is at least

$$(1-k^{-n})|B(0,k^{-i})| - \frac{1-2\delta}{1-\delta}|B(0,k^{-i})| = \frac{\delta}{2(1-\delta)}|B(0,k^{-i})|.$$

We fix $s = \big(\frac{2-3\delta}{2(1-\delta)}\big)^{1/n} \in (0,1)$ so that $1 - s^n = \frac{\delta}{2(1-\delta)}$. Since $t \mapsto t^{-n} \log(1/t)^{-1}$ is a decreasing function (for small t), we find that

$$\begin{aligned}\int_{(B(0,k^{-i}) \setminus B(0,k^{-i-1})) \cap A} |y|^{-n} \log(1/|y|)^{-1}\, dy &\geqslant \int_{(B(0,k^{-i}) \setminus B(0,s\,k^{-i}))} |y|^{-n} \log(1/|y|)^{-1}\, dy \\ &= \int_{sk^{-i}}^{k^{-i}} \log(1/t)^{-1} \frac{dt}{t} = \log \frac{\log(1/(sk^{-i}))}{\log(1/k^{-i})}.\end{aligned}$$

This estimate holds for every $i \geqslant i_0$ since $i_0 \gg 1$, and so we get

$$\varrho_{L^{p(\cdot)} \log L^{q(\cdot)}(\Omega)}(f) \geqslant \sum_{i \geqslant i_0} \log\left(1 + \frac{\log(1/s)}{i \log k}\right) = \infty. \qquad \square$$

6. The boundedness of the maximal operator implies that the exponent is bounded away from one

In this section we improve Theorem 1.7 by Cruz-Uribe, Fiorenza and Neugebauer [8] and show without any a priori assumptions on the exponent that the boundedness of the maximal operator implies that $p^- > 1$.

Since our method is based on a partition, we work with cubes instead of balls in this section. By a cube we always mean a cube with sides paralell to the coordinate axes. By $\bar{p}_Q$ we denote the harmonic average of p over Q,

$$\bar{p}_Q := \left(⨍_Q \frac{1}{p(x)}\, dx \right)^{-1}.$$

Lemma 6.1. *Let p be a variable exponent and let $Q \subset \mathbf{R}^n$ with $|Q| > 0$. Then*

$$\Big(\frac{t}{2}\Big)^{\bar{p}_Q} \leqslant \fint_Q t^{p(y)}\,dy \quad \text{and} \quad t^{\frac{1}{\bar{p}_Q}} \leqslant \fint_Q t^{\frac{1}{p(y)}}\,dy$$

for all $t \geqslant 0$.

Proof. For the second claim we define $F_t(s) = t^s$. Then the claim can be written as

$$F_t\bigg(\fint_Q \frac{1}{p(x)}\,dx\bigg) \leqslant \fint_Q F_t\bigg(\frac{1}{p(x)}\bigg)\,dx.$$

Since F_t is convex, this follows from Jensen's inequality.

For the first claim we define $G_t(s) = st^{1/s}$. Another application of Jensen's inequality gives

$$\frac{1}{\bar{p}_Q} t^{\bar{p}_Q} \leqslant \fint_Q \frac{1}{p(x)} t^{p(x)}\,dx.$$

The claim follows from this since $\big(\frac{t}{2}\big)^p \leqslant \frac{1}{p}t^p \leqslant t^p$. $\square$

We can now calculate $\|\chi_Q\|_{p(\cdot)}$. The novelty lies in including the case $p^+ = \infty$ and not assuming that p is log-Hölder continuous.

Lemma 6.2. *Let p be an exponent for which the maximal operator is bounded from $L^{p(\cdot)}$ to itself with constant $A > 0$. Then*

$$\tfrac{1}{6}|Q|^{\frac{1}{\bar{p}_Q}} \leqslant \|\chi_Q\|_{p(\cdot)} \lesssim A\,|Q|^{\frac{1}{\bar{p}_Q}}.$$

Proof. Only the proof of the upper bound is included here. The lower bound is not needed in the sequel, and its proof can be found in [11, Lemma 3.40].

Fix a cube Q and define $f(x) := \chi_Q(x)\,|Q|^{-1/p(x)}$. Note that $\varrho_{p(\cdot)}(f) \leqslant 1$ and hence $\|f\|_{p(\cdot)} \leqslant 1$. Since $|f|_Q \leqslant cMf(x)$ for $x \in Q$, we get with Lemma 6.1 for the first inequality,

$$\|\chi_Q\|_{p(\cdot)}\,|Q|^{-\frac{1}{\bar{p}_Q}} \leqslant \|\chi_Q\|_{p(\cdot)} \fint_Q |Q|^{-\frac{1}{p(x)}}\,dx = \|\chi_Q\,|f|_Q\|_{p(\cdot)} \lesssim \|Mf\|_{p(\cdot)} \lesssim A. \qquad \square$$

The next proof relies on the following fact: although M is continuous from $L^q(\Omega)$ to $L^q(\Omega)$ for all $q \in (1,\infty]$, the constant of continuity blows up as $q \to 1$. We will show that $p^- = 1$ also implies a blow up of the continuity constant of M.

Theorem 6.3. *Let $\Omega \subset \mathbf{R}^n$ be an open set. Let p be such a variable exponent that $M\colon L^{p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is bounded. Then $p^- > 1$.*

Proof. Assume for a contradiction that $p^- = 1$ and that $M\colon L^{p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is bounded with embedding constant $A \geqslant 1$.

Fix $\varepsilon \in (0,1)$. Since $p^- = 1$, the set $\big\{\frac{1}{p} > \frac{1}{1+\varepsilon/2}\big\}$ has positive measure and therefore has some point of density z_0, i.e. a point for which

$$\lim_{Q \to \{z_0\}} \frac{\Big|Q \cap \big\{\frac{1}{p} > \frac{1}{1+\varepsilon/2}\big\}\Big|}{|Q|} = 1,$$

where the limit is taken over all cubes containing z_0. Therefore, there exists a cube $Q_0 \subset \Omega$ with $z_0 \in Q_0$ and $\ell(Q_0) \leqslant 1$ such that $\frac{1}{\bar{p}_{Q_0}} = \fint_{Q_0} \frac{1}{p(y)}\,dy \geqslant 1/(1+\varepsilon)$, i.e. $\bar{p}_{Q_0} \leqslant 1+\varepsilon$. Here $\ell(Q_0)$ denotes the side length of Q_0. Let $m \in \mathbf{N}$ be large and split

Q_0 into $N := 2^{mn}$ disjoint cubes $Q_1, \dots, Q_N$ of side length $\ell(Q_j) = 2^{-m}\,\ell(Q_0)$. By renumbering we assume without loss of generality that

$$\bar{p}_{Q_1} = \min_{1 \leqslant j \leqslant N} \bar{p}_{Q_j}.$$

In particular $\bar{p}_{Q_1} \leqslant \bar{p}_{Q_0} \leqslant 1 + \varepsilon < 2$. Define $f \in L^1(\Omega)$ by

$$f := A^{-2}\,|Q_1|^{-1/\bar{p}_{Q_1}}\,\chi_{Q_1}.$$

Then Lemma 6.2 implies that $\|f\|_{p(\cdot)} = A^{-2}\,|Q_1|^{-1/\bar{p}_{Q_1}}\,\|\chi_{Q_1}\|_{p(\cdot)} \lesssim A^{-1}$. Especially, we have $f \in L^{p(\cdot)}(\mathbf{R}^n)$ and $\|Mf\|_{p(\cdot)} \leqslant A\,\|f\|_{p(\cdot)} \lesssim 1$. We arrive at a contradiction by showing that $\varrho_{p(\cdot)}(\beta\,Mf)$ will be large if $\varepsilon > 0$ is small enough and $m \in \mathbf{N}$ is large enough.

Let x_j denote the center of Q_j for $j = 1, \dots, N$. Then for $j = 2, \dots, N$, and for all $y \in Q_j$ one easily checks that

$$\beta\,Mf(y) \geqslant \underbrace{c\,\beta\,A^{-2}\,|Q_1|^{1-1/\bar{p}_{Q_1}}}_{=:A_2}\,|x_j - x_1|^{-n},$$

where c depends only on the dimension n. Therefore, by Lemma 6.1,

$$\begin{aligned}\int_{Q_0} \big(\beta\,Mf(y)\big)^{p(y)}\,dy &\geqslant \sum_{j\geqslant 2} \int_{Q_j} \Big(A_2\,|x_j - x_1|^{-n}\Big)^{p(y)}\,dy \\ &\geqslant \sum_{j\geqslant 2} |Q_j|\,\Big(\tfrac{1}{2} A_2\,|x_j - x_1|^{-n}\Big)^{\bar{p}_{Q_j}}.\end{aligned}$$

Since $\bar{p}_{Q_j} \geqslant \bar{p}_{Q_1}$, we have $t^{\bar{p}_{Q_1}} \leqslant t^{\bar{p}_{Q_j}} + 1$. Hence,

$$\begin{aligned}\int_{Q_0} \big(\beta\,Mf(y)\big)^{p(y)}\,dy &\geqslant \sum_{j\geqslant 2} |Q_j|\,\Big(\big(\tfrac{1}{2} A_2\,|x_j - x_1|^{-n}\big)^{\bar{p}_{Q_1}} - 1\Big) \\ &\geqslant -|Q_0| + \sum_{j\geqslant 2} |Q_j|\,\big(\tfrac{1}{2} A_2\,|x_j - x_1|^{-n}\big)^{\bar{p}_{Q_1}} \\ &\geqslant -1 + C \int_{Q_0 \setminus Q_1} \big(A_2\,|y - x_1|^{-n}\big)^{\bar{p}_{Q_1}}\,dy.\end{aligned}$$

We can essentially calculate the integral in the previous estimate:

$$\begin{aligned}\int_{Q_0 \setminus Q_1} |y - x_1|^{-n\bar{p}_{Q_1}}\,dy &\approx \int_{\ell(Q_1)}^{\ell(Q_0)/2} r^{n-1-n\bar{p}_{Q_1}}\,dr \\ &\approx \frac{1}{(\bar{p}_{Q_1} - 1)n}\big(|Q_1|^{1-\bar{p}_{Q_1}} - (|Q_0|/2^n)^{1-\bar{p}_{Q_1}}\big).\end{aligned}$$

We use this and the expression of A_2 in our previous estimate and conclude that

$$\begin{aligned}1 + \int_{Q_0} \big(\beta\,Mf(y)\big)^{p(y)}\,dy &\gtrsim \beta\,A^{-2\bar{p}_{Q_1}} \frac{|Q_1|^{\bar{p}_{Q_1}-1}}{\bar{p}_{Q_1} - 1}\big(|Q_1|^{1-\bar{p}_{Q_1}} - (|Q_0|/2^n)^{1-\bar{p}_{Q_1}}\big) \\ &= \frac{\beta\,A^{-2\bar{p}_{Q_1}}}{\bar{p}_{Q_1} - 1}\big(1 - 2^{-n(m-1)(\bar{p}_{Q_1}-1)}\big),\end{aligned}$$

where we used that $|Q_1|/|Q_0| = 2^{-nm}$ in the last step. Now we choose m so large that $2^{-n(m-1)(\bar{p}_{Q_1}-1)} \leqslant 1/2$ and recall that $1 \leqslant \bar{p}_{Q_1} \leqslant 1+\varepsilon < 2$. Then we have that

$$1 + \int_{Q_0} \big(\beta\, Mf(y)\big)^{p(y)}\, dy \gtrsim \beta\, A^{-4}\varepsilon^{-1}.$$

As $\varepsilon \to 0$ this contradicts $\|Mf\|_{p(\cdot)} \leqslant 1$, which means that the assumption $p^- = 1$ was wrong, as was to be shown. □

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Received 23 September 2008