Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 41–50

ON OPEN AND DISCRETE MAPPINGS WITH A MODULUS CONDITION

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Abstract. It is proved that sense preserving continuous mappings $f\colon D\to \mathbf{R}^n$ of a domain D in $\mathbf{R}^n$, $n\geqslant 2$, satisfying some general inequalities for p-modulus of families of curves are open and discrete.

1. Preliminaries

The paper is devoted to the study of quasiregular mappings and their natural generalizations investigated long time, see e.g. [AC, Cr_1, Cr_2, Gol_1, Gol_2, GRSY, HK, IM, KO, MRV, MRSY, Re, Ri, UV] and further references therein.

Let us give some definitions. Everywhere further D is a domain in $\mathbf{R}^n$, $n\ge 2$, m is the Lebesgue measure in $\mathbf{R}^n$, $m(A)$ the Lebesgue measure of a set $A\subset\mathbf{R}^n$. A mapping $f\colon D\to\mathbf{R}^n$ is called *discrete* if $f^{-1}(y)$ consists of isolated points for each $y\in\mathbf{R}^n$, and f is said to be *open* if it maps open sets onto open sets. The notation $f\colon D\to\mathbf{R}^n$ assumes that f is continuous. A mapping f is said to be *sense-preserving* if the topological index $\mu(y,f,G)>0$ for an arbitrary domain $G\subset D$ such that $\overline{G}\subset D$ and $y\in f(G)\setminus f(\partial G)$, see e.g. [Re, II.2]. Given a mapping $f\colon D\to\mathbf{R}^n$, a set $E\subset D$ and a point $y\in\mathbf{R}^n$, we define the multiplicity function $N(y,f,E)$ as the number of pre-images of y in E, i.e.,

$$N(y,f,E)=\operatorname{card}\{x\in E\colon f(x)=y\}$$

and

$$N(f,E)=\sup_{y\in\mathbf{R}^n}N(y,f,E).$$

A set $H\subset\overline{\mathbf{R}^n}$ is called *totally disconnected* if its every component degenerates to a point; in this case we write $\dim H=0$ where dim denotes the *topological dimension* of H (see [HW, Section 1, Ch. II]). A mapping $f\colon D\to\overline{\mathbf{R}^n}$ is said to be *light* if $\dim\{f^{-1}(y)\}=0$ for every $y\in\overline{\mathbf{R}^n}$. Set

$$B(x_0,r)=\{x\in\mathbf{R}^n\colon |x-x_0|<r\},\quad \mathbf{B}^n:=B(0,1),\quad \mathbf{S}^{n-1}:=S(0,1),$$

Ω_n is a volume of the unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$, and ω_{n-1} is an area of the unit sphere $\mathbf{S}^{n-1}$ in $\mathbf{R}^n$.

A curve γ in $\mathbf{R}^n$ is a continuous mapping $\gamma\colon\Delta\to\mathbf{R}^n$ where Δ is an interval in $\mathbf{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family Γ of curves γ in $\mathbf{R}^n$, a Borel function $\rho\colon\mathbf{R}^n\to[0,\infty]$ is called *admissible* for Γ, abbr. $\rho\in\operatorname{adm}\Gamma$, if

$$\int_\gamma \rho(x)|dx|\ge 1$$

doi:10.5186/aasfm.2016.4105
2010 Mathematics Subject Classification: Primary 30C65; Secondary 30C62.
Key words: Mappings with bounded and finite distortion, moduli of families of curves, discrete and open mappings.

for each (locally rectifiable) $\gamma \in \Gamma$. Given $p \geq 1$, the *p-modulus* of Γ is defined by

$$M_p(\Gamma) := \inf_{\rho \in \operatorname{adm} \Gamma} \int_{\mathbf{R}^n} \rho^p(x)\, dm(x)$$

interpreted as $+\infty$ if $\operatorname{adm} \Gamma = \varnothing$. Note that $M_p(\varnothing) = 0$; $M_p(\Gamma_1) \leq M_p(\Gamma_2)$ whenever $\Gamma_1 \subset \Gamma_2$, and $M_p\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) \leq \sum_{i=1}^{\infty} M_p(\Gamma_i)$, see [Va, Theorem 6.2].

Denote $\Gamma(E, F, D)$ a family of all paths $\gamma\colon [a, b] \to \overline{\mathbf{R}^n}$, which join sets E and F in D, i.e., $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$.

The following fact has been established in [Sev]. Let f be a sense-preserving mapping of a domain $D \subset \mathbf{R}^n$, $n \geqslant 2$, into $\mathbf{R}^n$ obeying a condition

$$M(\Gamma) \leqslant \int_{f(D)} Q(y) \cdot \rho_*^n(y)\, dm(y) \tag{1.1}$$

for every $\rho_* \in \operatorname{adm} f(\Gamma)$ with respect to the conformal modulus $M(\Gamma) := M_n(\Gamma)$ and a given function $Q\colon \mathbf{R}^n \to [0, \infty]$. Then f is open and discrete whenever Q satisfies some conditions. Given $y_0 \in f(D)$ and numbers $0 < r_1 < r_2 < \infty$, we denote

$$A(r_1, r_2, y_0) = \{y \in \mathbf{R}^n\colon r_1 < |y - y_0| < r_2\}. \tag{1.2}$$

The goal of the present paper is to prove a similar result for $n - 1 < p \leqslant n$. Namely, given $y_0 \in f(D)$ and $0 < r_1 < r_2 < \infty$, let $\Gamma(y_0, r_1, r_2)$ be the family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(r_1, r_2, y_0))$. Instead of (1.1), assume that f satisfies the inequality

$$M_p(\Gamma(y_0, r_1, r_2)) \leqslant \int_{f(D)} Q(y) \cdot \eta^p(|y - y_0|) dm(y) \tag{1.3}$$

for some $p \in (n - 1, n]$, every $y_0 \in f(D)$, any $0 < r_1 < r_2 < \infty$, and any nonnegative Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with

$$\int_{r_1}^{r_2} \eta(r)\, dr \geqslant 1. \tag{1.4}$$

Observe that the inequality (1.3) is much weaker than (1.1) even for $p = n$. In fact, let $\rho_* \in \operatorname{adm} f(\Gamma)$, and assume that the relation (1.1) holds. We show that the inequality (1.3) is true. To this end, pick arbitrary $y_0 \in f(D)$ and set $\rho_*(y) := \eta(|y - y_0|)$, where η satisfies (1.4). Note that $\rho_* \in \Gamma(S_1, S_2, A)$ because $\int_\gamma \rho_*(y)\, |dy| \geqslant \int_{r_1}^{r_2} \eta(t)\, dt \geqslant 1$ for every $\gamma \in \Gamma(S_1, S_2, A)$ (cf. [Va, theorem 5.7]). Thus, the inequality (1.1) becomes (1.3).

The present paper is devoted to the study of the following question:

What are the properties of the majorant Q which ensure for mappings f obeying (1.3) for some $n - 1 < p \leqslant n$ to be discrete and open?

Following [IR], we say that a function $\varphi\colon \mathbf{R}^n \to \mathbf{R}$ has *a finite mean oscillation* at a point $x_0 \in \mathbf{R}^n$, write $\varphi \in FMO(x_0)$, if

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{\Omega_n \varepsilon^n} \int_{B(x_0, \varepsilon)} |\varphi(x) - \varphi_\varepsilon|\, dm(x) < \infty,$$

where Ω_n is the volume of the unit ball in $\mathbf{R}^n$ and

$$\varphi_\varepsilon = \frac{1}{\Omega_n \varepsilon^n} \int_{B(x_0, \varepsilon)} \varphi(x)\, dm(x).$$

Given a Lebesgue measurable function $Q\colon \mathbf{R}^n \to [0,\infty]$, $q_{x_0}(r)$ denotes the integral average of $Q(x)$ over the sphere $S(x_0, r)$, i.e.

$$q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x-x_0|=r} Q(x)\, dS, \tag{1.5}$$

where dS is an area element of S.

The main result of the present paper is the following

Theorem 1.1. *Let $p \in (n-1, n]$, and $Q\colon \mathbf{R}^n \to (0,\infty)$ be a Lebesgue measurable function. Assume that $f\colon D \to \mathbf{R}^n$ is a sense-preserving mapping satisfying (1.3) for every $y_0 \in f(D)$, any $0 < r_1 < r_2 < \infty$, and any nonnegative Lebesgue measurable function $\eta\colon (r_1, r_2) \to [0,\infty]$ obeying (1.4). Then f is discrete and open whenever the function Q satisfies at least one of the following conditions:*

1) *$Q \in FMO(y_0)$ for every $y_0 \in f(D)$,*
2) *$q_{y_0}(r) = O\left(\left[\log \frac{1}{r}\right]^{n-1}\right)$ as $r \to 0$ for every $y_0 \in f(D)$,*
3) *for every $y_0 \in f(D)$ there exists $\delta(y_0) > 0$ such that for every sufficiently small $\varepsilon > 0$*

$$\int_{\varepsilon}^{\delta(y_0)} \frac{dt}{t^{\frac{n-1}{p-1}} q_{y_0}^{\frac{1}{p-1}}(t)} < \infty, \qquad \int_{0}^{\delta(y_0)} \frac{dt}{t^{\frac{n-1}{p-1}} q_{y_0}^{\frac{1}{p-1}}(t)} = \infty. \tag{1.6}$$

Remark 1.1. Theorem 1.1 can be extended to the mappings $f\colon D \to \overline{\mathbf{R}^n}$. In this case, for $y = \infty$, we must require that the conditions 1)–3) hold for $\widetilde{Q} = Q \circ \varphi$ at 0, where $\varphi(y) = \frac{y}{|y|^2}$, $\varphi(0) := \infty$.

2. Main lemma

A connected compactum $C \subset \overline{\mathbf{R}^n}$ is called a *continuum*. We say that a family of paths Γ_1 *is minorized* by a family Γ_2, write $\Gamma_1 > \Gamma_2$, if for every $\gamma \in \Gamma_1$ there exists a subpath which belongs to Γ_2. In this case, $M_p(\Gamma_1) \leqslant M_p(\Gamma_2)$ (see [Va, Theorem 6.4]).

Let (X, μ) be a metric space with measure μ. For each real number $n \geq 1$, we define *the Loewner function* $\phi_n\colon (0,\infty) \to [0,\infty)$ on X as

$$\phi_n(t) = \inf\{M_n(\Gamma(E, F, X))\colon \Delta(E,F) \leqslant t\},$$

where the infimum is taken over all disjoint nondegenerate continua E and F in X and

$$\Delta(E,F) := \frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}.$$

A pathwise connected metric measure space (X,μ) is said to be a *Loewner space* of exponent n, or an n-Loewner space, if the Loewner function $\phi_n(t)$ is positive for all $t > 0$ (see [MRSY, Section 2.5] or [He, Ch. 8]). Observe that $\mathbf{R}^n$ and $\mathbf{B}^n \subset \mathbf{R}^n$ are Loewner spaces (see [He, Theorem 8.2 and Example 8.24(a)]). As known, a condition $\mu(B(x_0,r)) \geqslant C \cdot r^n$ holds in Loewner spaces X for a constant $C > 0$, every point $x_0 \in X$ and all $r < \operatorname{diam} X$. A space X is called *geodesic* if every pair of points in X can be joined by a curve whose length is equal to the distance between the points. In particular, $\mathbf{B}^n$ is a geodesic space. A following definition can be found in [He, Section 1.4, Ch. I] or [AS, Section 1]. A measure μ in a metric space is called doubling if all balls have finite and positive measure and there is a constant $C \geqslant 1$ such that $\mu(B(x_0, 2r)) \leq C \cdot \mu(B(x_0, r))$ for every $x_0 \in X$ and all $r > 0$. We also call a metric measure space (X,μ) *doubling* if μ is a doubling measure. Following [He, Section 7.22], given a real-valued function u in a metric space X, a

Borel function $\rho\colon X\to[0,\infty]$ is said to be an *upper gradient* of a function $u\colon X\to\mathbf{R}$ if $|u(x)-u(y)|\leqslant\int_\gamma\rho\,|dx|$ for each rectifiable curve γ joining x and y in X. Let (X,μ) be a metric measure space and let $1\leqslant p<\infty$. We say that X admits *a* $(1;p)$*-Poincare inequality* if there is a constant $C\geqslant 1$ such that

$$\frac{1}{\mu(B)}\int_B|u-u_B|\,d\mu(x)\leqslant C\cdot(\operatorname{diam}B)\left(\frac{1}{\mu(B)}\int_B\rho^p\,d\mu(x)\right)^{1/p}$$

for all balls B in X, for all bounded continuous functions u on B, and for all upper gradients ρ of u. Metric measure spaces where the inequalities

$$\frac{1}{C}R^n\leqslant\mu(B(x_0,R))\leqslant CR^n$$

hold for a constant $C\geqslant 1$, every $x_0\in X$ and all $R<\operatorname{diam}X$, are called *Ahlfors* n*-regular.*

We need the following statement.

Proposition 2.1. *The unit ball* $\mathbf{B}^n$ *is an Ahlfors* n*-regular metric space in which* $(1;p)$*-Poincare inequality holds. Moreover, the estimate*

$$M_p(\Gamma(E,F,\mathbf{B}^n))>0 \tag{2.1}$$

holds for any continua $E,F\subset\mathbf{B}^n$ *and every* $p\in(n-1,n]$.

Proof. By comments given above, the unit ball $\mathbf{B}^n$ is Ahlfors n-regular, moreover, the space $\mathbf{B}^n$ is geodesic and and also a Loewner space. By [He, Theorems 9.8 and 9.5], the $(1;p)$-Poincare inequality holds in $\mathbf{B}^n$. Thus, (2.1) holds by [AS, Corollary 4.8]. □

Let $A(\varepsilon,\varepsilon_0,y_0)$ be defined by (1.2) with $r_1=\varepsilon$ and $r_2=\varepsilon_0$. The following lemma provides the main tool for establishing openness and discreteness in the most general situation.

Lemma 2.1. *Let* $p\in(n-1,n]$, *and* $Q\colon\mathbf{R}^n\to(0,\infty)$ *be a Lebesgue measurable function. Assume that* $f\colon D\to\mathbf{R}^n$ *is a sense-preserving mapping satisfying (1.3) for every* $y_0\in f(D)$, *any* $0<r_1<r_2<\infty$, *and any nonnegative Lebesgue measurable function* $\eta\colon(r_1,r_2)\to[0,\infty]$ *obeying (1.4). Suppose that, in addition, for every* $y_0\in f(D)$ *and some* $\varepsilon_0>0$,

$$\int\limits_{A(\varepsilon,\varepsilon_0,y_0)}Q(y)\cdot\psi^p(|y-y_0|)\,dm(y)=o\left(I^p(\varepsilon,\varepsilon_0)\right)\quad \textit{as } \varepsilon\to 0, \tag{2.2}$$

where $\psi(t)\colon(0,\infty)\to[0,\infty]$ *is a nonnegative Lebesgue measurable function such that*

$$0<I(\varepsilon,\varepsilon_0):=\int_\varepsilon^{\varepsilon_0}\psi(t)\,dt<\infty\quad\forall\,\varepsilon\in(0,\varepsilon_0). \tag{2.3}$$

Then f *is open and discrete.*

Remark 2.1. In Lemma 2.1, we may assume only that $\int_\varepsilon^{\varepsilon_0}\psi(t)\,dt>0$ for some ε and $\varepsilon_*\in(0,\varepsilon_0)$ instead of the condition $I(\varepsilon,\varepsilon_0)>0$ for all $\varepsilon\in(0,\varepsilon_0)$. Note also that the integral (2.2) is increasing under decreasing ε. Thus, since $Q(x)>0$, $I(\varepsilon,\varepsilon_0)\to\infty$ as $\varepsilon\to 0$ follows from (2.2) and (2.3).

Proof of Lemma 2.1. Without any loss of generality, we may asssume that $D=\mathbf{B}^n$. Since every light sense-preserving mapping $f\colon D\to\mathbf{R}^n$ is open and discrete in D, see e.g., [TY, Corollary, p. 333], it is sufficient to prove that f is light. Let us assume the contrary. Then there exists $y_0\in\mathbf{R}^n$ such that the set $\{f^{-1}(y_0)\}$ is not totally disconnected, i.e., there exists a nondegenerate continuum $C \subseteq \{f^{-1}(y_0)\}$. Since f is sense-preserving, $f \not\equiv y_0$. Then by the continuity of f there exist $x_0 \in D$ and $\delta_0 > 0$ such that $\overline{B(x_0, \delta_0)} \subset D$ and

$$f(x) \neq y_0 \quad \forall\, x \in \overline{B(x_0, \delta_0)}. \tag{2.4}$$

By [Na, Lemma 1.15] for $p = n$, and Proposition 2.1 for $p \in (n-1, n)$,

$$M_p\left(\Gamma\left(C, \overline{B(x_0, \delta_0)}, \mathbf{B}^n\right)\right) > 0. \tag{2.5}$$

By (2.4), since $f(C) = \{y_0\}$, every path of $\Delta = f(\Gamma(C, \overline{B(x_0, \delta_0)}, \mathbf{B}^n))$ does not degenerate to a point. On the other hand, one of the endpoints of every path of Δ is y_0. Let Γ_i be a family of all paths $\alpha_i(t)\colon [0,1] \to \mathbf{R}^n$ such that $\alpha_i(0) = y_0$ and $\alpha_i(1) \in S(y_0, r_i)$, $r_i \in (0, \varepsilon_0)$, $i = 1, 2, \ldots$, $r_i \to 0$ as $i \to \infty$. Note that

$$\Gamma\left(C, \overline{B(x_0, \delta_0)}, \mathbf{B}^n\right) = \bigcup_{i=1}^{\infty} \Gamma_i^*, \tag{2.6}$$

where Γ_i^* is the family of all paths γ in $\Gamma(C, \overline{B(x_0, \delta_0)}, \mathbf{B}^n)$ such that $f(\gamma)$ has a subpath in Γ_i. Observe that

$$\Gamma_i^* > \Gamma(\varepsilon, r_i, y_0) \tag{2.7}$$

for every $\varepsilon \in (0, r_i)$ where $\Gamma(\varepsilon, r_i, y_0)$ is the family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, \varepsilon), S(y_0, r_i), A(\varepsilon, r_i, y_0))$. Set

$$\eta_{i,\varepsilon}(t) = \begin{cases} \psi(t)/I(\varepsilon, r_i), & t \in (\varepsilon, r_i), \\ 0, & t \notin (\varepsilon, r_i), \end{cases}$$

where $I(\varepsilon, r_i) = \int_\varepsilon^{r_i} \psi(t)\,dt$. Observe that $\int_\varepsilon^{r_i} \eta_{i,\varepsilon}(t)\,dt = 1$. Now we can apply (1.3). By (1.3) and (2.7),

$$M_p(\Gamma_i^*) \leqslant M_p(\Gamma(r_i, \varepsilon, y_0)) \leqslant \int_{A(\varepsilon, \varepsilon_0, y_0)} Q(y) \cdot \eta_{i,\varepsilon}^p(|y - y_0|)\,dm(y) \leqslant \mathfrak{F}_i(\varepsilon) \tag{2.8}$$

where $\mathfrak{F}_i(\varepsilon) = \frac{1}{I(\varepsilon, r_i)^p} \int_{A(\varepsilon, \varepsilon_0, y_0)} Q(y)\,\psi^p(|y - y_0|)\,dm(y)$ and $I(\varepsilon, r_i) = \int_\varepsilon^{r_i} \psi(t)\,dt$. By (2.2),

$$\int_{A(\varepsilon, \varepsilon_0, y_0)} Q(y)\psi^p(|y - y_0|)\,dm(y) = G(\varepsilon) \cdot \left(\int_\varepsilon^{\varepsilon_0} \psi(t)\,dt\right)^p,$$

where $G(\varepsilon) \to 0$ as $\varepsilon \to 0$ by the assumptions of the lemma. Note that $\mathfrak{F}_i(\varepsilon) = G(\varepsilon) \cdot \left(1 + \frac{\int_{r_i}^{\varepsilon_0} \psi(t)\,dt}{\int_\varepsilon^{r_i} \psi(t)\,dt}\right)^p$, where $\int_{r_i}^{\varepsilon_0} \psi(t)\,dt < \infty$ and $\int_\varepsilon^{r_i} \psi(t)\,dt \to \infty$ as $\varepsilon \to 0$, because the left-hand side in (2.2) is increasing under decreasing ε. Thus, $\mathfrak{F}_i(\varepsilon) \to 0$ as $\varepsilon \to 0$, and by (2.8) $M_p(\Gamma_i^*) = 0$ for all i. Finally, by (2.6) and the inequality $M_p\left(\bigcup_{i=1}^{\infty} \Gamma_i^*\right) \leqslant \sum_{i=1}^{\infty} M_p(\Gamma_i^*)$ [Va, Theorem 6.2], we obtain that $M_p\left(\Gamma\left(C, \overline{B(x_0, \delta_0)}, \mathbf{B}^n\right)\right) = 0$, which contradicts to (2.5). Thus, f is light and, therefore, f is open and discrete (see [TY, Corollary, p. 333]). □

3. Proof of the main result

In this section we show that the assertion of Theorem 1.1 follows from Lemma 2.1. For this goal, given $Q \in FMO(y_0)$, we may apply the function $\psi(t) = \left(t\,\log\frac{1}{t}\right)^{-n/p}$. Indeed, by [IR, Corollary 2.3], see also [MRSY, Corollary 6.3, Ch. 6], we obtain

$$\int\limits_{\varepsilon<|y-y_0|<\varepsilon_0} Q(y)\cdot\psi^p(|y-y_0|)\,dm(y) = O\left(\log\log\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon\to 0 \tag{3.1}$$

and

$$I(\varepsilon,\varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt > \log\frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon_0}} > \log\log\frac{1}{\varepsilon} \tag{3.2}$$

for sufficiently small $\varepsilon_0>0$, and hence

$$\frac{1}{I^p(\varepsilon,\varepsilon_0)}\int\limits_{\varepsilon<|y-y_0|<\varepsilon_0} Q(y)\cdot\psi^p(|y-y_0|)\,dm(y)\leqslant C\left(\log\log\frac{1}{\varepsilon}\right)^{1-p}\to 0 \quad \text{as } \varepsilon\to 0.$$

The latter yields the desired conclusion for the case 1) because (2.2)–(2.3) hold.

Note that the case 2) is a consequence of case 3) and therefore, we may restrict ourselves by checking the condition 3). For this case we pick the function

$$\psi(t)=\begin{cases}1/[t^{\frac{n-1}{p-1}}q_{y_0}^{\frac{1}{p-1}}(t)], & t\in(\varepsilon,\varepsilon_0),\\ 0, & t\notin(\varepsilon,\varepsilon_0),\end{cases} \tag{3.3}$$

in Lemma 2.1 and thus,

$$I(\varepsilon,\varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dr}{r^{\frac{n-1}{p-1}}q_{y_0}^{\frac{1}{p-1}}(r)}. \tag{3.4}$$

Applying the Fubini theorem ([Sa, Theorem 8.1, Ch. III]), we obtain

$$\int\limits_{\varepsilon<|y-y_0|<\varepsilon_0} Q(y)\cdot\psi^p(|y-y_0|)\,dm(y)=\omega_{n-1}\cdot I(\varepsilon,\varepsilon_0)=o(I^p(\varepsilon,\varepsilon_0)) \quad \text{as } \varepsilon\to 0,$$

where ω_{n-1} denotes the area of the unit sphere $\mathbf{S}^{n-1}$ in $\mathbf{R}^n$. □

4. Corollaries

The following statements can be derived from Lemma 2.1.

Corollary 4.1. *Let $p\in(n-1,n]$, and $Q\colon\mathbf{R}^n\to(0,\infty)$ be a Lebesgue measurable function. Assume that $f\colon D\to\mathbf{R}^n$ is a sense-preserving mapping satisfying (1.3) for every $y_0\in f(D)$, any $0<r_1<r_2<\infty$, and any nonnegative Lebesgue measurable function $\eta\colon(r_1,r_2)\to[0,\infty]$ obeying (1.4). Then f is discrete and open whenever the function Q satisfies the both conditions:*

1) $\int\limits_{\varepsilon}^{\delta_0}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}<\infty$ *for every $y_0\in f(D)$, some $\delta_0=\delta_0(y_0)$ and sufficiently small $\varepsilon>0$,*
2) $\int\limits_{0}^{\delta_0}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}=\infty$ *for every $y_0\in f(D)$ and some $\delta_0=\delta_0(y_0)$.*

Proof. Arguing similarly to the proof of the case 3) in Theorem 1.1 and choosing

$$\psi(t)=\begin{cases}\left(1/[tq_{y_0}^{\frac{1}{n-1}}(t)]\right)^{n/p}, & t\in(\varepsilon,\varepsilon_0),\\ 0, & t\notin(\varepsilon,\varepsilon_0),\end{cases}$$

in Lemma 2.1, we obtain the desired conclusion. □

Corollary 4.2. *Let $p\in(n-1,n)$, and $Q\colon \mathbf{R}^n\to(0,\infty)$ be a Lebesgue measurable function. Assume that $f\colon D\to\mathbf{R}^n$ is a sense-preserving mapping satisfying (1.3) for every $y_0\in f(D)$, any $0<r_1<r_2<\infty$, and any nonnegative Lebesgue measurable function $\eta\colon(r_1,r_2)\to[0,\infty]$ obeying (1.4). Then f is discrete and open whenever $Q\in L^s_{\rm loc}$ for some $s\geqslant n/(n-p)$.*

Proof. Given $\varepsilon_0\in(0,\infty)$ and $x_0\in D$, set $G:=B(x_0,\varepsilon_0)$. Note that the function $\psi(t):=1/t$ satisfies (2.3). Thus, in order to apply Lemma 2.1 it remains to verify (2.2). Indeed, by the Hölder inequality we obtain

$$\begin{aligned}&\int_{\varepsilon<|x-x_0|<\varepsilon_0}\frac{Q(x)}{|x-x_0|^p}\,dm(x)\\ &\leqslant\left(\int_{\varepsilon<|x-x_0|<\varepsilon_0}\frac{1}{|x-x_0|^{pq}}\,dm(x)\right)^{\frac{1}{q}}\left(\int_G Q^{q'}(x)\,dm(x)\right)^{\frac{1}{q'}}\end{aligned}\tag{4.1}$$

where $q=n/p$, $1/q+1/q'=1$, i.e., $q'=n/(n-p)$. Observe that the first integral in the right-hand side of (4.1) can be implicitly calculated. Namely, by the Fubini theorem

$$\int_{\varepsilon<|x-x_0|<\varepsilon_0}\frac{1}{|x-x_0|^{pq}}\,dm(x)=\omega_{n-1}\int_{\varepsilon}^{\varepsilon_0}\frac{dt}{t}=\omega_{n-1}\log\frac{\varepsilon_0}{\varepsilon}.$$

Following notation of Lemma 2.1, we have

$$\frac{1}{I^p(\varepsilon,\varepsilon_0)}\int_{\varepsilon<|x-x_0|<\varepsilon_0}\frac{Q(x)}{|x-x_0|^p}\,dm(x)\leqslant\omega_{n-1}^{\frac{p}{n}}\|Q\|_{L^{\frac{n}{n-p}}(G)}\left(\log\frac{\varepsilon_0}{\varepsilon}\right)^{-p+\frac{p}{n}}\to 0\ \text{ as }\varepsilon\to 0$$

that implies (2.2). Thus, the desired conclusion follows from Lemma 2.1. □

5. Examples

First of all, let us give some examples of mappings obeying (1.1) and (1.3). It is known that, for an arbitrary quasiregular mapping $f\colon D\to\mathbf{R}^n$, one has

$$M(\Gamma)\le N(f,A)K_O(f)M(f(\Gamma))\tag{5.1}$$

for a constant $K_O(f)\geqslant 1$, for any Borel set A in the domain D such that $N(f,A)<\infty$ and any family Γ of curves γ in A (see [MRV, Theorem 3.2] or [Ri, Theorem 6.7, Chap. II]). Thus, for any such quasiregular mapping, the inequalities (1.1) and (1.3) hold.

Let us give other examples. Set at points $x\in D$ of differentiability of f

$$\|f'(x)\|=\max_{h\in\mathbf{R}^n\setminus\{0\}}\frac{|f'(x)h|}{|h|},\quad J(x,f)=\det f'(x),$$

and define for any $x \in D$ and $p \geqslant 1$

$$K_{O,p}(x,f) = \begin{cases} \frac{\|f'(x)\|^p}{|J(x,f)|}, & J(x,f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

We say that a property P holds for *p-almost every* (p-*a.e.*) curves γ in a family Γ if the subfamily of all curves in Γ, for which P fails, has p-modulus zero. Recall that a mapping $f\colon D \to \mathbf{R}^n$ is said to have *N-property* (*by Luzin*) if $m\,(f\,(S)) = 0$ whenever $m(S) = 0$ for $S \subset \mathbf{R}^n$. Similarly, f has the *N^{-1}-property* if $m\,(f^{-1}(S)) = 0$ whenever $m(S) = 0$.

If $\gamma\colon \Delta \to \mathbf{R}^n$ is a locally rectifiable curve, then there is the unique nondecreasing length function l_γ of Δ onto a length interval $\Delta_\gamma \subset \mathbf{R}$ with a prescribed normalization $l_\gamma(t_0) = 0 \in \Delta_\gamma$, $t_0 \in \Delta$, such that $l_\gamma(t)$ is equal to the length of the subcurve $\gamma|_{[t_0,t]}$ of γ if $t > t_0$, $t \in \Delta$, and $l_\gamma(t)$ is equal to minus length of $\gamma|_{[t,t_0]}$ if $t < t_0$, $t \in \Delta$. Let $g\colon |\gamma| \to \mathbf{R}^n$ be a continuous mapping, and suppose that the curve $\widetilde{\gamma} = g \circ \gamma$ is also locally rectifiable. Then there is a unique non–decreasing function $L_{\gamma,g}\colon \Delta_\gamma \to \Delta_{\widetilde{\gamma}}$ such that $L_{\gamma,g}\,(l_\gamma(t)) = l_{\widetilde{\gamma}}(t)$ for all $t \in \Delta$. We say that a mapping $f\colon D \to \mathbf{R}^n$ is *absolutely continuous on paths with respect to p-modulus,* write $f \in ACP_p$, if for p-a.e. curve $\gamma\colon \Delta \to D$ the function $L_{\gamma,f}$ is locally absolutely continuous on Δ.

The following result is a generalization of the above classical inequality (5.1) for quasiregular mappings and provides an example of the inequality (1.3) (see [Ri, Theorem 2.4, Ch. II]).

Theorem 5.1. *Let a mapping $f\colon D \to \mathbf{R}^n$ be differentiable a.e. in D, have N- and N^{-1}-properties and possess ACP_p-property for some $p \geqslant 1$. Let A be a Borel set in D, and let Γ be a family of paths in A. Suppose that $K_{O,p}(x,f) \leqslant \alpha(f(x))$ a.e. for a Borel function $\alpha\colon \mathbf{R}^n \to [0,\infty]$. Then*

$$M_p(\Gamma) \leq \int_{\mathbf{R}^n} \rho'^p(y) N(y,f,A)\alpha(y)\, dm(y) \quad \forall\, \rho' \in \operatorname{adm} f(\Gamma).$$

Recall that $N(y,f,A)$ is Lebesgue measurable for any Borel measurable set A (see [RR, Theorem of Section IV.1.2]).

Proof. Let $\rho' \in \operatorname{adm} f(\Gamma)$. Set $\rho(x) = \rho'(f(x))\|f'(x)\|$ for $x \in A$ and $\rho(x) = 0$ otherwise. Let Γ_0 be a family of all locally rectifiable curves of Γ where f is locally absolutely continuous. Since $f \in ACP_p$, $M_p(\Gamma) = M_p(\Gamma_0)$. Now, by [Ri, Lemma 2.2, Ch. II], $\int_\gamma \rho(x)|dx| = \int_\gamma \rho'(f(x))\|f'(x)\||dx| \geq \int_{f\circ\gamma} \rho'(y)|dy| \geq 1$, and, consequently, $\rho \in \operatorname{adm}\Gamma_0$. By the change of variables formula

$$\begin{aligned} M_p(\Gamma) = M_p(\Gamma_0) &\leq \int_{\mathbf{R}^n} \rho^p(x)\, dm(x) = \int_A \frac{\rho'^p(f(x))\|f'(x)\|^p|J(x,f)|}{|J(x,f)|}\, dm(x) \\ &\leq \int_A \rho'^p(f(x))\alpha(f(x))|J(x,f)|\, dm(x) = \int_{\mathbf{R}^n} \rho'^p(y)N(y,f,A)\alpha(y)\, dm(y), \end{aligned}$$

see [MRSY, Proposition 8.3]. Here we take into account, that $J(x,f) \neq 0$ a.e., see [MRSY, Proposition 8.3]. The theorem is proved. □

By Theorem 5.1 we obtain the following

Corollary 5.1. *Let $f\colon D \to \mathbf{R}^n$ be differentiable a.e. in D, have N- and N^{-1}-properties, and $f \in W^{1,p}_{\mathrm{loc}}$ for some $p \geqslant 1$. Let A in D be a Borel set and Γ be a family of paths in A. Suppose that $K_{O,p}(x,f) \leqslant \alpha(f(x))$ a.e. for a Borel function*

$\alpha\colon \mathbf{R}^n \to [0,\infty]$. *Then*

$$M_p(\Gamma) \le \int_{\mathbf{R}^n} \rho'^p(y) N(y,f,A)\alpha(y)\, dm(y) \quad \text{for all } \rho' \in \operatorname{adm} f(\Gamma).$$

Indeed, as known, $W^{1,p}_{\rm loc} = ACL^p$ (see [Maz, Theorems 1 and 2, section 1.1.3]). On the other hand, $ACL^p \subset ACP_p$ by the Fuglede lemma (see [Va, Theorem 28.2]). Thus, by Theorem 5.1 we obtain the inequality of type (1.3). □

The next statement follows from Theorems 1.1 and 5.1, Corollaries 4.1 and 4.2.

Corollary 5.2. *Let $p \in (n-1,n]$, $f\colon D \to \mathbf{R}^n$ be differentiable a.e. in D, have N- and N^{-1}-properties, ACP_p-property for some $p \geqslant 1$ and $K_{O,p}(x,f) \leqslant \alpha(f(x))$ a.e. for a Borel function $\alpha\colon \mathbf{R}^n \to [0,\infty]$. Let Γ be a family of paths in a Borel set $A \subseteq D$ and let $Q(y) = N(y,f,D)\cdot \max\{\alpha(y),1\}$. Then f is open and discrete whenever Q satisfies at least one of the following conditions:*

1) *$Q \in FMO(y_0)$ for every $y_0 \in f(D)$;*
2) *for every $y_0 \in f(D)$ $q_{y_0}(r) = O\left(\left[\log \frac{1}{r}\right]^{n-1}\right)$ as $r \to 0$, where $q_{y_0}(r)$ is defined in (1.5);*
3) *for all $y_0 \in f(D)$ there exists $\delta > 0$: for all $\varepsilon \in (0,\delta)$: $\displaystyle\int_\varepsilon^\delta \frac{dt}{t^{\frac{n-1}{p-1}} q_{y_0}^{\frac{1}{p-1}}(t)} < \infty$ and $\displaystyle\int_0^\delta \frac{dt}{t^{\frac{n-1}{p-1}} q_{y_0}^{\frac{1}{p-1}}(t)} = \infty$;*
4) *for all $y_0 \in f(D)$ there exists $\delta > 0$: for all $\varepsilon \in (0,\delta)$: $\displaystyle\int_\varepsilon^\delta \frac{dt}{t q_{y_0}^{\frac{1}{n-1}}(t)} < \infty$ and $\displaystyle\int_0^\delta \frac{dt}{t q_{y_0}^{\frac{1}{n-1}}(t)} = \infty$;*
5) *$p \in (n-1,n)$ and $Q \in L^s_{\rm loc}(\mathbf{R}^n)$ for some $s \geqslant \frac{n}{n-p}$.*

In particular, the assertion of Corollary 5.2 holds if $f \in W^{1,p}_{\rm loc}$ instead of $f \in ACP_p$.

Note that the preserving orientation is essential condition for mappings f in all statements given above. An example of a mapping f with finite length distortion that does not preserve orientation and such that $M(f(\Gamma)) = M(\Gamma)$, i.e., $Q \equiv 1$, but is neither discrete nor open, was given in [MRSY, Section 8.10].

We also give another simple example which shows that the preserving orientation can not be dropped. Let $x = (x_1,\ldots,x_n)$. We define f as the identical mapping in the closed domain $\{x_n \geqslant 0\}$ and set $f(x) = (x_1,\ldots,-x_n)$ for $x_n < 0$. Note that the mapping f preserves the lengths of paths. Therefore, f satisfies the inequality (1.3) with $Q \equiv 1$. This mapping is discrete but not open. In fact, under the mapping f the ball $\mathbf{B}^n$ is mapped onto the set $\{y = (y_1,\ldots,y_n) \in \mathbf{R}^n\colon |y| < 1, y_n \geqslant 0\}$ which is not open in $\mathbf{R}^n$.

Remark 5.1. Results obtained in the paper can be applied to various classes of plane and space mappings (see e.g. [GRSY] and [MRSY]).

Note that we discuss the case while p ranges between $n-1$ and n. The question on discreteness and openness of the mappings obeying the same modulus conditions for $1 \le p \le n-1$ remains open.

References

[AS] Adamowicz, T., and N. Shanmugalingam: Non-conformal Loewner type estimates for modulus of curve families. - Ann. Acad. Sci. Fenn. Math. 35, 2010, 609–626.

[AC] Andreian Cazacu, C.: On the length-area dilatation. - Complex Var. Theory Appl. 50, 2005, 765–776.

[Cr_1] Cristea, M.: Local homeomorphisms having local ACL^n inverses. - Complex Var. Elliptic Equ. 53:1, 2008, 77–99.

[Cr_2] Cristea, M.: Open discrete mappings having local ACL^n inverses. - Complex Var. Elliptic Equ. 55:1-3, 2010, 61–90.

[Gol_1] Golberg, A.: Homeomorphisms with finite mean dilatations. - Contemp. Math. 382, 2005, 177–186.

[Gol_2] Golberg, A.: Differential properties of (α, Q)-homeomorphisms. - In: Further Progress in Analysis, World Scientific Publ., 2009, 218–228.

[GRSY] Gutlyanskii, V. Ya., V. I. Ryazanov, U. Srebro, and E. Yakubov: The Beltrami equation: A geometric approach. - Dev. Math. 26, New York etc., Springer, 2012.

[He] Heinonen, J.: Lectures on analysis on metric spaces. - Springer, New York, 2001.

[HK] Hencl, S., and P. Koskela: Lectures on mappings of finite distortion. - Lecture Notes in Math. 2096, Springer, 2014.

[HW] Hurewicz, W., and H. Wallman: Dimension theory. - Princeton Univ. Press, 1948.

[IR] Ignat'ev, A., and V. Ryazanov: Finite mean oscillation in the mapping theory. - Ukr. Math. Bull. 2:3, 2005, 403–424.

[IM] Iwaniec, T., and G. Martin: Geometric function theory and non-linear analysis. - Clarendon Press, Oxford, 2001.

[KO] Koskela, P., and J. Onninen: Mappings of finite distortion: Capacity and modulus inequalities. - J. Reine Angew. Math. 599, 2006, 1–26.

[MRV] Martio, O., S. Rickman, and J. Väisälä: Definitions for quasiregular mappings. Ann. Acad. Sci. Fenn. Ser. A I Math. 448, 1969, 1–40.

[MRSY] Martio, O., V. Ryazanov, U. Srebro, and E. Yakubov: Moduli in modern mapping theory. - Springer, New York, 2009.

[Maz] Maz'ya, V.: Sobolev spaces. - Springer-Verlag, Berlin, 1985.

[Na] Näkki, R.: Boundary behavior of quasiconformal mappings in n-space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 484, 1970, 1–50.

[RR] Rado, T., and P V. Reichelderfer: Continuous transformations in analysis. - Springer-Verlag, Berlin etc., 1955.

[Re] Reshetnyak, Yu. G.: Space mappings with bounded distortion. English transl. - Amer. Math. Soc., Providence, RI, 1989.

[Ri] Rickman, S.: Quasiregular mappings. - Springer-Verlag, Berlin etc., 1993.

[Sa] Saks, S.: Theory of the integral. - Dover Publ. Inc., New York, 1964.

[Sev] Sevost'yanov, E. A.: On the openess and discreteness of mappings with unbounded characteristic of quasiconormality. - Ukrainian Math. J. 63:8, 2012, 1298–1305.

[TY] Titus, C. J., and G. S. Young: The extension of interiority with some applications. - Trans. Amer. Math. Soc. 103, 1962, 329–340.

[UV] Ukhlov, A. D., and S. K. Vodop'yanov: Sobolev spaces and mappings with bounded $(P; Q)$-distortion on Carnot groups. - Bull. Sci. Mat. 54:4, 2009, 349–370.

[Va] Väisälä, J.: Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin etc., 1971.

Received 26 January 2015 • Accepted 8 May 2015