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ASSOCIATE SPACES OF LOGARITHMIC INTERPOLATION SPACES AND GENERALIZED LORENTZ–ZYGMUND SPACES

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Abstract. We determine the associate space of the logarithmic interpolation space $(X_0,X_1)_{1,q,\mathbf{A}}$ where X_0 and X_1 are Banach function spaces over a σ-finite measure space (Ω,μ). Particularizing the results for the case of the couple (L_1,L_∞) over a non-atomic measure space, we recover results of Opic and Pick on associate spaces of generalized Lorentz–Zygmund spaces $L_{(\infty,q;\mathbf{A})}$. We also establish the corresponding results for sequence spaces.

1. Introduction

Logarithmic spaces $(A_0,A_1)_{1,q,\mathbf{A}}$ are interpolation spaces which are quite close to the space A_1. This fact is useful in several situations (see, for example, [10, 6, 8, 3]). When $A_0\cap A_1$ is dense in A_0 and A_1, the dual of $(A_0,A_1)_{1,q,\mathbf{A}}$ has been computed in [8] for $1\le q\le\infty$, and in [3] for the case $0<q<1$. Curiously, as it is pointed out in [3, Remark 4.5], although the couple (L_1,L_∞) does not satisfy that $L_1\cap L_\infty$ is dense in L_∞, writing down the duality results for (L_1,L_∞) the outcome coincides with the associate spaces of generalized Lorentz–Zygmund spaces $L_{(\infty,q;\mathbf{A})}$, determined by Opic and Pick in [16]. The aim of the present paper is to clarify this coincidence.

We compute the associate space of $(X_0,X_1)_{1,q,\mathbf{A}}$ where X_j are Banach function spaces on a σ-finite measure space (Ω,μ). This is done in Section 3 with the help of the description of logarithmic spaces in terms of the J-functional. Since there is no J-description in a certain range of the parameters (see [8, Proposition 3.4]), we show first in Section 2 that in such range the space $(A_0,A_1)_{1,q,\mathbf{A}}$ turns out to be equal to the sum of A_1 with a certain J-space modified. This result is of independent interest, it complements those of [8, 3] and it is useful in Section 3. Finally, in Section 4, we show some applications of the abstract results. We first consider a non-atomic σ-finite measure space and applying the results to the couple (L_1,L_∞) we recover the results of Opic and Pick [16] on associate spaces of generalized Lorentz–Zygmund

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spaces $L_{(\infty,q;\mathbf{A})}$. Then we establish the corresponding results for the sequence spaces $\ell_{(\infty,q;\alpha)}$. For this aim, we work with the measure space $(\mathbf{N},\#)$, where $\#$ is the counting measure, which is completely atomic. The sequence case has not been studied previously.

2. Logarithmic interpolation methods

By a *Banach couple* $\bar{A}=(A_0,A_1)$ we mean two Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. We set $\Sigma(\bar{A})=A_0+A_1$ and $\Delta(\bar{A})=A_0\cap A_1$. These spaces become Banach spaces when normed by

$$\|a\|_{A_0+A_1}=\|a\|_{\Sigma(\bar{A})}=\inf\{\|a_0\|_{A_0}+\|a_1\|_{A_1}: a=a_0+a_1, a_j\in A_j\}$$

and

$$\|a\|_{A_0\cap A_1}=\|a\|_{\Delta(\bar{A})}=\max\{\|a\|_{A_0},\|a\|_{A_1}\}.$$

For $t>0$, the *Peetre's K- and J-functionals* are defined by

$$K(t,a)=K(t,a;\bar{A})=\inf\{\|a_0\|_{A_0}+t\|a_1\|_{A_1}: a=a_0+a_1, a_j\in A_j\}, a\in A_0+A_1$$

and

$$J(t,a)=J(t,a;\bar{A})=\max\{\|a\|_{A_0},t\|a\|_{A_1}\},\ a\in A_0\cap A_1.$$

Note that $K(1,\cdot)=\|\cdot\|_{A_0+A_1}$ and $J(1,\cdot)=\|\cdot\|_{A_0\cap A_1}$.

The *Gagliardo completion* $A_j^{\sim}$ of A_j consists of all those $a\in\Sigma(\bar{A})$ for which there exists a bounded sequence (a_n) in A_j which converges to a in $\Sigma(\bar{A})$. The norm in $A_j^{\sim}$ is given by

$$\|a\|_{A_j^{\sim}}=\inf_{(a_n)}\left(\sup_{n\in\mathbf{N}}(\|a_n\|_{A_j})\right)=\sup_{0<t<\infty}\frac{K(t,a)}{t^j},\quad j=0,1$$

(see [1, Theorem V.1.4]). We have $A_j\hookrightarrow A_j^{\sim}\hookrightarrow\Sigma(\bar{A})$, where $\hookrightarrow$ means continuous embedding. Furthermore, for the Banach couple $\overline{A^{\sim}}=(A_0^{\sim},A_1^{\sim})$, we have

$$K(t,a;\overline{A^{\sim}})=K(t,a;\bar{A}),\quad t>0,\ a\in\Sigma(\bar{A}) \tag{2.1}$$

(see [1, Theorem V.1.5]). The couple $\bar{A}$ is said to be a *Gagliardo couple* if $A_0=A_0^{\sim}$ and $A_1=A_1^{\sim}$. Examples of Gagliardo couples are (L_1,L_∞) and (ℓ_1,ℓ_∞). See Section 4 for details.

For $t>0$, let $\ell(t)=1+|\log t|$ and $\ell\ell(t)=1+\log(1+|\log t|)$. For $\mathbf{A}=(\alpha_0,\alpha_\infty)\in\mathbf{R}^2$ write

$$\ell^{\mathbf{A}}(t)=\ell^{(\alpha_0,\alpha_\infty)}(t)=\begin{cases}\ell^{\alpha_0}(t) & \text{if } 0<t\le 1,\\ \ell^{\alpha_\infty}(t) & \text{if } 1<t<\infty,\end{cases}$$

and define $\ell\ell^{\mathbf{A}}(t)$ similarly.

Let $0<q\le\infty$ and $\mathbf{A}\in\mathbf{R}^2$. The *logarithmic interpolation space* $\bar{A}_{1,q,\mathbf{A}}=(A_0,A_1)_{1,q,\mathbf{A}}$ is formed of all $a\in\Sigma(\bar{A})$ which have a finite quasi-norm

$$\|a\|_{1,q,\mathbf{A}}=\left(\int_0^\infty[t^{-1}\ell^{\mathbf{A}}(t)K(t,a)]^q\frac{dt}{t}\right)^{1/q}$$

(as usual, the integral should be replaced by the supremum when $q=\infty$). See [11, 12, 8, 3]. The functional $\|\cdot\|_{1,q,\mathbf{A}}$ is a norm if $1\le q\le\infty$. We shall assume that

$$\begin{cases}\alpha_0+1/q<0 & \text{if } 0<q<\infty,\\ \alpha_0<0 & \text{if } q=\infty,\end{cases} \tag{2.2}$$

in order to avoid that $(A_0, A_1)_{1,q,\mathbf{A}} = \{0\}$ (see [12, Theorem 2.2]). The space $(A_0, A_1)_{1,q,\mathbf{A}}$ also makes sense when $q = \infty$ and $\alpha_0 = 0$. However we do not study this limit case here because the space $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$ has a different structure than the spaces $(A_0, A_1)_{1,q,\mathbf{A}}$ when q and $\mathbf{A}$ satisfy (2.2). We refer to [4] for the properties of the space $(A_0, A_1)_{1,\infty,(0,\alpha_\infty)}$.

This construction produces *exact interpolation spaces*. More precisely, if $\bar{B} = (B_0, B_1)$ is another Banach couple and $T\colon \Sigma(\bar{A}) \to \Sigma(\bar{B})$ is a linear operator whose restrictions to A_j define a bounded linear operator from A_j to B_j for $j = 0, 1$, then

$$T\colon\ (A_0, A_1)_{1,q,\mathbf{A}} \to (B_0, B_1)_{1,q,\mathbf{A}}$$

is also bounded and

$$\|T\|_{\bar{A}_{1,q,\mathbf{A}},\bar{B}_{1,q,\mathbf{A}}} \le \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}.$$

It is not hard to check that the quasi-norm of $(A_0, A_1)_{1,q,\mathbf{A}}$ is equivalent to

$$\|a\|_{\bar{A}_{1,q,\mathbf{A}}} = \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbf{A}}(2^m) K(2^m, a)\right]^q \right)^{1/q}.$$

Here the ℓ_q-quasi-norm should be replaced by the ℓ_∞-norm if $q = \infty$.

Next we introduce the corresponding J-spaces. We assume that

$$\text{(2.3)} \qquad \begin{cases} \alpha_\infty \ge 0 & \text{if } 0 < q \le 1, \\ \alpha_\infty - 1/q' > 0 & \text{if } 1 < q \le \infty, \end{cases}$$

where for $1 \le q \le \infty$ the parameter q' is given by equality $1/q + 1/q' = 1$. The space $\bar{A}^J_{1,q,\mathbf{A}} = (A_0, A_1)^J_{1,q,\mathbf{A}}$ consists of all $a \in \Sigma(\bar{A})$ for which there exists $(u_m)_{m\in\mathbf{Z}} \subseteq \Delta(\bar{A})$ such that

$$a = \sum_{m=-\infty}^{\infty} u_m \quad (\text{convergence in } \Sigma(\bar{A}))$$

and

$$\left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbf{A}}(2^m) J(2^m, u_m)\right]^q \right)^{1/q} < \infty.$$

The quasi-norm in $(A_0, A_1)^J_{1,q,\mathbf{A}}$ is

$$\|a\|_{\bar{A}^J_{1,q,\mathbf{A}}} = \inf\left\{ \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbf{A}}(2^m) J(2^m, u_m)\right]^q \right)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m \right\}.$$

If $1 < q \le \infty$ and $\alpha_\infty = 1/q'$, the J-space still makes sense if we replace $\ell^{\mathbf{A}}(t)$ by $\ell^{\mathbf{A}}(t)\ell\ell^{\mathbf{B}}(t)$ with $\mathbf{B} = (\beta_0, \beta_\infty)$ and $\beta_\infty - 1/q' > 0$. We denote the corresponding space by $\bar{A}^J_{1,q,\mathbf{A},\mathbf{B}} = (A_0, A_1)^J_{1,q,\mathbf{A},\mathbf{B}}$. The space $\bar{A}^J_{1,q,\mathbf{A},\mathbf{B}}$ is also well-defined for $0 < q \le 1$, $\alpha_\infty = 0$ and $\beta_\infty \ge 0$. The quasi-norm in $\bar{A}^J_{1,q,\mathbf{A},\mathbf{B}}$ is

$$\|a\|_{\bar{A}^J_{1,q,\mathbf{A},\mathbf{B}}} = \inf\left\{ \left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbf{A}}(2^m) \ell\ell^{\mathbf{B}}(2^m) J(2^m, u_m)\right]^q \right)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m \right\}.$$

It is easy to check that the J-spaces are also exact interpolation spaces.

According to [8, Theorems 3.5 and 3.6], if $\mathbf{A}$ and $1 \le q \le \infty$ satisfy (2.2) then we have with equivalence of norms

$$(2.4) \qquad (A_0, A_1)_{1,q,\mathbf{A}} = \begin{cases} (A_0, A_1)^J_{1,q,\mathbf{A}+1} & \text{if } \alpha_\infty + 1/q > 0, \\ (A_0, A_1)^J_{1,q,\mathbf{A}+1,(0,1)} & \text{if } \alpha_\infty = -1/q \text{ and } q < \infty. \end{cases}$$

Here for $\lambda \in \mathbf{R}$, we put $\mathbf{A} + \lambda = (\alpha_0 + \lambda, \alpha_\infty + \lambda)$.

If $0 < q < 1$ and (2.2) holds, then by [3, Theorem 3.2] we have with equivalence of quasi-norms

$$(2.5) \qquad (A_0, A_1)_{1,q,\mathbf{A}} = \begin{cases} (A_0^\sim, A_1^\sim)^J_{1,q,\mathbf{A}+1/q} & \text{if } \alpha_\infty + 1/q > 0, \\ (A_0^\sim, A_1^\sim)^J_{1,q,\mathbf{A}+1/q,(0,1/q)} & \text{if } \alpha_\infty = -1/q. \end{cases}$$

Moreover, if $0 < q < 1$ and we assume in addition that $A_0 \cap A_1$ is dense in A_0 and A_1, then

$$(A_0, A_1)_{1,q,\mathbf{A}} = (A_0^\sim, A_1^\sim)^J_{1,q,(\alpha_0+1/q,0)} \text{ for } \alpha_\infty + 1/q < 0.$$

In general there is no description for $(A_0, A_1)_{1,q,\mathbf{A}}$ as a J-space in the case

$$(2.6) \qquad \begin{cases} \alpha_\infty + 1/q < 0 & \text{if } q < \infty, \\ \alpha_\infty \le 0 & \text{if } q = \infty \end{cases}$$

(see [8, Proposition 3.4]). However, we show next that in this range $(A_0, A_1)_{1,q,\mathbf{A}}$ is the sum of A_1 with a modified J-space.

In what follows, if X and Y are quantities depending on certain parameters some of them being the significant parameters in our reasoning, we write $X \lesssim Y$ if $X \le cY$ with a constant $c > 0$ independent of the significant parameters. We put $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. Similarly, if $\|\cdot\|$ and $|||\cdot|||$ are quasi-norms on a space A, we put $\|a\| \lesssim |||a|||$ if there is a constant $c > 0$ such that $\|a\| \le c|||a|||$ for any $a \in A$. We write $\|a\| \sim |||a|||$ if $\|a\| \lesssim |||a|||$ and $|||a||| \lesssim \|a\|$.

Put $\mathbf{Z}^- = \{0, -1, -2, -3, \ldots\}$. If $0 < q \le \infty$ and $\alpha_0 \in \mathbf{R}$, we write $[\bar{A}]^J_{1,q,\alpha_0} = [A_0, A_1]^J_{1,q,\alpha_0}$ for the collection of all $a \in \Sigma(\bar{A})$ such that there exists $(u_n)_{n \in \mathbf{Z}^-} \subseteq \Delta(\bar{A})$ satisfying

$$a = \sum_{n=-\infty}^{0} u_n \text{ (convergence in } A_0 + A_1)$$

and

$$\left(\sum_{n=-\infty}^{0} \left[2^{-n} \ell^{\alpha_0}(2^n) J(2^n, u_n) \right]^q \right)^{1/q} < \infty.$$

We endow $[A_0, A_1]^J_{1,q,\alpha_0}$ with the quasi-norm

$$\|a\|_{[A_0,A_1]^J_{1,q,\alpha_0}} = \inf \left\{ \left(\sum_{n=-\infty}^{0} \left[2^{-n} \ell^{\alpha_0}(2^n) J(2^n, u_n) \right]^q \right)^{1/q} : a = \sum_{n=-\infty}^{0} u_n \right\}.$$

We claim that

$$A_0 \cap A_1 \hookrightarrow [A_0, A_1]^J_{1,q,\alpha_0} \hookrightarrow A_0 + A_1.$$

Indeed, take any $a \in A_0 \cap A_1$ and for any $n \in \mathbf{Z}^-$, put $u_n = \delta_n^0 a$ where δ_n^m is the Kronecker delta. So, $a = \sum_{n=-\infty}^{0} u_n$ and $\|a\|_{[A_0,A_1]^J_{1,q,\alpha_0}} \le J(1, a) = \|a\|_{A_0 \cap A_1}$. On the other hand, take any $a \in [A_0, A_1]^J_{1,q,\alpha_0}$ and let $a = \sum_{n=-\infty}^{0} u_n$ be a representation with

$$\left(\sum_{n=-\infty}^{0} \left[2^{-n} \ell^{\alpha_0}(2^n) J(2^n, u_n) \right]^q \right)^{1/q} \leq 2 \, \|a\|_{[A_0,A_1]^J_{1,q,\alpha_0}} .$$

Then

$$\|a\|_{A_0+A_1} = K(1,a) \leq \sum_{n=-\infty}^{0} K(1,u_n) \leq \sum_{n=-\infty}^{0} \min(1, 2^{-n}) J(2^n, u_n) = \sum_{n=-\infty}^{0} J(2^n, u_n).$$

If $1 \leq q \leq \infty$, applying Hölder's inequality, we obtain that

$$\begin{aligned} \|a\|_{A_0+A_1} &\leq \left(\sum_{n=-\infty}^{0} \left[2^{-n} \ell^{\alpha_0}(2^n) J(2^n, u_n) \right]^q \right)^{1/q} \left(\sum_{n=-\infty}^{0} \left[2^n \ell^{-\alpha_0}(2^n) \right]^{q'} \right)^{1/q'} \\ &\lesssim \left(\sum_{n=-\infty}^{0} \left[2^{-n} \ell^{\alpha_0}(2^n) J(2^n, u_n) \right]^q \right)^{1/q} \lesssim \|a\|_{[A_0,A_1]^J_{1,q,\alpha_0}} . \end{aligned}$$

If $0 < q \leq 1$, we get

$$\begin{aligned} \|a\|_{A_0+A_1} &\leq \left(\sum_{n=-\infty}^{0} J(2^n, u_n)^q \right)^{1/q} \\ &\leq \left(\sum_{n=-\infty}^{0} \left[2^{-n} \ell^{\alpha_0}(2^n) J(2^n, u_n) \right]^q \right)^{1/q} \sup_{n \in \mathbf{Z}^-} \left(2^n \ell^{-\alpha_0}(2^n) \right) \\ &\lesssim \|a\|_{[A_0,A_1]^J_{1,q,\alpha_0}} . \end{aligned}$$

This proves that $[A_0, A_1]^J_{1,q,\alpha_0} \hookrightarrow A_0 + A_1$.

It is also not hard to check that these modified J-spaces are exact interpolation spaces.

Lemma 2.1. *Let* $\mathbf{A} = (\alpha_0, \alpha_\infty) \in \mathbf{R}^2$ *and* $0 < q \leq \infty$ *satisfy (2.3). Given any Banach couple* $\bar{A} = (A_0, A_1)$*, we have with equivalent quasi-norms*

$$A_1 + (A_0, A_1)^J_{1,q,\mathbf{A}} = A_1 + [A_0, A_1]^J_{1,q,\alpha_0} .$$

Proof. Let $v = a_1 + a$ with $a_1 \in A_1$ and $a \in (A_0, A_1)^J_{1,q,\mathbf{A}}$. Find $(u_m)_{m \in \mathbf{Z}} \subseteq A_0 \cap A_1$ such that $a = \sum_{m=-\infty}^{\infty} u_m$ and

$$\left(\sum_{m=-\infty}^{\infty} \left[2^{-m} \ell^{\mathbf{A}}(2^m) J(2^m, u_m) \right]^q \right)^{1/q} \leq 2 \, \|a\|_{(A_0,A_1)^J_{1,q,\mathbf{A}}} .$$

Then $w = \sum_{m=1}^{\infty} u_m$ belongs to A_1. Indeed, if $0 < q \leq 1$, we have

$$\begin{aligned} \sum_{m=1}^{\infty} \|u_m\|_{A_1} \leq \sum_{m=1}^{\infty} 2^{-m} J(2^m, u_m) &\leq \left(\sum_{m=1}^{\infty} \left[2^{-m} J(2^m, u_m) \right]^q \right)^{1/q} \\ &\leq \left(\sum_{m=1}^{\infty} \left[2^{-m} \ell^{\alpha_\infty}(2^m) J(2^m, u_m) \right]^q \right)^{1/q} \sup_{m \in \mathbf{N}} \ell^{-\alpha_\infty}(2^m) \\ &\lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbf{A}}} \end{aligned}$$

where we have used that $\alpha_\infty \geq 0$ in the last inequality. If $1 < q \leq \infty$, we proceed using Hölder's inequality. We get

$$\begin{aligned}\sum_{m=1}^{\infty} \|u_m\|_{A_1} &\leq \sum_{m=1}^{\infty} 2^{-m} J(2^m, u_m) \\ &\leq \left(\sum_{m=1}^{\infty} \left[2^{-m}\ell^{\alpha_\infty}(2^m) J(2^m, u_m)\right]^q\right)^{1/q} \left(\sum_{m=1}^{\infty} \ell^{-\alpha_\infty q'}(2^m)\right)^{1/q'} \\ &\lesssim \|a\|_{(A_0,A_1)^J_{1,q,\mathbf{A}}}\end{aligned}$$

because $\alpha_\infty - 1/q' > 0$.

Therefore, $v = (a_1 + w) + \sum_{m=-\infty}^{0} u_m$ belongs to $A_1 + [A_0, A_1]^J_{1,q,\alpha_0}$ with

$$\begin{aligned}\|v\|_{A_1+[A_0,A_1]^J_{1,q,\alpha_0}} &\leq \|a_1 + w\|_{A_1} + \left(\sum_{m=-\infty}^{0} \left[2^{-m}\ell^{\alpha_0}(2^m) J(2^m, u_m)\right]^q\right)^{1/q} \\ &\lesssim \|a_1\|_{A_1} + \|a\|_{(A_0,A_1)^J_{1,q,\mathbf{A}}} .\end{aligned}$$

This yields that

$$A_1 + (A_0, A_1)^J_{1,q,\mathbf{A}} \hookrightarrow A_1 + [A_0, A_1]^J_{1,q,\alpha_0} .$$

The converse inclusion is clear. □

Now we are ready to show the announced result for the modified J-spaces.

Theorem 2.2. *Let* $\mathbf{A} = (\alpha_0, \alpha_\infty) \in \mathbf{R}^2$ *and* $0 < q \leq \infty$ *satisfy (2.2) and (2.6). Given any Banach couple* $\bar{A} = (A_0, A_1)$ *we have with equivalence of quasi-norms*

$$(A_0, A_1)_{1,q,\mathbf{A}} = \begin{cases} A_1 + [A_0, A_1]^J_{1,q,\alpha_0+1} & \text{if } 1 \leq q \leq \infty, \\ A_1^{\sim} + [A_0^{\sim}, A_1^{\sim}]^J_{1,q,\alpha_0+1/q} & \text{if } 0 < q < 1. \end{cases}$$

Proof. The argument in the proof of [8, Lemma 2.3] for $1 \leq q \leq \infty$ is still valid for $0 < q \leq \infty$ showing that in the assumption (2.6) we have

$$\|a\|_{\bar{A}_{1,q,\mathbf{A}}} \sim \left(\int_0^1 \left[t^{-1} K(t, a) \ell^{\alpha_0}(t)\right]^q \frac{dt}{t}\right)^{1/q} . \tag{2.7}$$

In particular, if $a \in A_1$, we obtain

$$\|a\|_{\bar{A}_{1,q,\mathbf{A}}} \lesssim \left(\int_0^1 \ell^{\alpha_0 q}(t) \frac{dt}{t}\right)^{1/q} \|a\|_{A_1} \lesssim \|a\|_{A_1} .$$

This yields that

$$A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbf{A}} . \tag{2.8}$$

Take any $\alpha \in \mathbf{R}$ with $\alpha + 1/q > 0$. We claim that

$$(A_0, A_1)_{1,q,\mathbf{A}} = A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)} \tag{2.9}$$

with equivalent quasi-norms. Indeed, by (2.7) and (2.8) we have that

$$(A_0, A_1)_{1,q,(\alpha_0,\alpha)} \hookrightarrow (A_0, A_1)_{1,q,\mathbf{A}} \quad \text{and} \quad A_1 \hookrightarrow (A_0, A_1)_{1,q,\mathbf{A}} .$$

So,

$$A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)} \hookrightarrow (A_0, A_1)_{1,q,\mathbf{A}} .$$

Conversely, if $a \in (A_0, A_1)_{1,q,\mathbf{A}}$, we can write $a = a_0 + a_1$ with $a_j \in A_j$ and

$$\|a_0\|_{A_0} + \|a_1\|_{A_1} \leq 2 \|a\|_{A_0+A_1} . \tag{2.10}$$

Now we check that a_0 belongs to $(A_0, A_1)_{1,q,(\alpha_0,\alpha)}$. We have

$$\begin{aligned}\|a_0\|_{\bar{A}_{1,q,(\alpha_0,\alpha)}} &\lesssim \left(\int_1^\infty \left[t^{-1}K(t,a_0)\ell^{\alpha}(t)\right]^q \frac{dt}{t}\right)^{1/q} \\ &\quad + \left(\int_0^1 \left[t^{-1}K(t,a_1)\ell^{\alpha_0}(t)\right]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^1 \left[t^{-1}K(t,a)\ell^{\alpha_0}(t)\right]^q \frac{dt}{t}\right)^{1/q} \\ &\lesssim \left(\int_1^\infty \left[t^{-1}\ell^{\alpha}(t)\right]^q \frac{dt}{t}\right)^{1/q} \|a_0\|_{A_0} + \left(\int_0^1 \ell^{\alpha_0 q}(t)\frac{dt}{t}\right)^{1/q} \|a_1\|_{A_1} + \|a\|_{\bar{A}_{1,q,\mathbf{A}}} \\ &\lesssim \|a\|_{A_0+A_1} + \|a_1\|_{A_1} + \|a\|_{\bar{A}_{1,q,\mathbf{A}}}\end{aligned}$$

where we have used (2.10) in the last inequality. Hence,

$$\|a_0\|_{\bar{A}_{1,q,(\alpha_0,\alpha)}} \lesssim \|a_1\|_{A_1} + \|a\|_{\bar{A}_{1,q,\mathbf{A}}}\,.$$

This shows that $(A_0, A_1)_{1,q,\mathbf{A}} \hookrightarrow A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)}$ with

$$\|a\|_{A_1+(A_0,A_1)_{1,q,(\alpha_0,\alpha)}} \le \|a_1\|_{A_1} + \|a_0\|_{\bar{A}_{1,q,(\alpha_0,\alpha)}} \lesssim \|a\|_{A_0+A_1} + \|a\|_{\bar{A}_{1,q,\mathbf{A}}} \lesssim \|a\|_{\bar{A}_{1,q,\mathbf{A}}}$$

which establishes (2.9).

Combining (2.9) with (2.4) and Lemma 2.1, we conclude for $1 \le q \le \infty$ that

$$\begin{aligned}(A_0, A_1)_{1,q,\mathbf{A}} &= A_1 + (A_0, A_1)_{1,q,(\alpha_0,\alpha)} = A_1 + (A_0, A_1)^J_{1,q,(\alpha_0+1,\alpha+1)} \\ &= A_1 + [A_0, A_1]^J_{1,q,\alpha_0+1}\,.\end{aligned}$$

If $0 < q < 1$, we use (2.1), (2.9), (2.5) and Lemma 2.1 to derive

$$\begin{aligned}(A_0, A_1)_{1,q,\mathbf{A}} &= (A_0^\sim, A_1^\sim)_{1,q,\mathbf{A}} = A_1^\sim + (A_0^\sim, A_1^\sim)_{1,q,(\alpha_0,\alpha)} \\ &= A_1^\sim + (A_0^\sim, A_1^\sim)^J_{1,q,(\alpha_0+1/q,\alpha+1/q)} = A_1^\sim + [A_0^\sim, A_1^\sim]^J_{1,q,\alpha_0+1/q}\,.\end{aligned}$$

The proof is complete. □

3. Associate spaces

In what follows, (Ω, μ) is a σ-finite measure space and $\mathcal{M}$ is the collection of all (equivalence classes of) scalar valued μ-measurable functions on Ω which are finite μ-almost everywhere. We endow $\mathcal{M}$ with the topology of convergence in measure on sets of finite measure.

The notion of Banach function space as described in [1] and [9] includes the Fatou property. However, in other books one can find a similar concept but leaving out the Fatou property. See [18], [14] and [15]. In this paper we follow this last point of view.

By a *Banach function space* we mean a Banach space $(X, \|\cdot\|_X)$ of functions in $\mathcal{M}$ satisfying the following three properties:

(i) Whenever $g \in \mathcal{M}$, $f \in X$ and $|g(x)| \le |f(x)|$ μ-a.e., then $g \in X$ and $\|g\|_X \le \|f\|_X$ (lattice property).
(ii) $\chi_E \in X$ for every $E \subseteq \Omega$ with $\mu(E) < \infty$.
(iii) For every $E \subseteq \Omega$ with $\mu(E) < \infty$ there is $c_E > 0$ such that $\int_E |f|\,d\mu \le c_E \|f\|_X$ for every $f \in X$.

Clearly, simple functions are contained in X and $|||f|||_X = \|f\|_X$ for $f \in X$.

The argument in [1, p. 4] based on (iii) can still be applied with the result that

$$X \hookrightarrow \mathcal{M}\,. \tag{3.1}$$

Lebesgue spaces L_p, Lorentz spaces $L_{p,q}$, Orlicz spaces L^Φ are examples of Banach function spaces. Other examples are the generalized Lorentz–Zygmund spaces $L_{(p,q;\mathbf{A})}$ formed by all those $f \in \mathcal{M}$ satisfying that

$$\|f\|_{L_{(p,q;\mathbf{A})}} = \left(\int_0^\infty \left[t^{1/p-1}\ell^{\mathbf{A}}(t)\int_0^t f^*(s)ds\right]^q \frac{dt}{t}\right)^{1/q} < \infty \tag{3.2}$$

(the outer integral should be replaced by the supremum if $q = \infty$). Here $1 \le p \le \infty$, $1 \le q \le \infty$, $\mathbf{A} = (\alpha_0, \alpha_\infty) \in \mathbf{R}^2$ and f^* stands for the decreasing rearrangement of f, defined by

$$f^*(t) = \inf\{\delta > 0\colon \mu(\{\omega \in \Omega\colon |f(\omega)| > \delta\}) \le t\},\ t > 0.$$

Spaces $L_{(p,q;\mathbf{A})}$ make also sense if $0 < q < 1$ but then (3.2) is no longer a norm but a quasi-norm. We refer to [16] and [9] for properties of generalized Lorentz–Zygmund spaces. In order to avoid that $L_{(p,q;\mathbf{A})} = \{0\}$ one should assume that any of the following conditions holds

$$\begin{cases} 1 < p < \infty; & & \\ p = \infty, & q < \infty, & \alpha_0 + 1/q < 0; \\ p = \infty, & q = \infty, & \alpha_0 \le 0; \\ p = 1, & q < \infty, & \alpha_\infty + 1/q < 0; \\ p = 1, & q = \infty, & \alpha_\infty \le 0; \end{cases}$$

(see [16, Lemma 3.5/(ii)]).

We write $L_{(p,q;\alpha_0)}(0,1)$ if the integral in (3.2) is taken only on the interval $(0,1)$ instead of $(0,\infty)$.

The *associate space* X' of the Banach function space X consists of all $g \in \mathcal{M}$ such that

$$\int_\Omega |fg|\, d\mu < \infty \quad \text{for every } f \in X.$$

It is also a Banach function space over Ω endowed with the norm

$$\|g\|_{X'} = \sup\Big\{\int_\Omega |fg|\, d\mu\colon\ \|f\|_X \le 1\Big\}.$$

Indeed, the arguments in the proof of [1, Theorem I.2.2] can be applied to show that $(X', \|\cdot\|_{X'})$ is a normed space of functions in $\mathcal{M}$ which satisfies the corresponding versions of (i), (ii) and (iii). Moreover, using the definition of $\|\cdot\|_{X'}$, it is not hard to check that if $(g_n) \subseteq X'$ and $\sum_{n=1}^\infty \|g_n\|_{X'} < \infty$ then the function $g = \sum_{n=1}^\infty g_n$ belongs to X' and $\|g - \sum_{j=1}^n g_j\|_{X'} \to 0$ as $n \to \infty$.

We also have that $\int_\Omega |fg|\, d\mu \le \|f\|_X \|g\|_{X'}$. If $(Y, \|\cdot\|_Y)$ is a quasi-Banach space of functions in $\mathcal{M}$ such that the corresponding versions of (i), (ii) and (iii) hold, then we define Y' as above.

We let ℓ_q be the usual space of scalar q-summable sequences with indices on $\mathbf{Z}$. It is known that $\ell_q' = \ell_{q'}$ for $1 \le q \le \infty$ where $1/q + 1/q' = 1$. For later use we compute now the associate space of the quasi-Banach space ℓ_q when $0 < q < 1$.

Lemma 3.1. *Let $0 < q < 1$. Then $\ell_q' = \ell_\infty$ with equality of norms.*

Proof. Take any $\eta = (\eta_m) \in \ell_\infty$ and $\xi = (\xi_m) \in \ell_q$. It follows from

$$\sum_{m=-\infty}^{\infty} |\xi_m \eta_m| \le \left(\sum_{m=-\infty}^{\infty} |\xi_m|^q\, |\eta_m|^q\right)^{1/q} \le \|\xi\|_{\ell_q} \|\eta\|_{\ell_\infty}$$

that $\ell_\infty \hookrightarrow \ell'_q$ and that the embedding has norm less than or equal to 1. Conversely, take any $\eta = (\eta_m) \in \ell'_q$ and for $n \in \mathbf{Z}$ let $e_n = (\delta^n_m)_{m\in\mathbf{Z}}$. We have

$$|\eta_n| = \sum_{m=-\infty}^{\infty} |\eta_m \delta^n_m| \le \|\eta\|_{\ell'_q} \|e_n\|_{\ell_q} = \|\eta\|_{\ell'_q}.$$

Hence, η belongs to ℓ_∞ and $\|\eta\|_{\ell_\infty} \le \|\eta\|_{\ell'_q}$. This completes the proof. □

Let X_0, X_1 be Banach function spaces over Ω. According to (3.1), we have that $X_j \hookrightarrow \mathcal{M}$. Hence $\bar{X} = (X_0, X_1)$ is a Banach couple. Subsequently, we are going to determine the associate space of $\bar{X}_{1,q,\mathbf{A}}$ and $\bar{X}^J_{1,q,\mathbf{A}}$. We work under different assumptions on $\bar{X}$ than in [13] and [5], but ideas of those papers will be useful for our considerations.

Let $g \in \mathcal{M}$ and $f \in X_0 \cap X_1$ with $|g(x)| \le |f(x)|$ a.e., then it is clear that $g \in X_0 \cap X_1$ with $J(t, g; \bar{X}) \le J(t, f; \bar{X})$, $t > 0$. So, $X_0 \cap X_1$ is a Banach function space with the norm $J(t, \cdot; \bar{X})$. As for the K-functional, using that

$$K(t, f; \bar{X}) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : |f| \le f_0 + f_1, f_j \ge 0, f_j \in X_j\}$$

(see, for example, [7, Lemma 3.1]), it follows that $K(t, g; \bar{X}) \le K(t, f; \bar{X})$ provided that $|g| \le |f|$, $f \in X_0 + X_1$. Now it is not hard to check that $X_0 + X_1$ is also a Banach function space.

The properties above of the J- and K-functionals also yield that for $1 \le q \le \infty$ the spaces $\bar{X}_{1,q,\mathbf{A}}$, $\bar{X}^J_{1,q,\mathbf{A}}$ and $\bar{X}^J_{1,q,\mathbf{A},\mathbf{B}}$ are Banach function spaces. To check (3) one can rely on the exact interpolation property of the logarithmic interpolation methods applied to the operator $f \rightsquigarrow \int_E f\,d\mu$. If $0 < q < 1$, these spaces have also properties (i), (ii), (iii) but $\|\cdot\|_{\bar{X}_{1,q,\mathbf{A}}}$, $\|\cdot\|_{\bar{X}^J_{1,q,\mathbf{A}}}$ and $\|\cdot\|_{\bar{X}^J_{1,q,\mathbf{A},\mathbf{B}}}$ are only quasi-norms.

Let $\overline{X'} = (X'_0, X'_1)$ be the Banach couple formed by the associate spaces. If $f \in X_0 \cap X_1$, $g \in X'_0 + X'_1$ with $g = g_0 + g_1$, $g_j \in X'_j$ and $t > 0$, we have

$$\begin{aligned}\int_\Omega |fg|\,d\mu &\le \int_\Omega |fg_0|\,d\mu + \int_\Omega |fg_1|\,d\mu \le \|f\|_{X_0}\|g_0\|_{X'_0} + \|f\|_{X_1}\|g_1\|_{X'_1}\\ &\le J(t, f; \bar{X})\left(\|g_0\|_{X'_0} + t^{-1}\|g_1\|_{X'_1}\right).\end{aligned}$$

This yields

$$\int_\Omega |fg|\,d\mu \le J(t, f; \bar{X})K(t^{-1}, g; \overline{X'}),\ t > 0. \tag{3.3}$$

Furthermore, we have that

$$J(t, g; \overline{X'}) = \sup_{f\in X_0+X_1} \frac{\int_\Omega |fg|\,d\mu}{K(t^{-1}, f; \bar{X})}, \quad g \in X'_0 \cap X'_1, \quad t > 0. \tag{3.4}$$

Indeed, let $g \in X'_0 \cap X'_1$. Proceeding as to establish (3.3) we get

$$\sup_{f\in X_0+X_1} \frac{\int_\Omega |fg|\,d\mu}{K(t^{-1}, f; \bar{X})} \le J(t, g; \overline{X'}).$$

To check the converse inequality we write λX for the space X normed by $\lambda\|\cdot\|_X$. Since the embeddings $X_0 \hookrightarrow (X_0 + X_1, K(t^{-1}, \cdot))$ and $t^{-1}X_1 \hookrightarrow (X_0 + X_1, K(t^{-1}, \cdot))$ have norm less than or equal to 1, it follows that the imbeddings $(X_0+X_1, K(t^{-1}, \cdot))' \hookrightarrow X'_0$ and $(X_0 + X_1, K(t^{-1}, \cdot))' \hookrightarrow tX'_1$ have norm less than or equal to 1. Hence

$$(X_0 + X_1, K(t^{-1}, \cdot))' \hookrightarrow (X'_0 \cap X'_1, J(t, \cdot))$$

with

$$J(t, g; \overline{X'}) \leq \sup_{f \in X_0 + X_1} \frac{\int_\Omega |fg| \, d\mu}{K(t^{-1}, f; \bar{X})}.$$

This establishes (3.4).

Recall that a Banach function space X over Ω is said to have *absolutely continuous norm* if for any $f \in X$ and any decreasing sequence (E_n) of μ-measurable sets with empty intersection we have that $\|f\chi_{E_n}\|_X \downarrow 0$ as $n \to \infty$. If X has absolutely continuous norm then X' coincides with the dual space X^* of X (see [18, Theorem 15.72.5, p. 480]).

Subsequently, $\mathbf{K}$ stands for the scalar field, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

Lemma 3.2. *Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces over Ω. Suppose that X_0 or X_1 has absolutely continuous norm. Then*

$$K(t^{-1}, g; \overline{X'}) = \sup_{f \in X_0 \cap X_1} \frac{\int_\Omega |fg| \, d\mu}{J(t, f; \bar{X})}, \quad g \in X_0' + X_1', \quad t > 0. \tag{3.5}$$

Proof. Inequality

$$\sup_{f \in X_0 \cap X_1} \frac{\int_\Omega |fg| \, d\mu}{J(t, f; \bar{X})} \leq K(t^{-1}, g; \overline{X'}), \quad g \in X_0' + X_1', \quad t > 0,$$

follows from (3.3). To check the reverse inequality, consider $X_0 \cap X_1$ with the norm $J(t, \cdot; \bar{X})$, endow $X_0' + X_1'$ with the norm $K(t^{-1}, \cdot; \overline{X'})$ and take any $g \in (X_0 \cap X_1)'$. Then the functional T assigning to any $f \in X_0 \cap X_1$ the scalar $Tf = \int_\Omega fg d\mu$ belongs to $(X_0 \cap X_1)^*$ with

$$\|T\|_{(X_0 \cap X_1)^*} = \sup\left\{ \left| \int_\Omega fg \, d\mu \right| : J(t, f; \bar{X}) \leq 1 \right\} = \|g\|_{(X_0 \cap X_1)'} .$$

Consider the space $X_0 \times X_1$ normed by

$$\|(f_0, f_1)\|_{X_0 \times X_1} = \max\{\|f_0\|_{X_0}, t\,\|f_1\|_{X_1}\}$$

and put $A = \{(f_0, f_1) \in X_0 \times X_1 \colon f_0 = f_1\}$. The linear functional $F : A \to \mathbf{K}$ defined by

$$F(f_0, f_1) = T\left(\frac{f_0 + f_1}{2}\right) = \int_\Omega f_0 g \, d\mu, \ (f_0, f_1) \in A,$$

is bounded with

$$\|F\|_{A^*} = \sup\left\{ \left| \int_\Omega fg \, d\mu \right| : f \in X_0 \cap X_1 \text{ with } J(t, f; \bar{X}) \leq 1 \right\} = \|g\|_{(X_0 \cap X_1)'} .$$

According to the Hahn–Banach theorem, we can extend F to a bounded linear functional $\hat{F} \in (X_0 \times X_1)^*$ with $\|\hat{F}\|_{(X_0 \times X_1)^*} = \|g\|_{(X_0 \cap X_1)'}$. Hence, there are $L_j \in X_j^*$, $j = 0, 1$, such that

$$L_0 f_0 = \hat{F}(f_0, 0), \quad L_1 f_1 = \hat{F}(0, f_1) \ \text{ and } \ \hat{F}(f_0, f_1) = L_0 f_0 + L_1 f_1. \tag{3.6}$$

Assume that X_0 has absolutely continuous norm. Then $X_0^* = X_0'$ and so there is $g_0 \in X_0'$ such that $L_0 f_0 = \int_\Omega f_0 g_0 \, d\mu$ and $\|L_0\|_{X_0^*} = \|g_0\|_{X_0'}$. For any $f \in X_0 \cap X_1$, we have

$$\int_\Omega fg \, d\mu = \hat{F}(f, f) = L_0 f + L_1 f = \int_\Omega f g_0 \, d\mu + L_1 f.$$

Whence

$$L_1 f = \int_\Omega f(g-g_0)\,d\mu,\ \ f \in X_0 \cap X_1.$$

We claim that $g - g_0 \in X_1'$. Indeed, take any $f \in X_1$. We can find an increasing sequence of simple functions (f_n) such that $0 \le f_n \uparrow |f|$. Since $(f_n) \subseteq X_0 \cap X_1$, we get

$$\begin{aligned}\int_\Omega f_n\,|g-g_0|\,d\mu &= \left|\int_\Omega (\operatorname{sgn}(g-g_0)f_n)(g-g_0)\,d\mu\right| \\ &= |L_1(\operatorname{sgn}(g-g_0)f_n)| \le \|L_1\|_{X_1^*}\,\|f_n\|_{X_1} \le \|L_1\|_{X_1^*}\,\|f\|_{X_1}.\end{aligned}$$

Whence, using the monotone convergence theorem, we derive that

$$\int_\Omega |f|\,|g-g_0|\,d\mu = \lim_{n\to\infty}\int_\Omega f_n\,|g-g_0|\,d\mu < \infty.$$

This shows that $g_1 = g - g_0$ belongs to X_1' with $\|g_1\|_{X'} \le \|L_1\|_{X_1^*}$. So, $g = g_0 + g_1 \in X_0' + X_1'$. Moreover, given any $\varepsilon > 0$, we have

$$\begin{aligned}\|g\|_{X_0'+X_1'} - \varepsilon(1+t^{-1}) &\le \|g_0\|_{X_0'} - \varepsilon + t^{-1}(\|g_1\|_{X_1'} - \varepsilon) \\ &\le \|L_0\|_{X_0^*} - \varepsilon + t^{-1}(\|L_1\|_{X_1^*} - \varepsilon) \le |L_0 f_0| + t^{-1}\,|L_1 f_1|\end{aligned}$$

for some $f_j \in X_j$ with $\|f_j\|_{X_j} \le 1$. Therefore, using (3.6), we get

$$\begin{aligned}\|g\|_{X_0'+X_1'} - \varepsilon(1+t^{-1}) &\le L_0\left(\frac{|L_0 f_0|}{L_0 f_0} f_0\right) + L_1\left(t^{-1}\frac{|L_1 f_1|}{L_1 f_1} f_1\right) \\ &= \hat{F}\left(\frac{|L_0 f_0|}{L_0 f_0} f_0, t^{-1}\frac{|L_1 f_1|}{L_1 f_1} f_1\right) \\ &\le \|\hat{F}\|_{(X_0\times X_1)^*}\max\left(\left\|\frac{|L_0 f_0|}{L_0 f_0} f_0\right\|_{X_0}, tt^{-1}\left\|\frac{|L_1 f_1|}{L_1 f_1} f_1\right\|_{X_1}\right) \\ &\le \|\hat{F}\|_{(X_0\times X_1)^*} = \|g\|_{(X_0\cap X_1)'}.\end{aligned}$$

Letting $\varepsilon \to 0$, this yields that

$$K(t^{-1}, g; \overline{X'}) = \|g\|_{X_0'+X_1'} \le \|g\|_{(X_0\cap X_1)'} = \sup_{f\in X_0\cap X_1}\frac{\int_\Omega |fg|\,d\mu}{J(t,f;\bar{X})}$$

and completes the proof. The case when X_1 has absolutely continuous norm can be treated analogously. □

Next we determine the associate space of $\bar{X}^J_{1,q,\mathbf{A}}$. If $\mathbf{A} = (\alpha_0, \alpha_\infty) \in \mathbf{R}^2$, we put $-\mathbf{A} = (-\alpha_0, -\alpha_\infty)$ and we write $\tilde{\mathbf{A}} = (\alpha_\infty, \alpha_0)$ for the reverse pair.

Theorem 3.3. *Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces over Ω. Suppose that X_0 or X_1 has absolutely continuous norm. Let $\mathbf{A} = (\alpha_0, \alpha_\infty) \in \mathbf{R}^2$ and $0 < q \le \infty$ satisfying (2.3). Put*

$$q^* = \begin{cases} q' & \text{if } 1 \le q \le \infty, \\ \infty & \text{if } 0 < q < 1. \end{cases}$$

Then $(\bar{X}^J_{1,q,\mathbf{A}})' = \overline{X'}_{1,q^*,-\tilde{\mathbf{A}}}$.

Proof. We proceed following the lines of [13, Theorem 3.4]. Take any $g \in (\bar{X}^J_{1,q,\mathbf{A}})'$ and any $\varepsilon > 0$. According to (3.5), for any $m \in \mathbf{Z}$, there is $f_m \in X_0 \cap X_1$ such that

$$(1-\varepsilon)K(2^{-m}, |g|\,; \overline{X'}) \leq J(2^m, |f_m|\,; \bar{X})^{-1} \int_\Omega |f_m g|\, d\mu.$$

Take any sequence (δ_m) of non-negative scalars such that

$$\left(\sum_{m=-\infty}^{\infty} \left(2^{-m}\ell^{\mathbf{A}}(2^m)\delta_m\right)^q \right)^{1/q} \leq 1.$$

Put $u_m = J(2^m, |f_m|\,; \bar{X})^{-1}\delta_m\,|f_m|$. Then the function $f = \sum_{m=-\infty}^{\infty} u_m$ belongs to $\bar{X}^J_{1,q,\mathbf{A}}$ and

$$\|f\|_{\bar{X}^J_{1,q,\mathbf{A}}} \leq \left(\sum_{m=-\infty}^{\infty} \left[2^{-m}\ell^{\mathbf{A}}(2^m)J(2^m, u_m)\right]^q \right)^{1/q} \leq \left(\sum_{m=-\infty}^{\infty} \left[2^{-m}\ell^{\mathbf{A}}(2^m)\delta_m\right]^q \right)^{1/q} \leq 1.$$

Moreover,

$$\begin{aligned}(1-\varepsilon) &\sum_{m=-\infty}^{\infty} 2^m\ell^{-\mathbf{A}}(2^m)K(2^{-m}, |g|\,; \overline{X'})2^{-m}\ell^{\mathbf{A}}(2^m)\delta_m \\ &\leq \sum_{m=-\infty}^{\infty} \delta_m J(2^m, |f_m|\,; \bar{X})^{-1} \int_\Omega |f_m g|\, d\mu = \int_\Omega |fg|\, d\mu \leq \|g\|_{(\bar{X}^J_{1,q,\mathbf{A}})'}\,.\end{aligned}$$

Using that $\ell_q' = \ell_{q^*}$, we derive that $g \in \overline{X'}_{1,q^*,-\tilde{\mathbf{A}}}$ and that $\|g\|_{\overline{X'}_{1,q^*,-\tilde{\mathbf{A}}}} \leq \|g\|_{(\bar{X}^J_{1,q,\mathbf{A}})'}$.

Conversely, take any $g \in \overline{X'}_{1,q^*,-\tilde{\mathbf{A}}}$, let $f \in \bar{X}^J_{1,q,\mathbf{A}}$ and take any J-representation $f = \sum_{m=-\infty}^{\infty} f_m$ of f. By (3.3), we have

$$\int_\Omega |f_m g|\, d\mu \leq J(2^m, f_m; \bar{X})K(2^{-m}, g; \overline{X'}), \quad m \in \mathbf{Z}.$$

Hence, if $1 \leq q \leq \infty$, it follows by using Hölder's inequality that

$$\begin{aligned}\int_\Omega |fg|\, d\mu &\leq \sum_{m=-\infty}^{\infty} J(2^m, f_m; \bar{X})K(2^{-m}, g; \overline{X'}) \\ &\leq \left(\sum_{m=-\infty}^{\infty} \left[2^{-m}\ell^{\mathbf{A}}(2^m)J(2^m, f_m; \bar{X})\right]^q \right)^{1/q} \left(\sum_{m=-\infty}^{\infty} \left[2^m\ell^{-\mathbf{A}}(2^m)K(2^{-m}, g; \overline{X'})\right]^{q'} \right)^{1/q'}.\end{aligned}$$

If $0 < q < 1$, we obtain

$$\begin{aligned}\int_\Omega |fg|\, d\mu &\leq \sum_{m=-\infty}^{\infty} J(2^m, f_m; \bar{X})K(2^{-m}, g; \overline{X'}) \\ &\leq \left(\sum_{m=-\infty}^{\infty} \left[J(2^m, f_m; \bar{X})K(2^{-m}, g; \overline{X'})\right]^q \right)^{1/q} \\ &\leq \left(\sum_{m=-\infty}^{\infty} \left[2^{-m}\ell^{\mathbf{A}}(2^m)J(2^m, f_m; \bar{X})\right]^q \right)^{1/q} \sup_{m\in\mathbf{Z}}\{2^m\ell^{-\mathbf{A}}(2^m)K(2^{-m}, g; \bar{X})\}.\end{aligned}$$

Therefore, for any $0 < q \leq \infty$, we get that

$$\int_\Omega |fg|\, d\mu \leq \|f\|_{\bar{X}^J_{1,q,\mathbf{A}}}\, \|g\|_{\overline{X'}_{1,q^*,-\tilde{\mathbf{A}}}}\,.$$

This shows that g belongs to $(\bar{X}^J_{1,q,\mathbf{A}})'$ with $\|g\|_{(\bar{X}^J_{1,q,\mathbf{A}})'} \leq \|g\|_{\overline{X'}_{1,q^*,-\tilde{\mathbf{A}}}}$. The proof is complete. $\square$

Remark 3.4. For $\mathbf{A} = (\alpha_0, \alpha_\infty)$, $\mathbf{B} = (\beta_0, \beta_\infty) \in \mathbf{R}^2$ and $1 < q \leq \infty$ with $\alpha_\infty = 1/q'$ and $\beta_\infty > 1/q'$, or $0 < q \leq 1$, $\alpha_\infty = 0$ and $\beta_\infty \geq 0$, one can determine the associate space of $(X_0, X_1)^J_{1,q,\mathbf{A},\mathbf{B}}$ proceeding in a similar way. If X_0 or X_1 has absolutely continuous norm, the outcome is

$$\Big((X_0, X_1)^J_{1,q,\mathbf{A},\mathbf{B}}\Big)' = (X_0', X_1')_{1,q^*,-\tilde{\mathbf{A}},-\tilde{\mathbf{B}}} \tag{3.7}$$

where the quasi-norm in the K-space is given by

$$\|g\|_{\overline{X'}_{1,q^*,-\tilde{\mathbf{A}},-\tilde{\mathbf{B}}}} = \left(\sum_{m=-\infty}^{\infty} \Big[2^{-m}\ell^{-\tilde{\mathbf{A}}}(2^m)\ell\ell^{-\tilde{\mathbf{B}}}(2^m)K(2^m, g; \overline{X'})\Big]^{q^*}\right)^{1/q^*}.$$

Remark 3.5. The same techniques are useful to determine the associate space of the modified J-space $[X_0, X_1]^J_{1,q,\alpha_0}$. The relevant K-spaces are now

$$[X_0, X_1]^K_{1,q,\alpha} = \{f \in X_0 + X_1 \colon \|f\|_{[\bar{X}]^K_{1,q,\alpha}} = \left(\sum_{m=0}^{\infty} [2^{-m}\ell^{\alpha}(2^m)K(2^m, f; \bar{X})]^q\right)^{1/q} < \infty\}.$$

We have

$$\Big([X_0, X_1]^J_{1,q,\alpha_0}\Big)' = [X_0', X_1']^K_{1,q^*,-\alpha_0} \tag{3.8}$$

provided that X_0 or X_1 has absolutely continuous norm.

Remark 3.6. On the contrary to the duality formulae of [8] and [3] where it is essential that $A_0 \cap A_1$ is dense in A_0 and A_1, such assumption is not needed in Theorem 3.3. It suffices that X_0 or X_1 has absolutely continuous norm. Moreover, the parameter q can also take the value ∞ in the Theorem 3.3.

Next we determine the associate spaces of K-spaces. We start with the case $1 \leq q \leq \infty$. We put $1/q + 1/q' = 1$.

Theorem 3.7. *Let $\bar{X} = (X_0, X_1)$ be a couple of Banach function spaces over the space Ω. Suppose that X_0 or X_1 has absolutely continuous norm. Let $1 \leq q \leq \infty$ and $\mathbf{A} = (\alpha_0, \alpha_\infty) \in \mathbf{R}^2$ satisfy (2.2). Then we have with equivalence of norms:*

(i) *If $\alpha_\infty + 1/q > 0$, then $(\bar{X}_{1,q,\mathbf{A}})' = \overline{X'}_{1,q',-\tilde{\mathbf{A}}-1}$.*
(ii) *If $q < \infty$ and $\alpha_\infty = -1/q$, then $(\bar{X}_{1,q,\mathbf{A}})' = \overline{X'}_{1,q',-\tilde{\mathbf{A}}-1,(-1,0)}$.*
(iii) *If $\alpha_\infty + 1/q < 0$ and $q < \infty$, or $\alpha_\infty \leq 0$ and $q = \infty$, then*

$$(\bar{X}_{1,q,\mathbf{A}})' = X_1' \cap [X_0', X_1']^K_{1,q',-\alpha_0-1}\,.$$

Proof. Statements (i) and (ii) follow from (2.4), Theorem 3.3 and Remark 3.4. To prove (iii) we use Theorem 2.2, (3.4) and Remark 3.5. $\square$

Next we deal with the case $0 < q < 1$.

Theorem 3.8. *Let $\bar{X} = (X_0, X_1)$ be a Gagliardo couple of Banach function spaces over Ω. Suppose that X_0 or X_1 has absolutely continuous norm. Let $0 < q < 1$ and $\mathbf{A} = (\alpha_0, \alpha_\infty) \in \mathbf{R}^2$ satisfying (2.2). Then we have with equivalence of norms:*

(i) *If $\alpha_\infty + 1/q > 0$, $(\bar{X}_{1,q,\mathbf{A}})' = \overline{X'}_{1,\infty,-\tilde{\mathbf{A}}-1/q}$.*
(ii) *If $\alpha_\infty + 1/q = 0$, $(\bar{X}_{1,q,\mathbf{A}})' = \overline{X'}_{1,\infty,-\tilde{\mathbf{A}}-1/q,(-1/q,0)}$.*
(iii) *If $\alpha_\infty + 1/q < 0$, $(\bar{X}_{1,q,\mathbf{A}})' = X_1' \cap [X_0', X_1']^K_{1,\infty,-\alpha_0-1/q}$.*

Proof. Since $\bar{X}$ is a Gagliardo couple, we have that $X_j^\sim = X_j$, $j = 0, 1$. Statements follow from (2.5), Theorems 2.2 and 3.3, and Remarks 3.4 and 3.5. □

4. Generalized Lorentz–Zygmund spaces

First we assume that (Ω, μ) is a non-atomic σ-finite measure space and we deal with the spaces $L_{(\infty,q;\mathbf{A})}$ (see (3.2)). Their associate spaces have been determined by Opic and Pick [16, Theorem 6.2/(ii),(v) and Theorem 6.6/(ii),(iv)] by means of direct calculations. In this section we derive them from the abstract results obtained in Section 3 as an specific example.

Consider the Banach couple (L_1, L_∞) . It is well-known that

$$K(t, f; L_1, L_\infty) = \int_0^t f^*(s)\, ds, \quad t > 0,$$

(see, for example, [2, Theorem 5.2.1]). From this equality it is not hard to check that (L_1, L_∞) is a Gagliardo couple (see the comment after [1, Theorem V.1.6]). Besides, the norm of L_1 is absolutely continuous.

Theorem 4.1. *Let $1 \le q \le \infty$, $1/q + 1/q' = 1$ and $\mathbf{A} = (\alpha_0, \alpha_\infty) \in \mathbf{R}^2$ satisfying (2.2). We have with equivalence of norms:*

(i) *If $\alpha_\infty + 1/q > 0$, then $(L_{(\infty,q;\mathbf{A})})' = L_{(1,q';-\mathbf{A}-1)}$.*

(ii) *If $\alpha_\infty + 1/q < 0$ and $q < \infty$, or $\alpha_\infty \le 0$ and $q = \infty$, then*

$$(L_{(\infty,q;\mathbf{A})})' = \Big\{ g \in \mathcal{M} \colon \|g\| = \int_0^\infty g^*(t)\, dt + \|g\|_{L_{(1,q';-\alpha_0-1)}(0,1)} < \infty \Big\}.$$

(iii) *If $\alpha_\infty + 1/q = 0$ and $q < \infty$, then*

$$(L_{(\infty,q;\mathbf{A})})' = \left\{ g \in \mathcal{M} \colon \|g\| = \left(\int_0^\infty \left[\ell^{(-\alpha_0-1,-1/q')}(t) \ell\ell^{(0,-1)}(t) \int_0^t g^*(s)\, ds \right]^{q'} \frac{dt}{t} \right)^{1/q'} < \infty \right\}.$$

Proof. We have

$$(L_1, L_\infty)_{1,q,\mathbf{A}} = L_{(\infty,q;\mathbf{A})} \tag{4.1}$$

because

$$\|f\|_{(L_1,L_\infty)_{1,q,\mathbf{A}}} = \left(\int_0^\infty \left[t^{-1} \ell^{\mathbf{A}}(t) K(t, f) \right]^q \frac{dt}{t} \right)^{1/q} = \left(\int_0^\infty \left[t^{-1} \ell^{\mathbf{A}}(t) \int_0^t f^*(s)\, ds \right]^q \frac{dt}{t} \right)^{1/q} = \|f\|_{L_{(\infty,q;\mathbf{A})}}.$$

Besides, $L_1' = L_\infty$ and $L_\infty' = L_1$. Hence, it follows from Theorem 3.7/(i) that

$$\begin{aligned} \|g\|_{(L_{(\infty,q;\mathbf{A})})'} &\sim \left(\int_0^\infty \left[t^{-1} \ell^{-\tilde{\mathbf{A}}-1}(t) K(t, g; L_\infty, L_1) \right]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \left(\int_0^\infty \left[\ell^{-\tilde{\mathbf{A}}-1}(t) K(t^{-1}, g; L_1, L_\infty) \right]^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \left(\int_0^\infty \left[\ell^{-\mathbf{A}-1}(t) \int_0^t g^*(s)\, ds \right]^{q'} \frac{dt}{t} \right)^{1/q'} = \|g\|_{L_{(1,q';-\mathbf{A}-1)}}. \end{aligned}$$

This establishes (i). As for (ii), using Theorem 3.7/(iii), we get

$$\|g\|_{(L_{(\infty,q;\mathbf{A})})'} \sim \|g\|_{L_1} + \|g\|_{[L_\infty,L_1]^K_{1,q',-\alpha_0-1}}.$$

On the other hand, reversing the couple, we derive

$$\begin{aligned}\|g\|_{[L_\infty,L_1]^K_{1,q',-\alpha_0-1}} &= \left(\sum_{m=0}^{\infty}\left[2^{-m}\ell^{-\alpha_0-1}(2^m)K(2^m,g;L_\infty,L_1)\right]^{q'}\right)^{1/q'}\\ &= \left(\sum_{m=0}^{\infty}\left[\ell^{-\alpha_0-1}(2^m)K(2^{-m},g;L_1,L_\infty)\right]^{q'}\right)^{1/q'}\\ &\sim \left(\int_0^1\left[\ell^{-\alpha_0-1}(t)K(t,g;L_1,L_\infty)\right]^{q'}\frac{dt}{t}\right)^{1/q'}\\ &= \left(\int_0^1\left[\ell^{-\alpha_0-1}(t)\int_0^t g^*(s)\,ds\right]^{q'}\frac{dt}{t}\right)^{1/q'}\\ &= \|g\|_{L_{(1,q';-\alpha_0-1)}}(0,1).\end{aligned}$$

This completes the proof of (ii). Finally, for (iii), according to Theorem 3.7/(ii), we obtain

$$\begin{aligned}\|g\|_{(L_{(\infty,q,\mathbf{A})})'} &= \left(\int_0^\infty\left[t^{-1}\ell^{-\tilde{\mathbf{A}}-1}(t)\ell\ell^{(-1,0)}(t)K(t,g;L_\infty,L_1)\right]^{q'}\frac{dt}{t}\right)^{1/q'}\\ &= \left(\int_0^\infty\left[\ell^{-\tilde{\mathbf{A}}-1}(t)\ell\ell^{(-1,0)}(t)K(t^{-1},g;L_1,L_\infty)\right]^{q'}\frac{dt}{t}\right)^{1/q'}\\ &= \left(\int_0^\infty\left[\ell^{(-\alpha_0-1,-1/q')}(t)\ell\ell^{(0,-1)}(t)\int_0^t g^*(s)\,ds\right]^{q'}\frac{dt}{t}\right)^{1/q'}. \qquad \square\end{aligned}$$

Theorem 4.2. *Let $0<q<1$ and $\mathbf{A}=(\alpha_0,\alpha_\infty)\in\mathbf{R}^2$ satisfy (2.2). Then we have with equivalence of norms:*

(i) *If $\alpha_\infty+1/q>0$, $(L_{(\infty,q;\mathbf{A})})'=L_{(1,\infty;-\mathbf{A}-1/q)}$.*

(ii) *If $\alpha_\infty+1/q<0$,*

$$(L_{(\infty,q;\mathbf{A})})'=\left\{g\in\mathcal{M}\colon\ \|g\|=\int_0^\infty g^*(t)\,dt+\|g\|_{L_{(1,\infty;-\alpha_0-1/q)}(0,1)}<\infty\right\}.$$

(iii) *If $\alpha_\infty+1/q=0$,*

$$(L_{(\infty,q;\mathbf{A})})'=\left\{g\in\mathcal{M}\colon\ \|g\|=\sup_{0<t<\infty}\left(\ell^{(-\alpha_0-1/q,0)}(t)\ell\ell^{(0,-1/q)}(t)\int_0^t g^*(s)\,ds\right)<\infty\right\}.$$

Proof. One can proceed from (4.1) as in Theorem 4.1 but replacing Theorem 3.7 by Theorem 3.8. □

We close the paper by establishing the corresponding results to Theorems 4.1 and 4.2 for sequence spaces. This question has not been discussed in [16], but it can be treated as another specific application of the abstract results of Section 3.

Let $\Omega=\mathbf{N}$ and $\mu=\#$ the counting measure. Given any bounded sequence of scalars $\xi=(\xi_n)$, we put

$$\xi_n^*=\inf\{\tau>0\colon \#\{j\in\mathbf{N}\colon\ |\xi_j|\ge\tau\}<n\}.$$

The sequence (ξ_n^*) is the decreasing rearrangement of (ξ_n) by magnitude of modulus. If $\xi = (\xi_n)$ converges to zero, then

$$\xi_1^* = \max\{|\xi_n| : n \in \mathbf{N}\} = |\xi_{n_1}|, \quad \xi_2^* = \max\left\{ |\xi_n| : n \in \mathbf{N} \setminus \{n_1\}\right\} \quad \text{and so on.}$$

For $\alpha \in \mathbf{R}$, let $\ell_{(\infty,q;\alpha)}$ be the collection of all bounded sequences $\xi = (\xi_n)$ such that

$$\|\xi\|_{\ell_{(\infty,q;\alpha)}} = \left(\sum_{n=1}^{\infty} \left[\ell^{\alpha}(n) n^{-1} \sum_{j=1}^{n} \xi_j^*\right]^q n^{-1} \right)^{1/q} < \infty.$$

We put $\ell_{(1,q;\alpha)}$ for the set of all bounded sequences $\xi = (\xi_n)$ such that

$$\|\xi\|_{\ell_{(1,q;\alpha)}} = \left(\sum_{n=1}^{\infty} \left[\ell^{\alpha}(n) \sum_{j=1}^{n} \xi_j^*\right]^q n^{-1} \right)^{1/q} < \infty.$$

Replacing the weight $\ell^{\alpha}(n)$ by $\ell^{\alpha}(n)\ell\ell^{\beta}(n)$, where $\beta \in \mathbf{R}$, we obtain the spaces $\ell_{(1,q;\alpha,\beta)}$.

To determine the associate space of $\ell_{(\infty,q;\alpha)}$, we work with the Banach couple (ℓ_1, ℓ_∞). The K-functional for this couple is

$$K(n, \xi; \ell_1, \ell_\infty) = \sum_{j=1}^{n} \xi_j^*, \ n \in \mathbf{N} \tag{4.2}$$

(see [17, p. 126]). Since

$$\|\xi\|_{\ell_1^{\sim}} = \sup_{0<t<\infty} K(t, \xi) = \lim_{n\to\infty} \sum_{j=1}^{n} \xi_j^* = \|\xi\|_{\ell_1}$$

we have that (ℓ_1, ℓ_∞) is a Gagliardo couple. Moreover, ℓ_1 has absolutely continuous norm.

Theorem 4.3. *Let $0 < q \le \infty$ and $\alpha \in \mathbf{R}$. Then we have*

$$\ell'_{(\infty,q;\alpha)} = \begin{cases} \ell_{(1,q';-\alpha-1)} & \text{if } 1 \le q \le \infty \quad \text{and} \quad \alpha + 1/q > 0, \\ \ell_{(1,\infty;-\alpha-1/q)} & \text{if } 0 < q < 1 \quad \text{and} \quad \alpha + 1/q > 0, \\ \ell_{(1,q';-\alpha-1,-1)} & \text{if } 1 \le q < \infty \quad \text{and} \quad \alpha + 1/q = 0, \\ \ell_{(1,\infty;-\alpha-1/q,-1/q)} & \text{if } 0 < q < 1 \quad \text{and} \quad \alpha + 1/q = 0. \end{cases}$$

Proof. Choose $\alpha_0 \in \mathbf{R}$ with $\alpha_0 + 1/q < 0$. We have $K(t, \xi; \ell_1, \ell_\infty) \le t \|\xi\|_{\ell_\infty}$. Hence

$$\left(\int_0^1 \left[t^{-1}\ell^{\alpha_0}(t) K(t, \xi; \ell_1, \ell_\infty)\right]^q \frac{dt}{t} \right)^{1/q} \le \left(\int_0^1 \ell^{\alpha_0 q}(t) \frac{dt}{t} \right)^{1/q} \|\xi\|_{\ell_\infty} \lesssim \|\xi\|_{\ell_\infty}.$$

Therefore, using (4.2), we obtain

$$\begin{aligned} \|\xi\|_{(\ell_1,\ell_\infty)_{1,q,(\alpha_0,\alpha)}} &\sim \left(\int_1^\infty \left[t^{-1}\ell^{\alpha}(t) K(t, \xi; \ell_1, \ell_\infty)\right]^q \frac{dt}{t} \right)^{1/q} \\ &\sim \left(\sum_{n=1}^{\infty} \left[n^{-1}\ell^{\alpha}(n) \sum_{j=1}^{n} \xi_j^*\right]^q n^{-1} \right)^{1/q} \\ &= \|\xi\|_{\ell_{(\infty,q;\alpha)}}. \end{aligned}$$

Whence, if $1 \leq q \leq \infty$ and $\alpha + 1/q > 0$, according to Theorem 3.7/(i), we derive

$$\ell'_{(\infty,q;\alpha)} = (\ell_\infty, \ell_1)_{1,q',(-\alpha-1,-\alpha_0-1)}.$$

Furthermore,

$$\begin{aligned}&\left(\int_1^\infty \left[t^{-1}\ell^{-\alpha_0-1}(t)K(t,\xi;\ell_\infty,\ell_1)\right]^{q'} \frac{dt}{t}\right)^{1/q'} \\ &\lesssim \left(\int_1^\infty \left[t^{-1}\ell^{-\alpha_0-1}(t)\right]^{q'} \frac{dt}{t}\right)^{1/q'} \|\xi\|_{\ell_\infty} \lesssim \|\xi\|_{\ell_\infty}.\end{aligned}$$

This yields that

$$\begin{aligned}\|\xi\|_{(\ell_\infty,\ell_1)_{1,q',(-\alpha-1,-\alpha_0-1)}} &\sim \left(\int_0^1 \left[t^{-1}\ell^{-\alpha-1}(t)K(t,\xi;\ell_\infty,\ell_1)\right]^{q'} \frac{dt}{t}\right)^{1/q'} \\ &= \left(\int_0^1 \left[\ell^{-\alpha-1}(t)K(t^{-1},\xi;\ell_1,\ell_\infty)\right]^{q'} \frac{dt}{t}\right)^{1/q'} \\ &\sim \left(\int_1^\infty \left[\ell^{-\alpha-1}(t)K(t,\xi;\ell_1,\ell_\infty)\right]^{q'} \frac{dt}{t}\right)^{1/q'} \\ &\sim \left(\sum_{n=1}^\infty \left[\ell^{-\alpha-1}(n)\sum_{j=1}^n \xi_j^*\right]^{q'} n^{-1}\right)^{1/q'} = \|\xi\|_{\ell_{(1,q';-\alpha-1)}}.\end{aligned}$$

Consequently, $\ell'_{(\infty,q,\alpha)} = \ell_{(1,q';-\alpha-1)}$.

The case $\alpha + 1/q = 0$ with $1 \leq q < \infty$ follows similarly but using now Theorem 3.7/(ii). The remaining cases where $0 < q < 1$ can be derived analogously but replacing Theorem 3.7 by Theorem 3.8. □

In Theorem 4.3 we have not considered the case $\alpha + 1/q < 0$. The reason is that in this situation we have that $\ell_{(\infty,q;\alpha)} = \ell_\infty$. Indeed,

$$\begin{aligned}\|\xi\|_{\ell_\infty} \leq \|\xi\|_{\ell_{(\infty,q;\alpha)}} &= \left(\sum_{n=1}^\infty \left[\ell^\alpha(n)n^{-1}\sum_{j=1}^n \xi_j^*\right]^q n^{-1}\right)^{1/q} \\ &\leq \left(\sum_{n=1}^\infty \ell^{\alpha q}(n)n^{-1}\right)^{1/q} \|\xi\|_{\ell_\infty} \lesssim \|\xi\|_{\ell_\infty}.\end{aligned}$$

Therefore,

$$\ell'_{(\infty,q;\alpha)} = \ell'_\infty = \ell_1 \quad \text{for} \quad \alpha+1/q<0 \quad \text{and} \quad 0<q<\infty, \quad \text{or} \quad \alpha \leq 0 \quad \text{and} \quad q=\infty. \tag{4.3}$$

Still, this last result can be derived from Theorems 3.7/(iii) and 3.8/(iii) noting that in this case $X_1' = \ell_1$ and that $[\ell_\infty,\ell_1]^K_{1,q',-\alpha_0-1} = \ell_\infty = [\ell_\infty,\ell_1]^K_{1,\infty,-\alpha_0-1/q}$ because

$$\begin{aligned}\left(\int_1^\infty \left[t^{-1}\ell^{-\alpha_0-1}(t)K(t,\xi;\ell_\infty,\ell_1)\right]^{q'} \frac{dt}{t}\right)^{1/q'} &\leq \left(\int_1^\infty \left[t^{-1}\ell^{-\alpha_0-1}(t)\right]^{q'} \frac{dt}{t}\right)^{1/q'} \|\xi\|_{\ell_\infty} \\ &\lesssim \|\xi\|_{\ell_\infty}\end{aligned}$$

and similarly

$$\sup_{1<t<\infty} \left[t^{-1}\ell^{-\alpha_0-1/q}(t)K(t,\xi;\ell_\infty,\ell_1)\right] \lesssim \|\xi\|_{\ell_\infty}.$$

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