Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 881–896

ON THE EXTENSION OF QUASISYMMETRIC MAPS

Pekka Alestalo and Dmitry Alexandrovich Trotsenko

Aalto University School of Science, Department of Mathematics and Systems Analysis
PL 11100, FI-00076 Aalto, Finland; pekka.alestalo@aalto.fi

Sobolev Institute of Mathematics
Acad. Koptyug Avenue 4, 630090 Novosibirsk, Russia, and
Novosibirsk State University
Pirogova Street 2, 630090 Novosibirsk, Russia; trotsenk@yandex.ru

Abstract. We show that an ε-power-quasisymmetric map $f\colon A \to \mathbf{R}^n$ can be extended to a $C\varepsilon$-power-quasisymmetric map $F\colon \mathbf{R}^n \to \mathbf{R}^n$ if $A \subset \mathbf{R}^n$ satisfies a geometric thickness condition and ε is small enough. The constant C depends on c and n only.

1. Introduction

Let $A \subset \mathbf{R}^n$ and let $f\colon A \to \mathbf{R}^n$ be a mapping. A very general question is: Can f be extended to a function $F\colon \mathbf{R}^n \to \mathbf{R}^n$ having similar properties as the original function f? This question has been studied and solved in many different cases. Classical results include the extension theorems for continuous (Tietze; or Brouwer, Lebesgue in this setting) and smooth (Whitney, $n = 1$) functions f. In particular, the general Whitney extension problem "$F \in C^m(\mathbf{R}^n)$?" has been completely solved by Fefferman and his collaborators; cf. the Introduction and references in [DF].

However, in the case of continuous and injective f there are still many open problems, if the extension is required to be a homeomorphism. Sometimes, extensions may not exist for topological reasons, but the most interesting cases arise when the (geo)metric properties of A play a crucial role. Several results in the positive direction can be found, for example, in [AH, Jo, MS, Pa, PV, Re, Tr1, Vä]. The last reference also contains some basic counterexamples related to the quantitative properties of extension.

The present authors have studied this problem for $(1+\varepsilon)$-bilipschitz maps, and we present a similar result for quasisymmetric maps in this article. More precisely, it was proved by the authors and Väisälä in [ATV1] that $(1+\varepsilon)$-bilipschitz maps $f\colon A \to \mathbf{R}^n$ can be well approximated by isometries if the set A satisfies a geometric condition related to its thickness. This result was applied in [ATV2] to show that, under similar conditions, the map f has a $(1+C\varepsilon)$-bilipschitz extension $F\colon \mathbf{R}^n \to \mathbf{R}^n$. In [AT1], the present authors gave a geometric characterization for plane sets having this linear bilipschitz extension property. Note that here *linear* refers to the linear growth of the error term $C\varepsilon$, which is optimal.

Before stating our main theorem, we recall the definition of a quasisymmetric map.

doi:10.5186/aasfm.2016.4154
2010 Mathematics Subject Classification: Primary 30C65, 46B20.
Key words: Power quasisymmetric, extension, sturdy.
Acknowledgements: This cooperation was supported by the Vilho, Yrjö and Kalle Väisälä Fund, the Academy of Finland, and the Sobolev Institute of Mathematics.

Definition 1.1. Let X, Y be metric spaces with distance written as $|a-b|$, and let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism, called a growth function. An embedding $f\colon X \to Y$ is η-quasisymmetric (abbr. η-QS) if

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \le \eta\left(\frac{|x-y|}{|x-z|}\right)$$

for all $x, y, z \in X$ satisfying $x \neq z$. A function f is quasisymmetric if it is η-quasisymmetric with some growth function η.

Remark 1.2. An L-bilipschitz map is η-quasisymmetric with $\eta(t) = L^2 t$. Conversely, if the growth function η is of the form $\eta(t) = Ct$ for some constant C, then the corresponding map $f\colon X \to Y$ is a composition of a similarity and a bilipschitz map; see [TuV, 2.3].

It follows from [TuV, 3.12] and [TrV, 6.5] that every growth function η can be replaced by a power form

$$\eta(t) = C\left(t^{\alpha} \vee t^{1/\alpha}\right)$$

if the set X is relatively connected, a class of spaces containing all connected ones, and even self-similar Cantor sets. Furthermore, this power-quasisymmetry property was completely characterized in [TrV]. In the present paper, we shall use these maps in the following case.

Definition 1.3. Let $\varepsilon > 0$. An embedding $f\colon X \to Y$ is *ε-power-quasisymmetric* if it is η-QS with

$$\eta(t) = (1+\varepsilon)\left(t^{1+\varepsilon} \vee t^{1/(1+\varepsilon)}\right).$$

Examples of mappings satisfying this condition include quasiconformal maps with a small dilatation. In particular, suitable radial stretching maps provide examples that are not bilipschitz.

Another concept, called s-quasisymmetry, can also be used to study how close a mapping is to a similarity. We refer to [AT2] for details. However, it turns out that s-QS mappings are not suitable for extension results with sharp linear bounds, as in the following main theorem of the present article. See 2.2 for the definition of sturdiness.

Theorem 1.4. *Let $A \subset \mathbf{R}^n$ be c-sturdy and let $f\colon A \to \mathbf{R}^n$ be ε-power-QS with $0 < \varepsilon \le \delta(c,n)$. Then f has a $C\varepsilon$-power-QS extension $F\colon \mathbf{R}^n \to \mathbf{R}^n$, where $C = C(c,n)$.*

2. Preliminary results

Our notation is standard and the same as in [AT1]. However, we recall the abbreviation $A(a,r) = A \cap \overline{B}(a,r)$ for a subset $A \subset \mathbf{R}^n$ and the following geometric properties of sets that are needed in our main result.

Definition 2.1. Thickness. For each unit vector $e \in \mathbf{S}^{n-1}$ we define the projection $\pi_e\colon \mathbf{R}^n \to \mathbf{R}$ by $\pi_e x = x \cdot e$. Let $A \neq \emptyset$ be a bounded set in $\mathbf{R}^n$, and let $d(A)$ denote the diameter of A. The *thickness* of A is the number

$$\theta(A) = \inf\{d(\pi_e A)\colon e \in \mathbf{S}^{n-1}\}.$$

Alternatively, $\theta(A)$ is the infimum of all $t > 0$ such that A lies between two parallel hyperplanes F, F' with mutual distance $d(F,F') = t$. We always have $0 \le \theta(A) \le d(A)$.

Definition 2.2. Sturdiness. Let $A \subset \mathbf{R}^n$. For $a \in A$ we set $s(a) = s_A(a) = d(a, A \setminus \{a\})$, the distance from a to the rest of A. Then $s(a) > 0$ if and only if a is isolated in A. Let $c \geq 1$. We say that the set $A \subset \mathbf{R}^n$ is *c-sturdy* if

(1) $\theta(A(a,r)) \geq 2r/c$ whenever $a \in A,\ r \geq cs(a),\ A \not\subset B(a,r)$,
(2) $\theta(A) \geq d(A)/c$.

If A is unbounded, we omit (2), and the condition $A \not\subset B(a,r)$ of (1) is unnecessary.

Examples of sturdy sets in the plane include bounded Lipschitz-domains, $\mathbf{Z}^2$, and the snowflake curve. We recall the definition of a nearisometry from [ATV1, 1.1].

Definition 2.3. Let X and Y be metric spaces, let $f\colon X \to Y$, and let $\varepsilon > 0$. We say that f is an *ε-nearisometry* if

$$|x-y| - \varepsilon \leq |f(x) - f(y)| \leq |x-y| + \varepsilon$$

for all $x, y \in X$.

To shorten notation, we let a proper triple T in a metric space X consist of points $T = (x; y, z)$ such that $y \neq x \neq z$, and define the ratio of T as

$$|T| = \frac{|x-y|}{|x-z|}.$$

An injective map $f\colon X \to Y$ maps each proper triple $T = (x; y, z)$ in X to another proper triple $T' = (f(x); f(y), f(z))$ in Y.

We start with a couple of inequalities.

Lemma 2.4. *Let $0 < x \leq 1 \leq y$ and let $0 \leq \varepsilon \leq 1$. Then*

a) $x^{-\varepsilon} \leq 1 + \varepsilon(1-x)/x \leq 1 + \varepsilon/x$;
b) $y^{\varepsilon} \leq 1 + \varepsilon(y-1)$.

Proof. a) The mean value theorem, applied to $t \mapsto t^{\varepsilon}$ gives

$$1 - x^{\varepsilon} = \varepsilon t_1^{\varepsilon-1}(1-x) \leq \varepsilon x^{\varepsilon-1}(1-x),$$

since $t_1 \geq x$. Therefore

$$x^{-\varepsilon} - 1 = \frac{1-x^{\varepsilon}}{x^{\varepsilon}} \leq \frac{\varepsilon(1-x)}{x},$$

which implies the first inequality, and the second follows trivially.

b) This follows from the first part by substituting $x = 1/y$. □

Lemma 2.5. *Let $a, b > 0$. Then*

$$\frac{a+t}{b-t} \leq \frac{a}{b} + 2t\frac{a+b}{b^2}$$

for $0 \leq t \leq b/2$. Moreover, if $a/b \leq 1$ and $c = 4(a+b)/ab$, then

$$\frac{a+t}{b-t} \leq (1+ct)\left(\frac{a}{b}\right)^{1/(1+ct)}$$

for $0 \leq t \leq b/2$.

Proof. Let $g(t) = (a+t)/(b-t)$ and $h(t) = a/b + 2t(a+b)/b^2$. Then $g''(t) - h''(t) = 2(a+b)/(b-t)^3 \geq 0$, so that $g - h$ is convex. Since $g(0) = h(0)$ and $g(b/2) = h(b/2)$, the first inequality follows.

For the second one, we estimate

$$g'(t) = \frac{a+b}{(b-t)^2} \leq \frac{4(a+b)}{b^2}$$

for $t \le b/2$. On the other hand, for $f(t) = (1+ct)(a/b)^{1/(1+ct)}$ we obtain

$$\begin{aligned} f'(t) &= \frac{4(a+b)(ab+4t(a+b)+ab\ln(b/a)}{ab(ab+4t(a+b))}\left(\frac{a}{b}\right)^{ab/(ab+4t(a+b))} \\ &\ge \frac{4(a+b)}{ab}\cdot\frac{a}{b} = \frac{4(a+b)}{b^2} \ge g'(t), \end{aligned}$$

since the logarithmic term is nonnegative, the exponent is at most 1, and $a/b \le 1$. As $f(0) = g(0) = a/b$, we obtain $g(t) \le f(t)$ for $0 \le t \le b/2$, and the claim is proved. □

Lemma 2.6. *Let $0 \le \varepsilon \le 1$ and let $f\colon X \to Y$ be ε-power-QS. Suppose that there exist points $a, b \in X$ such that $|a-b| = d(X) = 1$ and $|f(a)-f(b)| = 1$. Then f is a 23ε-nearisometry.*

Proof. Let $x, y \in X$, $x \ne y$. To prove the nearisometry condition, we may assume that $|x-a| \ge 1/2$.

Let $T_1 = (x; y, a)$ and $T_2 = (a; x, b)$. Then $|x-y| = |T_1||T_2|$, $|f(x)-f(y)| = |T_1'||T_2'|$,

$$|T_1| = \frac{|x-y|}{|x-a|} \le 2 \quad \text{and} \quad |T_2| = \frac{|x-a|}{|a-b|} \le 1.$$

We shall obtain the upper bound for $|f(x)-f(y)|$ by considering Cases 1 and 2 below, and the lower bound in Cases 3 and 4 after that.

Case 1. $|T_1| \le 1$. Now we have

$$\begin{aligned} |f(x)-f(y)| &\le (1+\varepsilon)^2|T_1|^{1/(1+\varepsilon)}|T_2|^{1/(1+\varepsilon)} \le (1+3\varepsilon)(|T_1||T_2|)^{1-\varepsilon} \\ &= (1+3\varepsilon)|x-y||x-y|^{-\varepsilon} \le (1+3\varepsilon)(|x-y|+\varepsilon) \le |x-y|+7\varepsilon \end{aligned}$$

by 2.4.a.

Case 2. $1 \le |T_1| \le 2$. Now $|T_1|^{2\varepsilon} \le 2\varepsilon(|T_1|-1)+1 \le 2\varepsilon+1$ by 2.4.b, and we obtain

$$\begin{aligned} |f(x)-f(y)| &\le (1+\varepsilon)^2|T_1|^{1+\varepsilon}|T_2|^{1/(1+\varepsilon)} \le (1+3\varepsilon)|T_1|^{1+\varepsilon}|T_2|^{1-\varepsilon} \\ &= (1+3\varepsilon)|T_1|^{2\varepsilon}(|T_1||T_2|)^{1-\varepsilon} \\ &\le (1+3\varepsilon)(2\varepsilon+1)(|x-y|+\varepsilon) \le |x-y|+23\varepsilon, \end{aligned}$$

using 2.4.a again.

We have shown that $|f(x)-f(y)| \le |x-y|+23\varepsilon$ in both cases. It remains to show that $|f(x)-f(y)| \ge |x-y|-23\varepsilon$ as well.

Let $T_3 = (x; a, y)$ and $T_4 = (a; b, x)$. Then $|T_3||T_4| = 1/|x-y|$, $|T_3'||T_4'| = 1/|f(x)-f(y)|$,

$$|T_3| = \frac{|x-a|}{|x-y|} \ge \frac{1}{2} \quad \text{and} \quad |T_4| = \frac{|a-b|}{|a-x|} \in [1,2].$$

Case 3. $|T_3| \le 1$. Now $(1-|T_3|)/|T_3| \le 2$ and $|T_4|-1 \le 1$, and we get

$$\begin{aligned} \frac{1}{|f(x)-f(y)|} &= |T_3'||T_4'| \le (1+\varepsilon)^2|T_3|^{1-\varepsilon}|T_4|^{1+\varepsilon} \\ &\le \frac{1+3\varepsilon}{|x-y|}|T_3|^{-\varepsilon}|T_4|^{\varepsilon} \le \frac{1+3\varepsilon}{|x-y|}(1+2\varepsilon)(1+\varepsilon) \le \frac{1+23\varepsilon}{|x-y|} \end{aligned}$$

by 2.4. Thus

$$|f(x)-f(y)| \geq \frac{|x-y|}{1+23\varepsilon} \geq (1-23\varepsilon)|x-y| \geq |x-y|-23\varepsilon,$$

since $|x-y| \leq 1$.

Case 4. $|T_3| \geq 1$. Now $|T_3||T_4| \geq 1$, so that using 2.4.b, we get

$$\begin{aligned}\frac{1}{|f(x)-f(y)|} &\leq (1+\varepsilon)^2|T_3|^{1+\varepsilon}|T_4|^{1+\varepsilon} \leq \frac{1+3\varepsilon}{|x-y|}(1+\varepsilon(|T_3||T_4|-1)) \\ &\leq \frac{1+3\varepsilon}{|x-y|}\left(1+\frac{\varepsilon}{|x-y|}\right).\end{aligned}$$

Thus

$$\begin{aligned}|f(x)-f(y)| &\geq \frac{|x-y|}{(1+3\varepsilon)(1+\varepsilon/|x-y|)} \geq \frac{|x-y|}{1+7\varepsilon/|x-y|} \\ &\geq |x-y|\left(1-\frac{7\varepsilon}{|x-y|}\right) = |x-y|-7\varepsilon.\end{aligned}$$

This completes the proof for the lower bound, and the lemma is proved. □

3. Approximation by similarities

In this section we go through some preliminary results related to the question: How to approximate power-QS maps by similarities? The approximating similarities will be the main tool in constructing the extension needed for our main theorem.

Definition 3.1. For a similarity $S\colon \mathbf{R}^n \to \mathbf{R}^n$ let $\|S\|$ denote its similarity ratio.

Since S is affine, this is also the norm of the corresponding linear transformation. When approximating a function $f\colon A \to \mathbf{R}^n$ with a similarity, we employ two equivalent ways to express the error of approximation:

$$\|f-S\|_A = \sup\{|f(x)-Sx| \mid x \in A\} \leq C\|S\|\varepsilon \iff \|S^{-1}\circ f - \mathrm{id}\|_A \leq C\varepsilon.$$

Theorem 3.2. *Let $A \subset \mathbf{R}^n$ be compact and let $f\colon A \to l_2$ be ε-power-QS. Then there is a surjective similarity $S\colon l_2 \to l_2$ such that $\|S^{-1}\circ f - \mathrm{id}\|_A \leq c_n d(A)\sqrt{\varepsilon}$. Also, we can choose S so that $S\mathbf{R}^n = \mathbf{R}^n$.*

Proof. Choose points $a, b \in A$ such that $|a-b| = d(A)$ and let $A_0 = A/d(A)$, $M = |f(a)-f(b)|$. The map $g\colon A_0 \to l_2$, defined by $g(x) = f(d(A)x)/M$, is ε-power-QS. From Lemma 2.6 it follows that g is a 23ε-nearisometry. By Theorem [ATV1, 2.2], there is a surjective isometry $S_0\colon l_2 \to l_2$ satisfying $\|S_0 - g\|_{A_0} \leq c_n\sqrt{23\varepsilon}$. Setting $S^{-1}x = d(A)S_0^{-1}(x/M)$ we obtain a surjective similarity $S\colon l_2 \to l_2$ with $\|S^{-1}\circ f - \mathrm{id}\|_A \leq c_n\sqrt{23\varepsilon}d(A)$. □

In a similar way, using [ATV1, 3.3] instead of [ATV1, 2.2], we obtain the following result.

Theorem 3.3. *Suppose that $c \geq 1$ and $A \subset \mathbf{R}^n$ is a compact set such that $\theta(A) \geq d(A)/c$. Let $f\colon A \to \mathbf{R}^n$ be ε-power-QS with $\varepsilon \leq 1$. Then there is a similarity $S\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\|S^{-1}\circ f - \mathrm{id}\|_A \leq c_n c\varepsilon d(A)$.* □

For easy reference, we note the following corollary to the preceding theorem.

Corollary 3.4. *Let $A \subset \mathbf{R}^n$ be c-sturdy and let $f\colon A \to \mathbf{R}^n$ be ε-power-QS. Then for all $a \in A$ and $0 < r \leq d(A)$, there is a similarity $S = S_{a,r}$ such that $Sa = f(a)$ and $\|S - f\|_{A(a,r)} \leq c_1(c,n)\|S\|\varepsilon r$.*

Proof. Since A is c-sturdy, we have

$$\theta(A(a,r)) \geq 2r/c \geq d(A(a,r))/c.$$

Thus 3.3 gives a similarity S_1 satisfying

$$\|S_1 - f\|_{A(a,r)} \leq 2\|S_1\| c c_n \varepsilon r.$$

We claim that $S = S_1 - S_1 a + f(a)$ is the required similarity. If $x \in A(a,r)$, then

$$|f(x) - Sx| \leq |f(x) - S_1 x| + |S_1 a - f(a)| \leq 4\|S_1\| c c_n \varepsilon r.$$

Since $\|S\| = \|S_1\|$, this proves the claim with $c_1(c,n) = 4cc_n$. □

Definition 3.5. A similarity S satisfying the properties of the preceeding corollary is called a c_1-*special similarity* for (f,a,r).

Lemma 3.6. *Let $X \subset \mathbf{R}^n$ be a bounded set with diameter $r = d(X) > 0$, and let $f \colon X \to \mathbf{R}^n$ be a map that can be approximated by similarities S_1 and S_2 so that*

$$|S_i^{-1} f(x) - x| \leq \varepsilon r$$

for all $x \in X$, $i = 1,2$. If $0 \leq \varepsilon \leq 1/16$, then the similarity ratios satisfy the double inequality

$$(1 - 8\varepsilon)\|S_2\| \leq \|S_2\|/(1+8\varepsilon) \leq \|S_1\| \leq (1+8\varepsilon)\|S_2\|.$$

Proof. Choose points $u, v \in A$ satisfying $|u - v| = r$, and write $x_i = S_i^{-1} f(u)$, $y_i = S_i^{-1} f(v)$. By assumption, we have $|x_i - u| \leq \varepsilon r$ and $|y_i - v| \leq \varepsilon r$ for $i = 1,2$. Now

$$\frac{\|S_2\|}{\|S_1\|} = \frac{\|S_1^{-1}\|}{\|S_2^{-1}\|} = \frac{|x_1 - y_1|}{|f(u) - f(v)|} \frac{|f(u) - f(v)|}{|x_2 - y_2|} = \frac{|x_1 - y_1|}{|x_2 - y_2|} \leq \frac{r + 2\varepsilon r}{r - 2\varepsilon r} \leq 1 + 8\varepsilon$$

by 2.5. This proves the second inequality, and the last one follows by interchanging S_1 and S_2. □

The following result was essentially proved by the second author in [Tr2], but in a somewhat different context. We therefore give a reformulation which is better suited for our needs, and a complete proof. See also [Vä, 3.9].

Theorem 3.7. *Let $0 \leq \varepsilon \leq 1/100$ and let $F \colon \mathbf{R}^n \to \mathbf{R}^n$ be a mapping having the following approximation property: For every $x \in \mathbf{R}^n$ and $r > 0$, there is a similarity $S = S_{x,r}$ of $\mathbf{R}^n$ satisfying*

$$\|S \circ F - \mathrm{id}\|_{\overline{B}(x,r)} \leq \varepsilon r.$$

Then f is 50ε-power-quasisymmetric.

Proof. Let $x, y, z \in \mathbf{R}^n$ be distinct points, and let $T = (x; y, z)$, $T' = (f(x); f(y), f(z))$. We shall prove that

$$(1 - 25\varepsilon)|T|^{1+12\varepsilon} \leq |T'| \leq (1 + 25\varepsilon)|T|^{1-16\varepsilon} \tag{3.1}$$

in the case where $|x - y| \leq |x - z|$, i.e. $|T| \leq 1$. Because of the double inequality (3.1), the case $|T| \geq 1$ follows by interchanging y and z, and using $1/(1+2a) \leq 1 - a \leq 1/(1+a)$.

We thus assume that $|x - y| \leq |x - z| = r$, and then $y, z \in \overline{B}(x,r)$. Let $B_i = \overline{B}(x, r/2^i)$ and choose an integer k such that

$$\frac{r}{2^{k+1}} < |x - y| \leq \frac{r}{2^k}.$$

For every $0 \le i < k$, there is a similarity S_i such that

$$\|S_i f - \mathrm{id}\|_{B_i} \le \frac{\varepsilon r}{2^i} \quad \text{and} \quad \|S_{i+1} f - \mathrm{id}\|_{B_{i+1}} \le \frac{\varepsilon r}{2^{i+1}}.$$

Thus $\|S_j f - \mathrm{id}\|_{B_{i+1}} \le 2\varepsilon \cdot r/2^{i+1} = \varepsilon d(B_{i+1})$ for $j = i$ and $j = i+1$. Applying lemma 3.6 to the successive balls B_i and B_{i+1}, it follows that

$$(1-8\varepsilon)^k \|S_0\| \le \|S_k\| \le (1+8\varepsilon)^k \|S_0\|.$$

By construction, we have $k \le \log_2(1/|T|) < k+1$. Therefore

$$(1+8\varepsilon)^k \le |T|^{-12\varepsilon},$$

using the inequality $(1+t)^{\log_2 s} \le s^{1.5t}$, which is valid for $s \ge 1$ and $0 \le t \le 1/2$. Similarly,

$$(1-8\varepsilon)^k \ge |T|^{16\varepsilon},$$

using the inequality $(1-t)^{\log_2 s} \ge s^{-2t}$, valid for $s \ge 1$ and $0 \le t \le 1/2$. Combining the above inequalities, we obtain the estimates

$$|T|^{16\varepsilon} \|S_0\| \le \|S_k\| \le |T|^{-12\varepsilon} \|S_0\|. \tag{3.2}$$

We are now ready to estimate $|f(x) - f(y)|$ and $|f(x) - f(z)|$ separately, starting from the first expression. Since $|S_k f(y) - y| \le \varepsilon r/2^k$, we first obtain

$$|x-y| - \frac{\varepsilon r}{2^k} \le |S_k f(y) - x| \le |x-y| + \frac{\varepsilon r}{2^k}.$$

Using this and $|S_k f(x) - x| \le \varepsilon r/2^k \le 2\varepsilon |x-y|$, we have

$$(1-4\varepsilon)|x-y| \le |S_k f(x) - S_k f(y)| \le (1+4\varepsilon)|x-y|.$$

Combining this with the equation

$$\|S_k\| = \frac{|S_k f(x) - S_k f(y)|}{|f(x) - f(y)|}$$

and with (3.2), we get

$$\frac{(1-4\varepsilon)|x-y|}{|T|^{-12\varepsilon}\|S_0\|} \le |f(x) - f(y)| \le \frac{(1+4\varepsilon)|x-y|}{|T|^{16\varepsilon}\|S_0\|}. \tag{3.3}$$

Next we estimate $|f(x) - f(z)|$. Using the approximation S_0 for both $f(x)$ and $f(z)$, we obtain

$$(1-2\varepsilon)|x-z| \le |S_0 f(x) - S_0 f(z)| \le (1+2\varepsilon)|x-z|.$$

From this we get the double inequality

$$\frac{(1-2\varepsilon)|x-z|}{|f(x) - f(z)|} \le \|S_0\| = \frac{|S_0 f(x) - S_0 f(z)|}{|f(x) - f(z)|} \le \frac{(1+2\varepsilon)|x-z|}{|f(x) - f(z)|}.$$

Combining these estimates for $\|S_0\|$ with (3.3), we obtain the double inequality

$$\frac{(1-4\varepsilon)|x-y|}{|T|^{-12\varepsilon}(1+2\varepsilon)|x-z|} \le \frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le \frac{(1+4\varepsilon)|x-y|}{|T|^{16\varepsilon}(1-2\varepsilon)|x-z|}.$$

From this, the estimates (3.1) easily follow, and the proof is complete. □

Definition 3.8. Suppose that $A \subset \mathbf{R}^n$, $a \in A$, $r > 0$, and $c \ge 1$. We say that an n-simplex Δ is *c-special* for (A, a, r), or briefly a *c-special simplex* of A, if

(1) $\Delta^0 \subset A(a, r)$, and
(2) the smallest height $b(\Delta)$ of Δ satisfies $b(\Delta) \ge r/c$.

Since $d(\Delta) \le 2r$, we have $\rho(\Delta) = d(\Delta)/b(\Delta) \le 2c$ for every c-special simplex Δ of A.

Lemma 3.9. *Let $A \subset \mathbf{R}^n$ be closed, unbounded, and c-sturdy. If $a \in A$ and $r \ge cs(a)$, then there is a c-special simplex for (A, a, r).*

Proof. See [ATV2, 3.6]. □

The following lemma is almost identical with [ATV2, 3.12], and we do not repeat the proof.

Lemma 3.10. *Let $A \subset \mathbf{R}^n$ be closed, unbounded, and c-sturdy, and let $f\colon A \to \mathbf{R}^n$ be $(1+\varepsilon)$-power-QS. Suppose that $a, b \in A$ and $r_1 \ge cs(a)$, $r_2 \ge cs(b)$. If S and T are c_1-special similarities for (f, a, r_1) and (f, b, r_2), respectively, then they have the same orientation, provided that $0 \le \varepsilon \le \delta(c, n)$.*

Lemma 3.11. *Let $\Delta \subset \mathbf{R}^n$ be an n-simplex, and let $S, T\colon \Delta \to \mathbf{R}^n$ be similarities such that $\|S - T\|_{\Delta^0} \le \eta$. Then $\big|\|S\| - \|T\|\big| \le 2\eta/d(\Delta)$ and*

$$|Sx - Tx| \le \eta(1 + M|x - v|/d(\Delta))$$

for all $x \in \Delta$ and $v \in \Delta^0$, where $M = 4 + 6n\rho(\Delta)(1 + \rho(\Delta))^{n-1}$.

Proof. See [Vä, 2.11]. □

4. Proof of the main theorem

The following result reduces the extension problem to the case of unbounded sturdy sets. This makes it easier to handle the definition of sturdiness, because Condition 2.2(2) can be omitted.

Theorem 4.1. *Suppose that all unbounded c-sturdy sets $A \subset \mathbf{R}^n$ have the following property: There is $\delta = \delta(c, n) > 0$ such that every ε-power-QS map $f\colon A \to \mathbf{R}^n$ with $0 \le \varepsilon \le \delta$ extends to a $C\varepsilon$-power-QS map $F : \mathbf{R}^n \to \mathbf{R}^n$, where $C = C(c, n)$. Then all c-sturdy sets $A \subset \mathbf{R}^n$ have the same property with δ replaced by $\delta' = \delta(6c, n)/34c_nc$ and C replaced by $C' = 34c_ncC(6c, n))$. Here c_n is the constant from 3.3.*

Proof. Suppose that $A \subset \mathbf{R}^n$ is bounded and c-sturdy. Let $\varepsilon \le \delta'(c, n)$ and let $f\colon A \to \mathbf{R}^n$ be ε-power-quasisymmetric. Setting $R = d(A)$, we have $\theta(A) \ge R/c$ by sturdiness. By 3.3, there is a similarity $S\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\|S \circ f - \mathrm{id}\|_A \le c_nc\varepsilon R$.

We may assume that $0 \in A$, so that $A \subset \overline{B}(R)$. Let $A_1 = A \cup (\mathbf{R}^n \setminus B(2R))$. Then it follows from [ATV2, 4.1] that A_1 is $6c$-sturdy. We extend f to a map $f_1\colon A_1 \to \mathbf{R}^n$ by setting $f_1(x) = S^{-1}(x)$ in case $|x| \ge 2R$.

We shall prove below that f_1 is $34c_nc\varepsilon$-power-quasisymmetric. Since A_1 is unbounded and $6c$-sturdy, the assumptions give a $C'\varepsilon$-power-quasisymmetric extension $F\colon \mathbf{R}^n \to \mathbf{R}^n$ of f_1. This will be the required extension of f also.

It remains to prove that f_1 is $34c_nc\varepsilon$-power-quasisymmetric. Let thus $x, y, z \in A_1$ be distinct points and let $T = (x; y, z)$. We divide the proof into six nontrivial cases. Let $\sigma = \|S^{-1}\|$ be the similarity ratio of S^{-1}.

Case 1. $x \in A$, $y, z \notin A$. Using the approximation S^{-1}, we obtain

$$|f(x) - S^{-1}(y)| = \sigma|Sf(x) - y| \le \sigma(|Sf(x) - x| + |x - y|) \le \sigma(|x - y| + c_nc\varepsilon R).$$

We shall use similar estimates quite often, and also in the opposite direction in the form $|f(x)-S^{-1}(z)|\ge \sigma(|x-y|-c_nc\varepsilon R)$. Thus in this case

$$|T'|=\frac{|f(x)-S^{-1}(y)|}{|f(x)-S^{-1}(z)|}=\frac{|Sf(x)-y|}{|Sf(x)-z|}\le\frac{|x-y|+c_nc\varepsilon R}{|x-z|-c_nc\varepsilon R}.$$

If $|x-z|\le|x-y|$ and $\varepsilon\le 1/2c_nc$, then the first part of 2.5 gives

$$\begin{aligned}|T'|&\le|T|+2c_nc\varepsilon R\frac{|x-y|+|x-z|}{|x-z|^2}\\&\le|T|+4c_nc\varepsilon R\frac{|x-y|}{|x-z|^2}\le(1+4c_nc\varepsilon)|T|,\end{aligned}$$

since $|x-z|\ge R$. If $|x-y|\le|x-z|$, then $|T|\le 1$ and the second part of 2.5 gives

$$|T'|\le(1+Mc_nc\varepsilon R)|T|^{1/(1+Mc_nc\varepsilon R)}.$$

Here

$$Mc_nc\varepsilon R=4c_nc\varepsilon R\frac{|x-y|+|x-z|}{|x-y||x-z|}\le 8c_nc\varepsilon,$$

since $|x-y|\ge R$. This completes the proof of Case 1.

Case 2. $y\in A$, $x,z\notin A$. In this case, we have

$$|T'|=\frac{S^{-1}(x)-f(y)}{|S^{-1}(x)-S^{-1}(z)|}=\frac{|x-Sf(y)|}{|x-z|}\le\frac{|x-y|+c_nc\varepsilon R}{|x-z|}\le(1+c_nc\varepsilon)|T|,$$

since $R\le|x-y|$. Case 2 is thus proved.

Case 3. $z\in A$, $x,y\notin A$. In this case $|x-z|\ge R$, and we have

$$\begin{aligned}|T'|&=\frac{|S^{-1}(x)-S^{-1}(y)|}{|S^{-1}(x)-f(z)|}=\frac{|x-y|}{|x-Sf(z)|}\le\frac{|x-y|}{|x-z|-c_nc\varepsilon R}\\&\le\frac{|x-y|}{|x-z|}\left(1+\frac{2c_nc\varepsilon R}{|x-z|}\right)\le(1+2c_nc\varepsilon)\frac{|x-y|}{|x-z|}\end{aligned}$$

if $\varepsilon\le 1/2c_nc$. Case 3 is now proved.

Case 4. $y,z\in A$, $x\notin A$. Now

$$|T'|=\frac{|x-Sf(y)|}{|x-Sf(z)|}\le\frac{|x-y|+c_nc\varepsilon R}{|x-z|-c_nc\varepsilon R}.$$

From this, the proof goes on as in Case 1 with two subcases, since $|x-y|\ge R$ and $|x-z|\ge R$ also here.

Case 5. $x,z\in A$, $y\notin A$. We have

$$|T'|=\frac{|f(x)-S^{-1}(y)|}{|f(x)-f(z)|}=\frac{\sigma|Sf(x)-y|}{|f(x)-f(z)|}.$$

Choose $w\in A$ such that $|x-w|$ is maximal. Then $R/2\le|x-w|\le R$. Now

$$|f(x)-f(w)|=\sigma|Sf(x)-Sf(w)|\ge\sigma(|x-w|-2c_nc\varepsilon R),$$

and since

$$\frac{|f(x)-f(w)|}{|f(x)-f(z)|}\le(1+\varepsilon)\left(\frac{|x-w|}{|x-z|}\right)^{1+\varepsilon},$$

we obtain

$$\begin{aligned}\frac{1}{|f(x)-f(z)|} &\le (1+\varepsilon)\left(\frac{|x-w|}{|x-z|}\right)^{\varepsilon}\frac{1}{\sigma(1-2c_nc\varepsilon R/|x-w|)}\cdot\frac{1}{|x-z|}\\ &\le (1+\varepsilon)(1+8c_nc\varepsilon)\left(\frac{|x-y|}{|x-z|}\right)^{\varepsilon}\frac{1}{\sigma|x-z|},\end{aligned}$$

where we used $|x-w|^\varepsilon \le |x-y|^\varepsilon$ and $R \le 2|x-w|$.

Combining these estimates and using $R \le |x-y|$, we get

$$\begin{aligned}|T'| &\le \frac{|x-y|+c_nc\varepsilon R}{|x-z|}(1+\varepsilon)(1+8c_nc\varepsilon)\left(\frac{|x-y|}{|x-z|}\right)^{\varepsilon}\\ &\le (1+c_nc\varepsilon)(1+\varepsilon)(1+8c_nc\varepsilon)\left(\frac{|x-y|}{|x-z|}\right)^{1+\varepsilon} \le (1+34c_nc\varepsilon)\left(\frac{|x-y|}{|x-z|}\right)^{1+\varepsilon}.\end{aligned}$$

This completes the proof of case 5.

Case 6. $x, y \in A$, $z \notin A$. We have $|x-y| \le R \le |x-z|$ in this case. Choose again $w \in A$ such that $|w-x|$ is maximal; thus $R/2 \le |x-w| \le R$ and $|x-y|/|x-w| \le 2$.

Subcase 6a. $1 \le |x-y|/|x-w| \le 2$. Now

$$|f(x)-f(w)| \le \sigma(|x-w|+2c_nc\varepsilon R)$$

and

$$\left(\frac{|x-y|}{|x-w|}\right)^{1+\varepsilon} \le 2^\varepsilon\frac{|x-y|}{|x-w|} \le (1+\varepsilon)\frac{|x-y|}{|x-w|}.$$

Using ε-power-quasisymmetry, we obtain

$$\begin{aligned}|f(x)-f(y)| &\le (1+\varepsilon)\left(\frac{|x-y|}{|x-w|}\right)^{1+\varepsilon}|f(x)-f(w)|\\ &\le (1+\varepsilon)^2\sigma(|x-w|+2c_nc\varepsilon R)\frac{|x-y|}{|x-w|}\\ &\le (1+\varepsilon)^2\sigma(1+2c_nc\varepsilon)|x-y| \le (1+7c_nc\varepsilon)\sigma|x-y|.\end{aligned}$$

Therefore, we get

$$\begin{aligned}|T'| = \frac{|f(x)-f(y)|}{|f(x)-S^{-1}(z)|} &\le \frac{(1+7c_nc\varepsilon)\sigma|x-y|}{\sigma(|x-z|-c_nc\varepsilon R)}\\ &\le (1+7c_nc\varepsilon)(1+2c_nc\varepsilon)\frac{|x-y|}{|x-z|} \le (1+23c_nc\varepsilon)|T|.\end{aligned}$$

Subcase 6b. $|x-y| \le |x-w|$. We write $1-\varepsilon' = 1/(1+\varepsilon)$ to simplify notation, and then

$$\frac{|f(x)-f(y)|}{|f(x)-f(w)|} \le (1+\varepsilon)\left(\frac{|x-y|}{|x-w|}\right)^{1-\varepsilon'}.$$

Here

$$|f(x)-f(w)| \le \sigma(|x-w|+2c_nc\varepsilon R)$$

and

$$|f(x)-S^{-1}(z)| \ge \sigma(|x-z|-c_nc\varepsilon R).$$

Combining these estimates and using $|x-w|^{\varepsilon'} \le |x-z|^{\varepsilon'}$, we get

$$\begin{aligned}|T'| &\le (1+\varepsilon)\frac{|x-y|}{|x-w|}\frac{|x-w|^{\varepsilon'}}{|x-y|^{\varepsilon'}}\frac{|x-w|+2c_nc\varepsilon R}{|x-z|-c_nc\varepsilon R}\\ &\le (1+\varepsilon)\frac{|x-y|}{|x-w|}\frac{|x-z|^{\varepsilon'}}{|x-y|^{\varepsilon'}}\frac{1+2c_nc\varepsilon}{1-c_nc\varepsilon}\\ &\le (1+18c_nc\varepsilon)\frac{|x-y|^{1-\varepsilon'}}{|x-z|^{1-\varepsilon'}} \le (1+18c_nc\varepsilon)\left(\frac{|x-y|}{|x-z|}\right)^{1/(1+\varepsilon)}.\end{aligned}$$

This completes the proof of Subcase 6b.

Finally, comparing the constants and restrictions obtained in different cases, we obtain the expressions for δ' and C' given in the theorem. □

Proof of the main theorem 1.4. The proof will be carried out in several steps, some of which are similar to the ones used in the proof of [ATV2, Section 4]. We give here an essentially complete proof, but some technical details that can be found in the above-mentioned article are omitted. Also, some very similar cases and subcases are compressed in the last part of the proof.

By 4.1 we may assume that $A \subset \mathbf{R}^n$ is closed, unbounded, and c-sturdy, and $f\colon A \to \mathbf{R}^n$ is ε-power-QS. Let K be a decomposition of $G = \mathbf{R}^n \setminus A$ into Whitney cubes Q such that

(i) $1 \le d(Q,A)/d(Q) < 3$;
(ii) $1/2 \le d(Q)/d(Q') \le 2$ if $Q, Q' \in K$ and $Q \cap Q' \ne \emptyset$.

We first define the extension F in the set K^0 in the vertices of the cubes $Q \in K$. Let $v \in K^0$ and choose a point $a_v \in A$ such that $r_v = |v-a_v| = d(v,A)$. Let $t_v = s(a_v) \vee 8r_v$. By 3.5 there is a C_0-special similarity S_v for (f, a_v, ct_v) satisfying $S_v a_v = f(a_v)$ and

$$\|S_v - f\|_{A(a_v,ct_v)} \le C_0\|S_v\|t_v\varepsilon, \tag{4.1}$$

where $C_0 = 4c^2c_n$. We choose $S_u = S_v$ whenever $a_u = a_v = a$ and $u, v \in \overline{B}(a, s(a)/8)$. Furthermore, by 3.10 we may assume that all these similarities have positive orientation. We define $F(v) = S_v v$.

Next we triangulate each $Q \in K$ in a standard way to obtain a collection W of n-simplexes $W = \{\Delta \in Q \mid Q \in K\}$. These simplexes satisfy

$$d(\Delta) = \lambda\sqrt{n}, \quad b(\Delta) = \lambda/\sqrt{2}, \quad \rho(\Delta) = \sqrt{2n},$$

if $\Delta \in Q$ and Q has sides of length 2λ.

After this, we extend F to each $\Delta \in W$ in an affine way. Setting $F \mid A = f$, we obtain a map $F\colon \mathbf{R}^n \to \mathbf{R}^n$ that extends f.

Fact 1. *Let $Q \in K$ and $u, v \in Q \cap K^0$. Then*

(i) $|a_u - a_v| \le 3(r_u \vee r_v)$;
(ii) $r_v \le 2r_u$;
(iii) $\|S_u - S_v\|_Q \le C_1(\|S_u\| \vee \|S_v\|)(r_u \vee r_v)\varepsilon$;
(iv) $t_u \le 2t_v$ *and* $\overline{B}(a_u, t_u) \subset \overline{B}(a_v, 3t_v)$;
(v) $(1-16C_0\varepsilon)\|S_v\| \le \|S_u\| \le (1+16C_0\varepsilon)\|S_v\|$.

Proof. The first two inequalities are the same as in [ATV2, p. 965]. To prove (iii) we may assume that $r_u \le r_v$.

If $t_v = s(a_v) \geq 8r_v$, then a_v is isolated in A. We claim that $a_u = a_v$ in this case. If not, then (i) and (ii) imply

$$s(a_v) \leq |a_u - a_v| \leq 3r_v \leq 3s(a_v)/8,$$

a contradiction. Thus $a_u = a_v$, and since $r_u \leq r_v \leq 3s(a_v)/8$, we have $S_u = S_v$. Therefore, (iii) is trivially true in this case.

We now assume that $t_v = 8r_v \geq s(a_v)$.

Case 1. $r_u \geq cs(a_u)$. Now $t_u = 8r_u$ and $r_v \geq r_u \geq cs(a_u)$. By 3.9 there is a c-special simplex Δ_u for (A, a_u, r_v) satisfying

$$\Delta_u^0 \subset A(a_u, r_v), \quad b(\Delta_u) \geq r_v/c, \rho(\Delta_u) \leq 2c.$$

Since $r_v \leq 2r_u < t_u$ by (ii), we obtain by (4.1) the inequality

$$\|S_u - f\|_{\Delta_u^0} \leq C_0 \|S_u\| \varepsilon t_u \leq C_0 \|S_u\| \varepsilon t_v.$$

By (i), we also have $\overline{B}(a_u, r_v) \subset \overline{B}(a_v, 4r_v)$, and thus (4.1) implies that

$$\|S_v - f\|_{\Delta_u^0} \leq C_0 \varepsilon t_v \|S_v\|.$$

Consequently,

$$\|S_u - S_v\|_{\Delta_u^0} \leq 2C_0 \varepsilon t_v (\|S_u\| \vee \|S_v\|),$$

and 3.11 implies that

$$\begin{aligned} \big|\|S_u\| - \|S_v\|\big| &\leq 4C_0 \varepsilon t_v (\|S_u\| \vee \|S_v\|)/d(\Delta_u) = 32 C_0 \varepsilon r_v (\|S_u\| \vee \|S_v\|)/d(\Delta_v) \\ &\leq 192 C_0 \varepsilon (\|S_u\| \vee \|S_v\|). \end{aligned}$$

Let $x \in Q$. Choose a vertex $z \in \Delta_u^0$ and apply 3.11 to get

$$|S_u x - S_v x| \leq 2C_0 \varepsilon t_v (1 + M|x - z|/d(\Delta_u))(\|S_u\| \vee \|S_v\|).$$

Here

$$M = 4 + 6n\rho(\Delta_u)(1 + \rho(\Delta_u))^{n-1} \leq 4 + 12n(1 + 2c)^{n-1} \equiv M_1(c, n),$$

$|x - z| \leq |x - u| + |u - a_u| + |a_u - z| \leq 3r_v$, and $d(\Delta_u) \geq b(\Delta_u) \geq r_v/c$. Since $t_v = 8r_v$, these estimates imply (iii) with $C_1 = 16C_0(1 + 3cM_1)$.

Case 2. $r_u \leq cs(a_u)$. By 3.9 we can choose a c-special simplex Δ_u for $(A, a_u, cs(a_u))$. Since $s(a_u) \leq t_u$, we have

$$\|S_u - f\|_{\Delta_u^0} \leq C_0 \|S_u\| \varepsilon t_u$$

by (4.1).

We next show that

$$s(a_u) \leq 8r_v, \ \Delta_u^0 \subset A(a_v, 8cr_v). \tag{4.2}$$

Let $w \in \Delta_u^0$. If $a_u \neq a_v$, then $s(a_u) \leq |a_u - a_v| \leq 3r_v$ by (i), and

$$|w - a_v| \leq cs(a_u) + |a_u - a_v| \leq 6cr_v.$$

If $a_u = a_v$, then $s(a_u) = s(a_v) \leq 8r_v$, proving (4.2).

Since $t_v = 8r_v$, it follows from (4.2) that $t_u \leq t_v$ and $\|S_v - f\|_{\Delta_u^0} \leq 16C_0(\|S_v\| \vee \|S_u\|)\varepsilon r_v$. As in Case 1, we choose a vertex z of Δ_u and apply 3.11. For each $x \in Q$ we have

$$|x - z| \leq d(Q) + r_u + cs(a_u) \leq 3cs(a_u)$$

and $d(\Delta_u) \geq b(\Delta_u) \geq s(a_u)$. Thus we obtain (iii) with $C_1 = 16C_0(1 + 3cM_1)$.

We next prove (iv). If $s(a_u) \geq 8r_u$, then $a_u = a_v$ as above. This implies that $t_u = s(a_u) = s(a_v) \leq t_v$. If $s(a_u) < 8r_u$, then by (i) and (ii), we have $t_u = 8r_u \leq 16r_v \leq 2t_v$.

To prove the inclusion of balls in (iv), let $x \in \overline{B}(a_u, t_u)$. Then

$$|x - a_v| \le |x - a_u| + |a_u - a_v| \le t_u + 6r_v \le 2t_v + \frac{3}{4}t_v \le 3t_v.$$

This completes the proof of (iv).

To prove (v), we apply 3.6 with $X = A(a_u, t_u)$. Since $t_u \ge s(a_u)$, we have $t_u \le d(X)$, and by (iv), also $t_v \le 2t_u \le 2d(X)$. Assuming that $c \ge 3$, these and (iv) imply that the similarities S_u and S_v satisfy the approximation conditions of 3.6 in X, with ε replaced by $2C_0\varepsilon$, and (v) follows.

We have thus completed the proof of Fact 1. □

Fact 2. *There is a number $\delta_2(c, n) > 0$ such that if $\varepsilon \le \delta_2$ and $\Delta \in W$, then $F \mid \Delta$ is sense-preserving and L-bilipschitz with*

$$\frac{\|S_v\|}{1 + C_2\varepsilon} \le L \le \|S_v\|(1 + C_2\varepsilon),$$

for all vertices $v \in \Delta$.

Proof. Let $Q \in K$ be the cube containing Δ and let $v \in Q \cap K^0$ be such that $r_v = |v - a_v| = d(v, A)$ is maximal. From Fact 1(iii) and from the construction of F it follows that

$$\|S_v - F\|_\Delta \le C_1 r_v \varepsilon \|S_v\|.$$

Here

$$r_v \le d(Q, A) + d(Q) \le 4d(Q) = 8b(\Delta)\sqrt{2n},$$

and thus $\|S_v - F\|_\Delta \le \alpha b(\Delta)/(n + 1)$, where

$$\alpha = 8C_1\sqrt{2n}(n + 1)\varepsilon \le \frac{1}{2}$$

if $\varepsilon \le \delta_2(c, n) = (16\sqrt{2n}(n + 1)C_1)^{-1}$.

The claim now follows from [Vä, 2.7] for this particular vertex v. Also, $F \mid \Delta$ is L-bilipschitz with L satisfying

$$\|S_v\|/(1 + 2\alpha) \le L \le \|S_v\|(1 + 2\alpha).$$

Finally, from 3.6 the claim follows for all vertices of Δ. □

Finally, we show that F can be well approximated by similarities in all balls. Our main theorem then follows from 3.7. The most important case is dealt with in Fact 3 below, and the rest are postponed into Fact 4 because of many cases and subcases that complicate the proof.

Fact 3. *For each $a \in A$ and $r > 0$, there is a sense-preserving similarity S of $\mathbf{R}^n$ such that $Sa = f(a)$ and*

$$\|S^{-1}F - \mathrm{id}\|_{\overline{B}(a,r)} \le C_3\varepsilon r.$$

Proof. If $s(a) > 0$, then $F \mid K^0$ agrees with a similarity S in $\overline{B}(a, s(a)/8)$. This implies that $F = T$ in $\overline{B}(a, s(a)/16)$. We may thus assume that $s(a) \le 16r$.

By 3.5 there is a special similarity S for $(f, a, 20cr)$. The map S is sense-preserving and $Sa = f(a)$. Let $x \in \overline{B}(a, r)$. We show that $|Sx - F(x)| \le C_3\varepsilon\|S\|r$ for $x \in \overline{B}(a, r)$, which implies that S is the required similarity. Since

$$\|S - f\|_{A(a,20cr)} \le 20C_0\varepsilon\|S\|r$$

with $C_0 = 4c^2c_n$ as before, we may assume that $x \in G = \mathbf{R}^n \setminus A$.

Let $Q \in K$ be a cube containing x and let $v \in Q \cap K^0$. It suffices to find an estimate $|Sv - F(v)| \le C_3\varepsilon \|S\| r$. We have $F(v) = S_v v$ and

$$\|S_v - f\|_{A(a_v, ct_v)} \le C_0 \varepsilon t_v \|S_v\|$$

with $t_v = s(a_v) \vee 8r_v$. Since $ct_v \ge cs(a_v)$, there is a c-special simplex Δ_v for (A, a_v, ct_v). As $d(Q) \le d(Q, A) \le r$, we have

$$r_v \le |v - a| \le |v - x| + |x - a| \le 2r.$$

If $a_v \ne a$, this implies that $s(a_v) \le |a_v - a| \le 4r$. If $a_v = a$, then $s(a_v) = s(a) \le 16r$. Hence

$$t_v \le 16r \vee 8r_v = 16r,$$

and thus $\Delta_v^0 \subset A(a, 20cr)$. This implies that

$$\|S_v - S\|_{\Delta_v^0} \le 20C_0\varepsilon\|S\|r + C_0\varepsilon\|S_v\|t_v \le 68cC_0\varepsilon r\|S\|,$$

since we show in the lemma below that $t_v\|S_v\| \le 48cr\|S\|$. Fix a point $z \in \Delta_v^0$. Then

$$|F(v) - Sv| \le 136cC_0\varepsilon r\|S\|(1 + M_1|v - z|/d(\Delta_v)),$$

where $M_1 = M_1(c, n)$ is the constant from the proof of Fact 1, and $|v - z| \le r_v + ct_v \le 9ct_v/8$, $d(\Delta_v) \ge b(\Delta_v) \ge t_v$. Thus we obtain the required inequality with $C_3 = 136cC_0(1 + 2cM_1)$. □

Lemma. *Using the notation above, we have $t_v\|S_v\| \le 48cr\|S\|$.*

Proof. We have $t_v \le 16r$ from above. Choose an integrer N such that $2^N t_v \ge 24cr$ and $2^{N-1}t_v < 24cr$. Let S_k be a special similarity for $(f, a_v, 2^k ct_v)$ so that $S_0 = S_v$ and $S_N = S$, and let Δ_k be a special simplex for $A(a_v, 2^k ct_v)$. Then $d(\Delta_k) \ge b(\Delta_k) \ge 2^k t_v$, and we can use 3.6 with $r = 2^k t_v$ and $\varepsilon \mapsto 2C_0\varepsilon$. In the last step we also use

$$\overline{B}(a, 20cr) \subset \overline{B}(a_v, 24cr) \subset \overline{B}(a_v, 2^N t_v).$$

This implies that $\|S_k\| \le (1 + 16C_0\varepsilon)\|S_{k+1}\| \le 2\|S_{k+1}\|$ under the requirement $\varepsilon \le 1/16C_0$. It follows that $\|S_v\| \le 2^N\|S\| \le 48c(r/t_v)\|S\|$, which proves the lemma. □

Fact 4. *The corresponding result for Fact 3 is true for balls $\overline{B}(x, r)$ centered outside A.*

Proof. Let $x \in \mathbf{R}^n \setminus A$ and $r > 0$. The proof breaks up into several cases. All details are straightforward but rather long, cf. Fact 3, and therefore we omit most technicalities. As in the proof of Fact 3, the main idea in most cases is to find a suitable ball $\overline{B}(a_v, R) \supset \overline{B}(x, r)$ so that the special similarity for (f, a_v, R) is the required one.

Case 1. The ball $\overline{B}(x, r)$ does not contain any vertices of the triangulation.

Subcase 1a. The ball $\overline{B}(x, r)$ is contained in some simplex Δ of the triangulation. In this case the extension F is the convex combination of similarities S_v for $v \in \Delta^0$. The claim follows from Facts 1 and 2.

Subcase 1b. The ball $\overline{B}(x, r)$ is not included in a single simplex. In this case $\overline{B}(x, r)$ is contained in a finite union of adjacent Whitney cubes, whose number is bounded by a fixed constant depending on n. It follows from Facts 1 and 2 that the required similarity can be any of the similarities S_v, where v is a vertex of a simplex containing x.

Case 2. The ball $\overline{B}(x, r)$ contains vertices of the triangulation. Choose a vertex $v \in K^0 \cap \overline{B}(x, r)$ such that $d(v, A)$ is maximal. Excluding the trivial case (i) below, we claim that a special similarity S for (f, a_v, R) is the required one for a suitable radius R.

Subcase 2a(i). $s(a_v) > 0$, $|x - a_v| \le s(a_v)/16$, and $r \le s(a_v)/16$. In this case all the approximating similarities S_u used to define $F(u)$ in $\overline{B}(x,r) \cap K^0$ coincide. Thus F itself is a similarity in $\overline{B}(x,r)$.

Subcase 2a(ii). $s(a_v) > 0$, $|x - a_v| \le s(a_v)/16$, and $r \ge s(a_v)/16$. In this case

$$\overline{B}(x,r) \subset \overline{B}(a_v, 2r) \subset \overline{B}(a_v, 16cr).$$

Since $16cr \ge cs(a_v)$, we can use an approximating similarity S for $(f, a_v, 16cr)$.

Subcase 2a(iii). $s(a_v) > 0$, $|x - a_v| > s(a_v)/16$, and $r \ge r_v/2$. In this case $|x - a_v| \le |x - v| + |v - a_v| \le r + r_v \le 3r$, and thus

$$\overline{B}(x,r) \subset \overline{B}(a_v, 4r) \subset \overline{B}(a_v, 128cr).$$

Since $s(a_v) < 16|x - a_v| \le 64r_v$, we have $128cr \ge 64cr_v \ge cs(a_v)$. In this case we use a special similarity S for $(f, a_v, 128cr)$.

Subcase 2a(iv). $s(a_v) > 0$, $|x - a_v| > s(a_v)/16$, and $r < r_v/2$. In this case $|x - a_v| \le 3r_v/2$, so that

$$\overline{B}(x,r) \subset \overline{B}(x, 2r_v) \subset \overline{B}(a_v, 7r_v/2) \subset \overline{B}(a_v, 24cr_v).$$

Since $s(a_v) < 16|x - a_v| \le 24r_v$, we can use a special similarity S for $(f, a_v, 24cr_v)$.

Subcase 2b(i). $s(a_v) = 0$ and $r_v \le 2r$. Let S be a special similarity for $(f, a_v, 22cr)$. Then

$$\|S - f\|_{A(a_v, 22cr)} \le 22C_0\varepsilon\|S\|r.$$

For $u \in \overline{B}(x,r) \cap K^0$ we must estimate $|Su - F(u)| = |Su - S_u u|$, where S_u satisfies $\|S_u - f\|_{A(a_u, ct_u)} \le C_0\|S_u\|t_u$. If $a_u \ne a_v$, then

$$s(a_u) \le |a_u - a_v| \le r_u + 2r + r_v \le 6r,$$

so that $t_u = s(a_u) \vee 8r_u \le 6r \vee 8r_v \le 16r$. This is obviously true also in the case $a_u = a_v$, since then $s(a_u) = s(a_v) = 0$. Let w be a vertex of a special simplex Δ_u for (A, a_u, ct_u). Then

$$|w - a_v| \le ct_u + |a_u - a_v| \le 16cr + 6r \le 22cr.$$

This implies that $\Delta_u^0 \subset A(a_v, 22cr)$, and we can use Δ_u for estimating $|Su - S_u u|$. The details are essentially similar to the proof of Fact 3, and are therefore omitted.

Subcase 2b(ii). $s(a_v) = 0$ and $r_v \ge 2r$. This case is very similar to the previous one, but now we choose a similarity S satisfying

$$\|S - f\|_{A(a_v, 12cr_v)} \le 12C_0\varepsilon\|S\|r_v.$$

If $a_u \ne a_v$, then

$$s(a_u) \le |a_u - a_v| \le r_u + 2r + r_v \le 3r_v,$$

so that $t_u = s(a_u) \vee 8r_u \le 3r_v \vee 8r_u \le 8r_v$, which is true also in the case $a_u = a_v$. Since $\overline{B}(a_u, ct_u) \subset \overline{B}(a_v, 12cr_v)$ in this case, we can again proceed as in the proof of Fact 3.

This completes the proof of our main theorem. $\square$

References

[AT1] Alestalo, P., and D. A. Trotsenko: Plane sets allowing bilipschitz extensions. - Math. Scand. 105, 2009, 134–146.

[AT2] Alestalo, P., and D. A. Trotsenko: On mappings that are close to a similarity. - Math. Rep. (Bucur.) 15:4, 2013, 313–318.

[ATV1] Alestalo, P., D. A. Trotsenko, and J. Väisälä: Isometric approximation. - Israel J. Math. 125, 2001, 61–82.

[ATV2] Alestalo, P., D. A. Trotsenko, and J. Väisälä: Linear bilipschitz extension property. - Sibirsk. Mat. Zh. 44, 2003, 1226–1238. Translation in Sib. Math. J. 44, 2003, 959–968.

[AH] Astala, K., and J. Heinonen: On quasiconformal rigidity in space and plane. - Ann. Acad. Sci. Fenn. Ser. A I Math. 13, 1988, 81–92.

[DF] Damelin, S. B., and C. Fefferman: On extensions of ε differomorphisms. - arXiv: 1505.06950v1, 2015.

[Jo] John, F.: Rotation and strain. - Comm. Pure Appl. Math. 14, 1961, 391–413.

[MS] Martio, O., and J. Sarvas: Injectivity theorems in plane and space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 4, 1978/1979, 383–401.

[Pa] Partanen, J.: Invariance theorems for the bilischitz and quasisymmetric extension properties. - Ann. Acad. Sci. Ser. A I Math. Diss. 80, 1991, 1–40.

[PV] Partanen, J., and J. Väisälä: Extension of bilipschitz maps of compact polyhedra. - Math. Scand. 72, 1993, 235–264.

[Re] Reshetnyak, Yu. G.: Stability theorems in geometry and analysis. - Springer, 1994, 235–264.

[Tr1] Trotsenko, D. A.: Continuation from a domain and approximation of spatial quasiconformal mappings with a small distortion coefficient. - Soviet Math. Dokl. 27, 1983, 771–780.

[Tr2] Trotsenko, D. A.: Approximation by similarities of spatial mappings with bounded distortion. - Sib. Math. J. 27, 1986, 196–205.

[TrV] Trotsenko, D. A., and J. Väisälä: Upper sets and quasisymmetric maps. - Ann. Acad. Sci. Fenn. Math. 24, 1999, 465–488.

[TuV] Tukia, P., and J. Väisälä: Quasisymmetric embeddings of metric spaces. - Ann. Acad. Sci. Fenn. Math. 5, 1980, 97–114.

[Vä] Väisälä, J.: Bilipschitz and quasisymmetric extension properties. - Ann. Acad. Sci. Fenn. Math. 11, 1986, 239–274.

Received 2 March 2016 • Accepted 13 March 2016