BI-LIPSCHITZ EMBEDDINGS OF HEISENBERG SUBMANIFOLDS INTO EUCLIDEAN SPACES

Vasileios Chousionis, Sean Li, Vyron Vellis and Scott Zimmerman

University of Connecticut, Department of Mathematics
Storrs, CT 06269-1009, U.S.A.; vasileios.chousionis@uconn.edu

University of Connecticut, Department of Mathematics
Storrs, CT 06269-1009, U.S.A.; sean.li@uconn.edu

University of Connecticut, Department of Mathematics
Storrs, CT 06269-1009, U.S.A.; vyron.vellis@uconn.edu

University of Connecticut, Department of Mathematics
Storrs, CT 06269-1009, U.S.A.; scott.zimmerman@uconn.edu

Abstract. The Heisenberg group $\mathbf{H}$ equipped with a sub-Riemannian metric is one of the most well known examples of a doubling metric space which does not admit a bi-Lipschitz embedding into any Euclidean space. In this paper we investigate which *subsets* of $\mathbf{H}$ bi-Lipschitz embed into Euclidean spaces. We show that there exists a universal constant $L > 0$ such that lines L-bi-Lipschitz embed into $\mathbf{R}^3$ and planes L-bi-Lipschitz embed into $\mathbf{R}^4$. Moreover, $C^{1,1}$ 2-manifolds without characteristic points as well as all $C^{1,1}$ 1-manifolds locally L-bi-Lipschitz embed into $\mathbf{R}^4$ where the constant L is again universal. We also consider several examples of compact surfaces with characteristic points and we prove, for example, that Korányi spheres bi-Lipschitz embed into $\mathbf{R}^4$ with a uniform constant. Finally, we show that there exists a compact, porous subset of $\mathbf{H}$ which does not admit a bi-Lipschitz embedding into any Euclidean space.

1. Introduction

Given two metric spaces (X, d_X) and (Y, d_Y), we say that a mapping $f\colon X \to Y$ is *L-bi-Lipschitz* for $L \geq 1$ if

$$L^{-1} d_X(a, b) \leq d_Y(f(a), f(b)) \leq L d_X(a, b) \quad \text{for any } a, b \in X.$$

In this case we say that X *bi-Lipschitz embeds in* Y or that X *admits a bi-Lipschitz embedding into* Y. For two metric spaces X and Y, we define the quantity

$$\mathrm{Lip}(X, Y) := \inf\{L\colon \text{ there exists } L\text{-bi-Lipschitz } f\colon X \to Y\} \in [1, \infty]$$

so that X admits a bi-Lipschitz embedding into Y if and only if $\mathrm{Lip}(X, Y) < \infty$.

When does a metric space admit a bi-Lipschitz embedding into some (finite-dimensional) Euclidean space? Spaces admitting such embeddings can roughly be thought to live inside a Euclidean space, and this is very important for a comprehensive study of their geometry. The embedding problem has attracted considerable attention over the years due to its applications in theoretical computer science and, more specifically, in graphic imaging and storage and access issues for large data sets [KV15, Nao10].

It is well know that a metric space bi-Lipschitz embeds into some Euclidean space only if it is doubling. Recall that a space is *doubling* if every ball of radius r can be covered by at most N balls of radii $r/2$ for some fixed $N > 1$. Moreover Assouad [Ass77, Ass79, Ass83] showed that, if a metric space (X, d) is doubling, then, for any $\epsilon > 0$, the snowflaked space (X, d^ϵ) bi-Lipschitz embeds into some Euclidean space. Nevertheless, the doubling condition is not sufficient for the existence of a bi-Lipschitz embedding into a Euclidean space. The Heisenberg group endowed with a sub-Riemannian metric is probably the most well known example of a doubling space which does not admit bi-Lipschitz embedding into any Euclidean space. This follows from a deep theorem of Pansu [Pan89] and an observation of Semmes [Sem96]. In fact, the second author showed that $\mathbf{H}$ does not bi-Lipschitz embed in any Hilbert space [Li16]. We also record that bi-Lipschitz embeddability of sub-Riemannian manifolds, and especially of Carnot groups, has been studied by various authors [Sem96, Wu15a, RV17, Wu15b, Rom16].

Here, we are concerned with the following question: which subsets of $\mathbf{H}$ admit a bi-Lipschitz embedding into some Euclidean space? Clearly, every finite subset of the Heisenberg group admits such an embedding. On the other hand, following the arguments of Pansu and Semmes, it is easy to see that no open subset of the Heisenberg group admits such a bi-Lipschitz embedding. We are particularly interested in the bi-Lipschitz embeddability of submanifolds of $\mathbf{H}$, which have been studied extensively in the past 20 years in connection to geometric measure theory and geometric analysis, see e.g. [SC16, CDPT07].

Our first theorem focuses on affine subsets of $\mathbf{R}^3$. Here and in what follows, d_K is the Korányi metric defined in Section 2.

Theorem 1.1. *There exists a universal $L > 1$ with the following properties.*

(1) *If $\ell \subset \mathbf{R}^3$ is a line, then $\mathrm{Lip}((\ell, d_K), \mathbf{R}^3) \leq L$.*
(2) *If $P \subset \mathbf{R}^3$ is a plane, then $\mathrm{Lip}((P, d_K), \mathbf{R}^4) \leq L$.*

We note that the Euclidean dimensions in the preceding theorem are sharp. Indeed, the z-axis in the Heisenberg group is bi-Lipschitz homeomorphic to the snowflaked space $(\mathbf{R}, |\cdot|^{1/2})$ which cannot be embedded in $\mathbf{R}^2$. On the other hand, the plane $\{x = 0\}$ is bi-Lipschitz homeomorphic to the product space $(\mathbf{R}, |\cdot|) \times (\mathbf{R}, |\cdot|^{1/2})$ which can not be embedded in $\mathbf{R}^3$.

Using a bi-Lipschitz welding theorem [LP01], we obtain the following result as a corollary.

Theorem 1.2. *If M is a compact piecewise linear 2-manifold in $\mathbf{R}^3$, then there exists $N \in \mathbf{N}$ such that $\mathrm{Lip}((M, d_K), \mathbf{R}^N) < \infty$.*

It is natural now to ask if Theorem 1.2 holds if "piecewise linear" is replaced by some degree of differentiability. Our next theorem asserts that *any* $C^{1,1}$ 1-manifold can be locally bi-Lipschitz embedded in $\mathbf{R}^4$ with a uniform constant. Moreover we prove that the same holds true for $C^{1,1}$ 2-manifolds around non-characteristic points. Recall that a point p on a differentiable manifold in $\mathbf{R}^3$ is called *characteristic* if the horizontal distribution at that point is the same as the tangent space. Otherwise, the point is called $\mathbf{H}$-*regular*. By well known results of Balogh [Bal03], almost all points of a $C^{1,1}$ smooth 2-manifold in $\mathbf{R}^3$ are $\mathbf{H}$-regular. A 2-manifold is called $\mathbf{H}$-regular if all of its points are $\mathbf{H}$-regular.

Theorem 1.3. *There exists universal $L > 1$ with the following properties.*

(1) *If $\gamma \subset \mathbf{R}^3$ is a $C^{1,1}$ 1-manifold and $x \in \gamma$, then there exists a neighborhood $U \subset \gamma$ of x such that $\mathrm{Lip}((U, d_K), \mathbf{R}^4) \leq L$.*

(2) *If $M \subset \mathbf{R}^3$ is a $C^{1,1}$ 2-manifold and $x \in M$ is $\mathbf{H}$-regular, then there exists a neighborhood $U \subset M$ of x such that $\mathrm{Lip}((U, d_K), \mathbf{R}^4) \leq L$.*

The dimension for the second part is sharp. Moreover, the embeddings are only local as there exists a $C^{1,1}$ curve γ such that $(\gamma, d_\mathbf{H})$ does not bi-Lipschitz embed into any Euclidean space (simply take a $C^{1,1}$ curve that contains $\mathbf{Z}^3$).

The case of 2-manifolds with characteristic points is more complicated, and we only give partial results. In §5, we show that every $C^{1,1}$ surface obtained by revolving a curve around the z-axis admits local bi-Lipschitz embeddings into $\mathbf{R}^4$. In particular, we show that all Korányi spheres bi-Lipschitz embed in $\mathbf{R}^4$ with a uniform constant. Such surfaces contain at most 2 characteristic points. In §5.3 we show that the surface $z = \frac{1}{2}xy$, which contains infinitely many characteristic points, bi-Lipschitz embeds in $\mathbf{R}^{19}$.

Recall that a subset E of a metric space X is *porous* if there exists $c \geq 1$ such that, for every $x \in E$ and every $r > 0$, the set $B(x, r) \setminus E$ contains a ball of radius r/c. Following the techniques of Pansu and Semmes and using an Arzelá–Ascoli argument, one can show that a non-porous subset of $\mathbf{H}$ admits no bi-Lipschitz embedding in any Euclidean or Hilbert space. For relevant results on porous sets of Carnot groups, see [PS17a] and [PS17b]. As all the curves and surfaces considered above are porous sets in $\mathbf{H}$, it is natural to ask if all compact porous subsets of $\mathbf{H}$ admit a bi-Lipschitz embedding into some Euclidean space. In §6 we answer this question negatively.

Theorem 1.4. *There exists a compact porous subset $X \subset \mathbf{H}$ that does not admit a bi-Lipschitz embedding into ℓ_2.*

The paper is organized as follows. In Section 2 we will introduce the Heisenberg group and we will prove some basic estimates for the Korányi metric which will be used repeatedly in the following. In Section 3 we will prove Theorem 1.1, while in Section 4 we consider smooth curves and smooth regular surfaces and provide the proof of Theorem 1.3. In Section 5 we study several examples of smooth surfaces with characteristic points, and, in particular, we prove that Korányi spheres bi-Lipschitz embed into $\mathbf{R}^4$ with a uniform constant. Finally in Section 6 we use Laakso graphs in order to construct a porous compact subset of $\mathbf{H}$ which does not admit a bi-Lipschitz embedding into any Euclidean space.

2. Preliminaries

Given numbers $x, y \geq 0$ and parameters $a_1, \ldots, a_n$, we write $x \lesssim_{a_1,\ldots,a_n} y$ if there exists a positive and finite constant C depending on at most $a_1, \ldots, a_n$ such that $x \leq Cy$. We write $x \simeq_{a_1,\ldots,a_n} y$ when $x \lesssim_{a_1,\ldots,a_n} y$ and $y \lesssim_{a_1,\ldots,a_n} x$. Similarly, we write $x \lesssim y$ or $x \simeq y$ to denote that the implicit constants are universal.

Recall (see [BH04, Wu15a]) that there exists $L_\Phi > 1$ and an embedding $\Phi \colon \mathbf{R} \to \mathbf{R}^3$ such that

(1) for all $x, y \in \mathbf{R}$,
$$L_\Phi^{-1}|x - y|^{1/2} \leq |\Phi(x) - \Phi(y)| \leq L_\Phi|x - y|^{1/2},$$

(2) $\Phi([0, 1]) \subset [0, 1]^3$ with $\Phi(0) = (1/2, 1/2, 0)$ and $\Phi(1) = (1/2, 1/2, 1)$,

(3) for all $t \in [0, 1]$,
$$\mathrm{dist}(\Phi(t), \partial[0, 1]^3) \leq L_\Phi^{-1} \mathrm{dist}(\Phi(t), \{(1/2, 1/2, 0), (1/2, 1/2, 1)\}).$$

That is, Φ is a bi-Lipschitz embedding of $(\mathbf{R}, |\cdot|^{1/2})$ in $\mathbf{R}^3$.

Similarly, there exists $\phi\colon \mathbf{S}^1 \to \mathbf{R}^3$ such that, for all $x, y \in \mathbf{S}^1$,

$$(2.1) \qquad L_\Phi^{-1}|x - y|^{1/2} \le |\phi(x) - \phi(y)| \le L_\Phi |x - y|^{1/2}.$$

2.1. Heisenberg group. The *Heisenberg group* is the sub-Riemannian manifold $(\mathbf{R}^3, H, g)$ where H is the *horizontal distribution* generated by the vector fields

$$X = \partial_x - \frac{1}{2}y\partial_z \quad \text{and} \quad Y = \partial_y + \frac{1}{2}x\partial_z$$

and with g given by

$$g(aX + bY, cX + dY) = ac + bd.$$

The Heisenberg group can also be thought as a group $\mathbf{H} = (\mathbf{R}^3, \cdot)$ with the group law

$$(x, y, z) \cdot (x', y', z') = \left(x + x', y + y', z + z' + \tfrac{1}{2}(xy' - x'y)\right).$$

An absolutely continuous curve $\gamma(t) = (x(t), y(t), z(t))$ with $t \in [0, T]$ is *horizontal* if $\gamma'(t) \in H_{\gamma(t)}$ for almost all $t \in [0, T]$. That is,

$$z'(t) + \tfrac{1}{2}x'(t)y(t) - \tfrac{1}{2}x(t)y'(t) = 0 \quad \text{for almost all } t \in [0, T].$$

The *Carnot–Carathéodory metric* on $\mathbf{H}$ is defined by

$$d_{cc}(p, q) = \inf_\gamma \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt$$

where the infimum is taken over all horizontal curves $\gamma\colon [0, 1] \to \mathbf{R}^3$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Given points $p = (x, y, z)$ and $q = (x', y', z')$ in $\mathbf{H}$, define the *Korányi metric*

$$d_K(p, q) = \left(\left((x - x')^2 + (y - y')^2\right)^2 + 16(z - z' + \tfrac{1}{2}(xy' - x'y))^2\right)^{1/4}.$$

It is well known that d_K is a metric, and it is also bi-Lipschitz equivalent to d_{cc} in the sense that

$$C^{-1}d_K(p, q) \le d_{cc}(p, q) \le Cd_K(p, q) \quad \text{for all } p, q \in \mathbf{H}$$

for some constant $C \ge 1$; see [CDPT07]. More often, we consider the following distance $d_\mathbf{H}$ (which is not a true metric since there is a sub-additive constant present in the triangle inequality):

$$d_\mathbf{H}(p, q) = |x - x'| + |y - y'| + \left|z - z' + \tfrac{1}{2}(xy' - x'y)\right|^{\frac{1}{2}}.$$

Though $d_\mathbf{H}$ is not a metric, it is bi-Lipschitz equivalent to d_K in the above sense. Thus, it suffices to construct bi-Lipschitz embeddings of sets $A \subset \mathbf{R}^3$ endowed with $d_\mathbf{H}$. This provides a bi-Lipschitz embedding of the set A when considered as a subset of the metric space $(\mathbf{H}, d_K)$. The distance $d_\mathbf{H}$ is related to the Euclidean metric in the following way.

Lemma 2.1. *Suppose that $R > 0$ and $w, w' \in \mathbf{R}^3$ with $\max\{|w|, |w'|\} \le R$. Then*

$$\min\{R^{1/2}, R^{-1}\}|w - w'| \lesssim d_\mathbf{H}(w, w') \lesssim \max\{1, R^{1/2}\}|w - w'|^{1/2}.$$

Proof. Note that, if $w = (x, y, z)$ and $w' = (x', y', z')$, then

$$(2.2) \qquad d_\mathbf{H}(w, w') = |x - x'| + |y - y'| + |z - z' + \tfrac{1}{2}y'(x - x') - \tfrac{1}{2}x'(y - y')|^{1/2}.$$

Therefore, by (2.2) and the triangle inequality, we have

$$d_{\mathbf{H}}(w, w') \leq |z - z'|^{1/2} + |x - x'|^{1/2}(|y'|^{1/2} + |x - x'|^{1/2})$$
$$+ |y - y'|^{1/2}(|x'|^{1/2} + |y - y'|^{1/2}) \lesssim (1 + R^{1/2})|w - w'|^{1/2}.$$

If $\max\{|x - x'|, |y - y'|\} \leq (2R)^{-1}|z - z'|$, then by (2.2)

$$d_{\mathbf{H}}(w, w') \geq (|z - z'| - \tfrac{1}{2}R|x - x'| - \tfrac{1}{2}R|y - y'|)^{1/2} \gtrsim |z - z'|^{1/2}$$
$$\gtrsim \min\{1, R^{1/2}\}|w - w'|^{1/2} \geq \min\{R^{-1/2}, R^{1/2}\}|w - w'|.$$

If $\max\{|x - x'|, |y - y'|\} > (2R)^{-1}|z - z'|$, then by (2.2)

$$d_{\mathbf{H}}(w, w') \geq |x - x'| + |y - y'| \gtrsim \min\{1, R^{-1}\}|w - w'|.$$

The proof completes by noting that $\min\{1, R^{-1}, R^{1/2}, R^{-1/2}\} = \min\{R^{-1}, R^{1/2}\}$. $\quad\square$

The length of a horizontal curve between two points can be estimated by the following lemma.

Lemma 2.2. *Let $w\colon [t_1, t_2] \to \mathbf{H}$ be a horizontal curve with $w(t) = (x(t), y(t), z(t))$. Then,*

$$d_{\mathbf{H}}(w(t_1), w(t_2)) \lesssim (\|x'(t)\|_\infty + \|y'(t)\|_\infty)\,|t_1 - t_2|.$$

Proof. We have from the absolute continuity of x and y that

$$|x(t_1) - x(t_2)| + |y(t_1) - y(t_2)| = \left|\int_{t_1}^{t_2} x'(s)\,ds\right| + \left|\int_{t_1}^{t_2} y'(s)\,ds\right|$$
$$\leq (\|x'(t)\|_\infty + \|y'(t)\|_\infty)\,|t_1 - t_2|.$$

Moreover,

$$\left|z(t_1) - z(t_2) + \frac{1}{2}x(t_1)y(t_2) - \frac{1}{2}x(t_2)y(t_1)\right|^{1/2}$$
$$= 2^{-1/2}\left|\int_{t_1}^{t_2}[x'(t)y(t) - x(t)y'(t)]\,dt + x(t_1)y(t_2) - x(t_2)y(t_1)\right|^{1/2}$$
$$= 2^{-1/2}\left|\int_{t_1}^{t_2}[x'(t)y(t) - x(t)y'(t)]\,dt - y(t_2)(x(t_2) - x(t_1)) + x(t_2)(y(t_2) - y(t_1))\right|^{1/2}$$
$$= 2^{-1/2}\left|\int_{t_1}^{t_2}[x'(t)(y(t) - y(t_2))\,dt - \int_{t_1}^{t_2} y'(t)(x(t) - x(t_2))\,dt\right|^{1/2}$$
$$\leq \left(\int_{t_1}^{t_2}|x'(t)||y(t) - y(t_2)|\,dt + \int_{t_1}^{t_2}|y'(t)||x(t) - x(t_2)|\,dt\right)^{1/2}$$
$$\leq \|x'(t)\|_\infty^{1/2}\|y'(t)\|_\infty^{1/2}|t_1 - t_2|.$$

The proof completes by noting that $\|x'(t)\|_\infty^{1/2}\|y'(t)\|_\infty^{1/2} \lesssim \|x'(t)\|_\infty + \|y'(t)\|_\infty$. $\quad\square$

Given a $C^{1,1}$ 2-manifold $M \subset \mathbf{R}^3$ and a point $p \in M$, we denote by T_pM (resp. H_p) the tangent plane of M (resp. the horizontal distribution) at p.

Definition 2.3. Let M be a $C^{1,1}$ 2-manifold. A point $p \in M$ is *characteristic* if $T_pM = H_p$. A point $p \in M$ is *regular* if it is not characteristic.

In the example of vertical planes, all points are regular. In the example of the plane $z = 0$, all points are regular except for the origin which is characteristic.

Theorem 2.4. [Bal03, Theorem 1.2] *Let $\mathscr{C}(M)$ be the set of characteristic points of a 2-manifold M in $\mathbf{R}^3$.*

(1) *If M is $C^{1,1}$ then $\mathscr{C}(M)$ has (Euclidean) Hausdorff dimension less than 2.*

(2) *If M is C^2 then $\mathscr{C}(M)$ has (Euclidean) Hausdorff dimension less or equal to 1.*

3. Lines, planes and PL complexes

In §3.1 we show the first part of Theorem 1.1 and in §3.2 the second part. In §3.3 we show Theorem 1.2.

3.1. Lines in the Heisenberg group. Here we show the first part of Theorem 1.1 which we restate in the following proposition.

Proposition 3.1. *There exists universal $L > 1$ such that, for any straight line $\ell \subset \mathbf{R}^3$, the space $(\ell, d_{\mathbf{H}})$ L-bi-Lipschitz embeds into $\mathbf{R}^3$.*

Proof. Suppose that ℓ is given by the formula

$$w(t) = (a_1 t + c_1, a_2 t + c_2, a_3 t + c_3), \quad t \in \mathbf{R},$$

with $|a_1|^2 + |a_2|^2 + |a_3|^2 = 1$. Without loss of generality we may assume that $|a_2| \le |a_1|$. Moreover, since $d_{\mathbf{H}}$ is invariant under vertical translations, we may assume that $c_3 = 0$. Consider the following cases.

Case 1. Suppose that $a_1 = a_2 = 0$. Define the map $F \colon (\ell, d_{\mathbf{H}}) \to \mathbf{R}^3$ given by $F(w(t)) = \Phi(t)$. Since $|a_3| = 1$, we have for any $\tau, \tau' \in \mathbf{R}$,

$$d_{\mathbf{H}}(w(\tau), w(\tau')) = |\tau - \tau'|^{1/2} \quad \text{and} \quad |F(w(\tau)) - F(w(\tau'))| = |\Phi(\tau) - \Phi(\tau')|.$$

Thus F is L_Φ-bi-Lipschitz where L_Φ is the bi-Lipschitz constant of Φ.

Case 2. Suppose that $2a_3 + c_2 a_1 - c_1 a_2 = 0$. That is, the line is a horizontal curve. Define $F \colon (\ell, d_{\mathbf{H}}) \to \mathbf{R}$ by $F(w(t)) = (|a_1| + |a_2|)t$. Let $\tau, \tau' \in \mathbf{R}$. Then

$$d_{\mathbf{H}}(w(\tau), w(\tau')) = (|a_1| + |a_2|)|\tau - \tau'| = |F(w(\tau)) - F(w(\tau'))|,$$

so F is in fact an isometry.

Case 3. Suppose that none of the above cases applies. Set

$$\kappa = |2a_3 + c_2 a_1 - c_1 a_2| \ne 0, \quad \lambda = |a_1| + |a_2| \ne 0, \quad \text{and } \mu = \frac{2\lambda^2}{\kappa},$$

(compare with κ, λ in Lemma 4.5 and with κ in Lemma 4.2) and define $F \colon (\ell, d_{\mathbf{H}}) \to \mathbf{R}^3$ by

$$F(w(t)) = \frac{\lambda}{\mu} \Phi\left(\mu t - [\mu t]\right) + \frac{\lambda}{\mu}\left(0, 0, [\mu t]\right)$$

where $[a]$ denotes the integer part of a.

Let $\tau, \tau' \in \mathbf{R}$, $w = w(\tau)$ and $w' = w(\tau')$. Without loss of generality, assume that $\tau' > \tau$. We have that

$$d_{\mathbf{H}}(w, w') = \lambda |\tau - \tau'| + (\kappa/2)^{1/2} |\tau - \tau'|^{1/2}.$$

We now divide Case 3 into three sub-cases.

Case 3.1. Suppose that there exist at least two integers contained in the interval $(\mu\tau, \mu\tau']$. Let

$$m = \min \mathbf{Z} \cap (\mu\tau, \mu\tau'] \quad \text{and} \quad n = \max \mathbf{Z} \cap (\mu\tau, \mu\tau'].$$

Then, $|\tau - \tau'| \ge \mu^{-1}$ which yields $(\kappa/2)^{1/2} \le \lambda |\tau - \tau'|^{1/2}$, and thus

$$\lambda |\tau - \tau'| \le d_{\mathbf{H}}(w, w') \le 2\lambda |\tau - \tau'|.$$

On the other hand, since $|\mu\tau - \mu\tau'| > 1$,

$$\frac{\mu}{\lambda}|F(w) - F(w')| \leq |\Phi(\mu\tau' - n) - \Phi(\mu\tau - (m-1))| + (n - m + 1)$$

$$\lesssim |\mu\tau - \mu\tau'|^{1/2} + (n - m)^{1/2} + 1 + (n - m) + 1$$

$$\lesssim |\mu\tau - \mu\tau'| = \mu|\tau - \tau'|.$$

Notice that $\Phi([0,1]) \subset [0,1]^3$. Thus we have

$$\frac{\mu}{\lambda}|F(w) - F(w')| \geq (n - m + 1) - |\Phi(\mu\tau' - n) - \Phi(\mu\tau - (m-1))|$$

$$\gtrsim (n - m) \gtrsim |\mu\tau - \mu\tau'| = \mu|\tau - \tau'|,$$

since $3(n - m) \geq (n - m) + 2 > |\mu\tau - \mu\tau'|$. Hence, in this case, $|F(w) - F(w')| \simeq \lambda|\tau - \tau'| \simeq d_{\mathbf{H}}(w, w')$.

Case 3.2. Suppose that there exists no integer in $(\mu\tau, \mu\tau']$. Then, $|\tau - \tau'| < \mu^{-1}$ which yields

$$(\kappa/2)^{1/2}|\tau - \tau'|^{1/2} \leq d_{\mathbf{H}}(w, w') \leq (2\kappa)^{1/2}|\tau - \tau'|^{1/2}.$$

Moreover, $[\mu\tau] = [\mu\tau']$, so

$$|F(w) - F(w')| = \frac{\lambda}{\mu}|\Phi(\mu\tau - [\mu\tau]) - \Phi(\mu\tau' - [\mu\tau'])| \simeq \frac{\lambda}{\mu}|\mu\tau - \mu\tau'|^{1/2}$$

$$= (\kappa/2)^{1/2}|\tau - \tau'|^{1/2}.$$

Case 3.3. Suppose that there exists exactly one integer n in $(\mu\tau, \mu\tau']$. Then, $|\tau' - \tau| < \kappa/\lambda^2$ which yields

$$(\kappa/2)^{1/2}|\tau - \tau'|^{1/2} \leq d_{\mathbf{H}}(w, w') < (1 + \sqrt{2})(\kappa/2)^{1/2}|\tau - \tau'|^{1/2}.$$

Set $\tau'' = n/\mu \in (\tau, \tau')$ and $w'' = w(\tau'')$. Before estimating $|F(w) - F(w')|$, we make two observations. Firstly,

$$|F(w) - F(w')| \simeq |F(w) - F(w'')| + |F(w') - F(w'')|.$$

Indeed, if $C_1 = (\lambda/\mu)[0,1]^3 + (\lambda/\mu)(0, 0, [\mu\tau])$, $C_2 = (\lambda/\mu)[0,1]^3 + (\lambda/\mu)(0, 0, [\mu\tau'])$ are two cubes, then $F(w) \in C_1$, $F(w') \in C_2$ and by (3) in the definition of Φ,

$$|F(w) - F(w')| \geq \operatorname{dist}(F(w), \partial C_1) \gtrsim |F(w) - F(w'')|.$$

Similarly, $|F(w) - F(w')| \gtrsim |F(w') - F(w'')|$.

The second observation is that $t \mapsto \Phi(\mu t - [\mu t])$ is periodic with period $1/\mu$; in particular,

$$|F(w(t + 1/\mu)) - F(w(s + 1/\mu))| = |F(w(t)) - F(w(s))|.$$

Combining these two observations, we have

$$|F(w) - F(w')| \simeq |F(w) - F(w'')| + |F(w') - F(w'')|$$

$$= \left| F\left(w\left(\tau - \frac{n-1}{\mu}\right)\right) - F\left(w\left(\frac{1}{\mu}\right)\right) \right| + \left| F\left(w\left(\tau' - \frac{n}{\mu}\right)\right) - F(w(0)) \right|$$

$$= \frac{\lambda}{\mu}|\Phi(\mu\tau - (n-1)) - \Phi(1)| + \frac{\lambda}{\mu}|\Phi(\mu\tau' - n) - \Phi(0)|$$

$$\simeq \frac{\lambda}{\mu}\left(|n - \mu\tau|^{1/2} + |\mu\tau' - n|^{1/2}\right) \simeq (\kappa/2)^{1/2}|\tau - \tau'|^{1/2}. \qquad \square$$

3.2. Planes in the Heisenberg group. We now show the second part of Theorem 1.1 which we restate in the following proposition.

Proposition 3.2. *There exists a universal $L > 1$ such that, for any plane P in $\mathbf{R}^3$, there exists an L-bi-Lipschitz embedding of $(P, d_{\mathbf{H}})$ into $\mathbf{R}^4$.*

For the proof of Proposition 3.2 we distinguish two cases.

3.2.1. Planes parallel to z-axis. Suppose that

$$P = \{(x, y, z) \in \mathbf{R}^3 \colon y = bx + c\}$$

for some $b, c \in \mathbf{R}$. Without loss of generality we may assume that $|b| \leq 1$. Otherwise we consider the plane $P' = \{(x, y, z) \in \mathbf{R}^3 \colon x = b^{-1}y + b^{-1}c\}$ and the proof is similar. Define the map $f \colon P \to \mathbf{R}^4$ with

$$f(x, bx + c, z) = \left(x, \Phi\left(z + \tfrac{c}{2}x\right)\right).$$

Then, if $w = (x, bx + c, z)$ and $w' = (x', bx' + c, z')$ we have

$$|f(w) - f(w')| \lesssim |x - x'| + \left|\Phi\left(z - \tfrac{c}{2}x\right) - \Phi\left(z' - \tfrac{c}{2}x'\right)\right|$$

$$\leq |x - x'| + |b||x - x'| + L_\Phi \left|(z - z') - \tfrac{c}{2}(x - x')\right|^{\frac{1}{2}}$$

$$\leq L_\Phi d_{\mathbf{H}}(w, w').$$

Similarly we obtain

$$|f(w) - f(w')| \geq (L_\Phi|b| + L_\Phi)^{-1} d_{\mathbf{H}}(w, w') \geq (2L_\Phi)^{-1} d_{\mathbf{H}}(w, w').$$

3.2.2. Planes not parallel to z-axis. Suppose that

$$P = \{(x, y, z) \in \mathbf{R}^3 \colon z = ax + by + c\}$$

for some $a, b, c \in \mathbf{R}$.

Define first $P_0 = \{(x, y, z) \in \mathbf{R}^3 \colon z = 0\}$ and $g \colon P_0 \to P$ with

$$g(x, y, 0) = (x + 2b, y - 2a, a(x + 2b) + b(y - 2a) + c).$$

Note that g is an isometry with respect to $d_{\mathbf{H}}$. Indeed, if $w = (x, y, 0)$ and $w' = (x', y', 0)$ then

$$d_{\mathbf{H}}(g(w), g(w')) = |x - x'| + |y - y'| + \left|\tfrac{1}{2}(x'y - xy')\right|^{1/2} = d_{\mathbf{H}}(w, w').$$

Therefore, it suffices to prove Proposition 3.2 for the plane P_0.

Given a point $w \in \mathbf{R}^2$ there exist $t \geq 0$ and $s \in \mathbf{S}^1$ such that $w = ts$; if $w \neq (0, 0)$, then $t > 0$ and s are unique. Define now the embedding $F \colon P_0 \to \mathbf{R}^4$ by $F(ts, 0) = (t\,\phi(s), t)$ where ϕ is defined as in (2.1). The following two simple lemmas imply that F is bi-Lipschitz.

Lemma 3.3. *For each $i \in \{1, 2\}$ let $s_i \in \mathbf{S}^1$, $t_i \geq 0$ and $w_i = (s_i t_i, 0) \in \mathbf{R}^3$. Then,*

$$d_{\mathbf{H}}(w_1, w_2) \simeq |t_1 - t_2| + (t_1 \wedge t_2)|s_1 - s_2|^{1/2}.$$

Proof. Without loss of generality assume that $t_1 \leq t_2$. For $i = 1, 2$ write $s_i = (\cos(\theta_i), \sin(\theta_i))$ for some $\theta_i \in [0, 2\pi)$. Then,

$$d_{\mathbf{H}}(w_1, w_2) \simeq |s_1 t_1 - s_2 t_2| + (t_1 t_2)^{1/2}|\sin(\theta_1 - \theta_2)|^{1/2}$$

$$\simeq (t_2 - t_1) + t_1|s_1 - s_2| + (t_1 t_2)^{1/2}|\sin(\theta_1 - \theta_2)|^{1/2}.$$

If $t_2 \geq 2t_1$, then $t_1 \leq (t_2 - t_1)$, so $(t_1 t_2)^{1/2} \lesssim (t_2 - t_1)$ and the claim holds.

If $t_2 < 2t_1$ and $|\theta_1 - \theta_2| \in (\pi/2, 3\pi/2)$, then $|s_1 - s_2| \simeq 1$, $t_1 \simeq t_2$ and the claim holds.

If $t_2 < 2t_1$ and $|\theta_1 - \theta_2| \in [0, \pi/2] \cup [3\pi/2, 2\pi]$, then $|s_1 - s_2| \simeq |\sin(\theta_1 - \theta_2)|$, $t_1 \simeq t_2$, and $|s_1 - s_2| \lesssim |s_1 - s_2|^{1/2}$, and so the claim holds. $\qquad\square$

Lemma 3.4. *If $t_1, t_2 \geq 0$ and $s_1, s_2 \in \mathbf{S}^1$ then*

$$|(t_1\phi(s_1), t_1) - (t_2\phi(s_2), t_2)| \simeq |t_1 - t_2| + (t_1 \wedge t_2)|s_1 - s_2|^{1/2}.$$

Proof. Indeed,

$$\begin{aligned}
|(t_1\phi(s_1), t_1) - (t_2\phi(s_2), t_2)| &\simeq |t_1 - t_2| + |t_1\phi(s_1) - t_2\phi(s_2)| \\
&\simeq |t_1 - t_2| + (t_1 \wedge t_2)|\phi(s_1) - \phi(s_2)| \\
&\simeq |t_1 - t_2| + (t_1 \wedge t_2)|s_1 - s_2|^{1/2}. \qquad\square
\end{aligned}$$

3.3. Simplicial complexes. Recall that a 0-simplex is a point of $\mathbf{R}^3$, a 1-simplex $E \subset \mathbf{R}^3$ is the convex hull of two distinct points and a 2-simplex $E \subset \mathbf{R}^3$ is the convex hull of three distinct points in $\mathbf{R}^3$ that do not lie on the same line. The boundary of a k-simplex consists of j-faces ($j \leq k - 1$) which are j-simplices.

A (compact) simplicial 2-complex $K \subset \mathbf{R}^3$ is a finite union of simplices $S_1 \cup \cdots \cup S_k$ with the following properties

(1) each S_i is a k_i-simplex for some $k_1 \in \{0, 1, 2\}$;
(2) for each $i, j \in \{1, \ldots, n\}$, either $S_i \cap S_j$ is empty, or $S_i \cap S_j$ is a common edge of S_i and S_j, or $S_i = S_j$.

Theorem 3.5. *Let $K \subset \mathbf{R}^3$ be a finite union $K = D_1 \cup \cdots \cup D_k$ where each D_i is a simplicial 2-complex. Then, there exists $N \in \mathbf{N}$ and a bi-Lipschitz embedding of $(K, d_\mathbf{H})$ into $\mathbf{R}^N$.*

For the proof of Theorem 3.5 we use the following bi-Lipschitz welding theorem.

Lemma 3.6. [LP01, Theorem 3.2] *Let X be a metric space and let $X_1, X_2 \subset X$ be closed subsets such that $X = X_1 \cup X_2$. If X_1 bi-Lipschitz embeds in $\mathbf{R}^n$ and X_2 bi-Lipschitz embeds in $\mathbf{R}^m$ then X bi-Lipschitz embeds in $\mathbf{R}^{n+m+1}$.*

We now show Theorem 3.5.

Proof of Theorem 3.5. By Lemma 3.6 we only need to show that a simplicial 2-complex admits a bi-Lipschitz embedding into some $\mathbf{R}^N$. Let $K \subset \mathbf{R}^3$ be a simplicial 2-complex. Then, $K = \bigcup_{i=1}^n S_i$ where each S_i is a k_i-simplex for some $k_i \in \{0, 1, 2\}$. By Proposition 3.2, each $(S_i, d_\mathbf{H})$ L-bi-Lipschitz embeds into $\mathbf{R}^4$ for some universal $L > 1$. Applying Lemma 3.6 n times we obtain a bi-Lipschitz embedding of $(K, d_\mathbf{H})$ into $\mathbf{R}^{5n-1}$. $\qquad\square$

4. Smooth curves and H-regular surfaces

In §4.1 we show the first part of Theorem 1.3, and in §4.2 we prove the second part. For a set $A \subset \mathbf{R}^3$, we will say that $(A, d_\mathbf{H})$ *locally L-bi-Lipschitz embeds* in $\mathbf{R}^N$ for some $L > 1$ if, for any $p \in A$, there is a neighborhood U of p such that $(U \cap A, d_\mathbf{H})$ L-bi-Lipschitz embeds in $\mathbf{R}^N$.

4.1. Smooth curves. For the rest of §4.1, we denote by I a non-degenerate interval of $\mathbf{R}$. We prove the following proposition.

Proposition 4.1. *There is a universal constant $L > 1$ such that, for any $C^{1,1}$ curve $w : I \to \mathbf{R}^3$ with unit speed, $(w(I), d_\mathbf{H})$ locally L-bi-Lipschitz embeds into $\mathbf{R}^4$.*

The idea of the proof is to decompose the curve $\gamma = w(I)$ into two types of sub-arcs and embed each one in $\mathbf{R}^4$. In the following lemma we consider sub-arcs of γ that are "vertical" enough.

Lemma 4.2. *Let $w \colon [0, T] \to \mathbf{R}^3$ be a $C^{1,1}$ smooth curve with unit speed such that $|2z'(t) + x'(t)y(t) - y'(t)x(t)| \geq 1$ for all $t \in [0, T]$. Then, $(w([0, T]), d_{\mathbf{H}})$ locally L-bi-Lipschitz embeds into $\mathbf{R}^3$ for some universal $L > 1$.*

Proof. Let $M > 0$ be such that, for all $t \in [0, T]$,

$$1 \leq |2z'(t) + x'(t)y(t) - x(t)y'(t)| \leq M.$$

By compactness of $[0, T]$ and uniform continuity of $w'(t)$, there exists $\epsilon \in (0, 1)$ such that if $t_0, \ldots, t_5 \in [0, T]$ are such that $|t_i - t_j| \leq \epsilon$, then

$$(4.1) \qquad \frac{1}{2} \leq \frac{|2z'(t_1) + x'(t_2)y(t_3) - x(t_4)y'(t_5)|}{|2z'(t_0) + x'(t_0)y(t_0) - x(t_0)y'(t_0)|} \leq 2.$$

We claim that, for any $t, t' \in [0, T]$ with $|t - t'| < \epsilon$, there is an L-bi-Lipschitz map between $(w([t, t']), d_{\mathbf{H}})$ and $([t, t'], \lambda | \cdot |^{1/2})$ for some constant $\lambda \geq 1$ and a universal constant $L > 1$. Assuming the claim, we see that $(w([0, T]), d_{\mathbf{H}})$ locally L-bi-Lipschitz embeds into $\mathbf{R}^3$ for some universal L.

To prove the claim, fix $\tau_1, \tau_2 \in [0, T]$ such that $0 < \tau_2 - \tau_1 \leq \epsilon$. Fix also $t_0 \in [\tau_1, \tau_2]$ and set

$$\kappa = |2z'(t_0) + x'(t_0)y(t_0) - x(t_0)y'(t_0)| \in [1, M].$$

By the Mean Value Theorem, (4.1), and the fact that $\epsilon < 1 \leq \kappa^{1/2}$, if $t_1, t_2 \in [\tau_1, \tau_2]$, there exist $c_1, \ldots, c_5 \in (t_1, t_2)$ such that

$$d_{\mathbf{H}}(w(t_1), w(t_2))$$
$$= |x(t_1) - x(t_2)| + |y(t_1) - y(t_2)| + |z(t_1) - z(t_2) + \tfrac{1}{2}(x(t_1)y(t_2) - x(t_2)y(t_1))|^{1/2}$$
$$= (|x'(c_1)| + |y'(c_2)|)|t_1 - t_2| + |z'(c_3) + \tfrac{1}{2}x'(c_4)y(t_2) - \tfrac{1}{2}x(t_2)y'(c_5)||t_1 - t_2|^{1/2}$$
$$\simeq |z'(t_0) + \tfrac{1}{2}x'(t_0)y(t_0) - \tfrac{1}{2}x(t_0)y'(t_0)||t_1 - t_2|^{1/2}$$
$$= \kappa^{1/2}|t_1 - t_2|^{1/2}.$$

Hence, $(w([\tau_1, \tau_2]), d_{\mathbf{H}})$ is L-bi-Lipschitz equivalent to $([\tau_1, \tau_2], \kappa^{1/2}| \cdot |^{1/2})$ and the claim follows. $\qquad \square$

Proof of Proposition 4.1. Fix $t_1, t_2 \in I$ with $t_1 < t_2$. We show that the space $(w([t_1, t_2]), d_{\mathbf{H}})$ locally bi-Lipschitz embeds in $\mathbf{R}^4$.

Let $M > 1$ be such that

$$\max\{|x(t)|, |y(t)|\} \leq M \quad \text{for all } t \in [t_1, t_2].$$

Let $\epsilon_0 = (4M)^{-2}$. Note that if $|x'(t)|^2 + |y'(t)|^2 \leq \epsilon_0$, then, since w has unit speed in $\mathbf{R}^3$, we have

$$(4.2) \qquad |2z'(t) + x'(t)y(t) - x(t)y'(t)| \geq 2|z'(t)| - |x'(t)||y(t)| - |x(t)||y'(t)|$$
$$\geq 2|z'(t)| - 2M\sqrt{\epsilon_0}$$
$$= 2\sqrt{1 - |x'(t)|^2 - |y'(t)|^2} - 2M\sqrt{\epsilon_0}$$
$$\geq 2\sqrt{1 - \epsilon_0} - 2M\sqrt{\epsilon_0} \geq 1.$$

By uniform continuity of $w'|_{[t_1, t_2]}$ and compactness of $[t_1, t_2]$, there exist finitely many closed intervals $I_1, \ldots, I_k \subset [t_1, t_2]$ such that

(1) $\bigcup_{j=1}^{k} I_j = [t_1, t_2]$,

(2) for each $j \in \{1, \ldots, k\}$, either $|x'(t)|^2 + |y'(t)|^2 \leq \epsilon_0$ for all $t \in I_j$ or $|x'(t)|^2 + |y'(t)|^2 \geq \epsilon_0/2$ for all $t \in I_j$.

If $|x'(t)|^2 + |y'(t)|^2 \leq \epsilon_0$ for all $t \in I_j$, then (4.2) and Lemma 4.2 imply that $(w(I_j), d_{\mathbf{H}})$ locally bi-Lipschitz embeds in $\mathbf{R}^3$.

If $|x'(t)|^2 + |y'(t)|^2 \geq \epsilon_0/2$ for all $t \in I_j$, then the projection $\hat{w}(I_j)$ of $w(I_j)$ on $\mathbf{R}^2 \times \{0\}$ is a $C^{1,1}$ smooth curve which can be re-parameterized so that $|x'(t)|^2 + |y'(t)|^2 = 1$. The surface $(\hat{w}(I_j) \times \mathbf{R}, d_{\mathbf{H}})$ is $C^{1,1}$ and consists only of $\mathbf{H}$-regular points. Hence, we will see in Proposition 4.3 that $(\hat{w}(I_j) \times \mathbf{R}, d_{\mathbf{H}})$ locally bi-Lipschitz embeds in $\mathbf{R}^4$ with a universal constant. In particular, this means that $(w(I_j), d_{\mathbf{H}})$ locally bi-Lipschitz embeds in $\mathbf{R}^4$. $\qquad\square$

4.2. H-regular surfaces. Here we show the second part of Theorem 1.3 which we restate in the following proposition.

Proposition 4.3. *There exists a universal constant $L > 1$ satisfying the following. Let $M \subset \mathbf{R}^3$ be a $C^{1,1}$ 2-manifold in $\mathbf{R}^3$ and let p_0 be an $\mathbf{H}$-regular point of M. Then there exist a neighborhood $U \subset \mathbf{R}^3$ of p_0 and an L-bi-Lipschitz embedding of $(U \cap M, d_{\mathbf{H}})$ in $\mathbf{R}^4$.*

The proof of this proposition is comprised by the following two lemmas. In Lemma 4.4, we show that M can be foliated near p_0 by horizontal curves with C^1 dependence on parameters. In Lemma 4.5, we construct a local embedding of any surface which possesses such a foliation. In fact, Lemma 4.5 will be used in several other arguments throughout the rest of this paper when a foliation by horizontal curves can be constructed.

Lemma 4.4. *There exists a universal constant $L' > 1$ satisfying the following. Let M be a $C^{1,1}$ 2-manifold in $\mathbf{R}^3$ and let p_0 be an $\mathbf{H}$-regular point of M. Then there exist $\epsilon > 0$ and an L'-bi-Lipschitz map $G = (g_1, g_2, g_3) \colon [-\epsilon, \epsilon]^2 \to (M, |\cdot|)$ such that*

(1) $G(0,0) = p_0$
(2) $f = (g_1, g_2)$ *is a C^1 diffeomorphism on $[-\epsilon, \epsilon]^2$*
(3) *for all $v \in [-\epsilon, \epsilon]$, the curve $u \mapsto G(u, v)$ is a horizontal curve.*

Proof. In a neighborhood of $p_0 = (x_0, y_0, z_0)$, the manifold M can be written as the level set $H(x, y, z) = 0$ for some $C^{1,1}$ function H with $|\nabla H(x_0, y_0, z_0)| > 0$. Without loss of generality, we assume that

$$|H_z(x_0, y_0, z_0)| \geq \max\{|H_x(x_0, y_0, z_0)|, |H_y(x_0, y_0, z_0)|\}.$$

The other two cases are similar and are left to the reader.

Then, in a sufficiently small neighborhood U of p_0 we can write

$$M \cap U = \{(x, y, F(x, y)) \colon (x, y) \in V\}$$

for some open $V \subset \mathbf{R}^2$ where F is a $C^{1,1}$ function with $F(x_0, y_0) = z_0$ and

$$\sup_{(x,y) \in V} \{|F_x(x, y)|, |F_y(x, y)|\} \leq 2.$$

Since p_0 is $\mathbf{H}$-regular, either $x_0 - 2F_y(x_0, y_0) \neq 0$ or $y_0 + 2F_x(x_0, y_0) \neq 0$. Without loss of generality, we assume that $x_0 - 2F_y(x_0, y_0) \neq 0$ and that

(4.3) $$|y_0 + 2F_x(x_0, y_0)| \leq |x_0 - 2F_y(x_0, y_0)|.$$

There exists a neighborhood $V_1 \subset V$ of (x_0, y_0) such that $x - 2F_y \neq 0$ in V_1. Thus the horizontal curves in a neighborhood of p_0 in M are given by the ODE

$$(4.4) \qquad \frac{dy}{dx} = \frac{y + 2F_x}{x - 2F_y}.$$

By standard existence and uniqueness theorems (e.g. [CL55, Theorem 1.2] and [CL55, Theorem 2.2]), there is some $\epsilon_1 > 0$ such that, for each $v \in [-\epsilon_1, \epsilon_1]$, there exists a unique solution $y = g_v(x)$ of (4.4) defined on $[x_0 - \epsilon_1, x_0 + \epsilon_1]$ such that $g_v(x_0) = y_0 + v$. Moreover, the function

$$f \colon [-\epsilon_1, \epsilon_1]^2 \to \mathbf{R}^2 \quad \text{with} \quad f(u, v) := (x_0 + u, g_v(x_0 + u))$$

is C^1 [CL55, Theorem 7.1]. By existence and uniqueness of solutions, there is a neighborhood of (x_0, y_0) in which the graphs of g_v do not intersect, and every point in this neighborhood is contained in the graph of some g_v. Moreover, the derivative of f at $(0, 0)$ satisfies

$$Df(0,0) = \begin{bmatrix} 1 & 0 \\ \frac{d}{du} g_0(x_0 + u)|_{u=0} & \frac{d}{dv} g_v(x_0)|_{v=0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{y_0 + 2F_x(x_0, y_0)}{x_0 - 2F_y(x_0, y_0)} & 1 \end{bmatrix}.$$

Thus, for some $\epsilon_2 > 0$, f is a C^1 diffeomorphism in $[-\epsilon_2, \epsilon_2]^2$, and, by (4.3), there exists a universal constant $L_1 > 1$ such that

$$(4.5) \qquad L_1^{-1} |(u, v) - (u', v')| \le |f(u, v) - f(u', v')| \le L_1 |(u, v) - (u', v')|$$

for all $(u, v), (u', v') \in [-\epsilon_2, \epsilon_2]^2$.

Define now $G \colon [-\epsilon_2, \epsilon_2]^2 \to M$ by $G(u, v) = (f(u, v), F(f(u, v)))$ which has been defined so that the curves $u \mapsto G(u, v)$ are horizontal curves contained in M for any $v \in [-\epsilon_2, \epsilon_2]$. Moreover

$$\begin{aligned} L_1^{-1} |(u, v) - (u', v')| &\le |G(u, v) - G(u', v')| \\ &\lesssim (L_1 + \|F_x\|_\infty + \|F_y\|_\infty) |(u, v) - (u', v')| \\ &\le (L_1 + 2) |(u, v) - (u', v')|, \end{aligned}$$

and hence G is L'-bi-Lipschitz for some universal $L' > 1$. $\qquad \square$

Lemma 4.5. *There exists a universal constant $L > 1$ satisfying the following. Fix $M \subset \mathbf{R}^3$. Suppose there exist $\epsilon > 0$ and an L'-bi-Lipschitz map $G = (g_1, g_2, g_3) \colon [-\epsilon, \epsilon]^2 \to M$ (where L' is the universal constant from Lemma 4.4) such that*

(1) $G(0, 0) = p_0$,
(2) $f = (g_1, g_2)$ *is a C^1 diffeomorphism on $[-\epsilon, \epsilon]^2$,*
(3) *for all $v \in [-\epsilon, \epsilon]$, the curve $u \mapsto G(u, v)$ is a horizontal curve.*

Set

$$\begin{aligned} \kappa &:= |2\partial_v g_3(0, 0) + g_2(0, 0)\partial_v g_1(0, 0) - g_1(0, 0)\partial_v g_2(0, 0)|, \\ \lambda &:= |\partial_u g_1(0, 0)| + |\partial_u g_2(0, 0)|. \end{aligned}$$

Define the map

$$\Psi \colon (G([-\epsilon, \epsilon]^2), d_{\mathbf{H}}) \to \mathbf{R}^4 \quad \text{with} \quad \Psi(G(u, v)) = (\lambda u, \kappa^{1/2}\Phi(v)).$$

Then there is a neighborhood $U \subset \mathbf{R}^3$ of p_0 such that $\Psi \colon (U \cap M, d_{\mathbf{H}}) \to \mathbf{R}^4$ is L-bi-Lipschitz.

Proof. We first note that $\lambda > 0$ since the curve $\gamma_1(u) = G(u, 0)$ is horizontal. Moreover, since G is L'-bi-Lipschitz, λ is bounded from above by a constant C depending only on L'. On the other hand, the curve $\gamma_2(v) = G(0, v)$ intersects γ_1 transversely at p_0 and hence cannot be horizontal. Thus, $\kappa > 0$. Fix $(u_1, v_1), (u_2, v_2) \in [-\epsilon, \epsilon]^2$ and set

$$p_1 = G(u_1, v_1), \quad p_2 = G(u_2, v_2), \quad p_3 = G(u_1, v_2).$$

That is, p_2 and p_3 lie along the same horizontal curve $u \mapsto G(u, v_2)$.

Choosing ϵ possibly smaller, we have for some $c_1, c_2, c_3 \in [-\epsilon, \epsilon]$

$$d_{\mathbf{H}}(p_1, p_3) \simeq |g_1(u_1, v_1) - g_1(u_1, v_2)| + |g_2(u_1, v_1) - g_2(u_1, v_2)|$$
$$+ \left|g_3(u_1, v_1) - g_3(u_1, v_2) + \tfrac{1}{2}g_1(u_1, v_1)g_2(u_1, v_2) - \tfrac{1}{2}g_2(u_1, v_1)g_1(u_1, v_2)\right|^{1/2}$$
$$= (|\partial_v g_1(u_1, c_1)| + |\partial_v g_2(u_1, c_2)|)|v_1 - v_2|$$
$$+ \left|\partial_v g_3(u_1, c_3) + \tfrac{1}{2}\partial_v g_1(u_1, c_1)g_2(u_1, v_2) - \tfrac{1}{2}\partial_v g_2(u_1, c_2)g_1(u_1, v_2)\right|^{1/2}$$
$$\cdot |v_1 - v_2|^{1/2} \lesssim \kappa^{1/2}|v_1 - v_2|^{1/2}.$$

Additionally, by Lemma 2.2 we have

$$(4.6) \qquad d_{\mathbf{H}}(p_2, p_3) \lesssim (\|\partial_u g_1\|_\infty + \|\partial_u g_2\|_\infty)|u_1 - u_2| \lesssim \lambda|u_1 - u_2|$$

assuming that ϵ is sufficiently small. By the triangle inequality,

$$d_{\mathbf{H}}(p_1, p_2) \lesssim \lambda|u_1 - u_2| + \kappa^{1/2}|v_1 - v_2|^{1/2} \simeq |\Psi(p_1) - \Psi(p_2)|.$$

We now show that $d_{\mathbf{H}}(p_1, p_2) \gtrsim |\Psi(p_1) - \Psi(p_2)|$. By (4.6), there exists a universal constant C_0 such that

$$(4.7) \qquad \begin{aligned} &\left|g_3(u_1, v_2) - g_3(u_2, v_2) + \tfrac{1}{2}g_1(u_1, v_2)g_2(u_2, v_2) - \tfrac{1}{2}g_2(u_1, v_2)g_1(u_2, v_2)\right|^{1/2} \\ &\leq C_0\lambda|u_1 - u_2|. \end{aligned}$$

Consider the following two cases.

Case 1. Suppose that $C_0\lambda|u_1 - u_2| \geq (\kappa/16)^{1/2}|v_1 - v_2|^{1/2}$. Since G is L'-bi-Lipschitz,

$$|\Psi(p_1) - \Psi(p_2)| \simeq \lambda|u_1 - u_2| + \kappa^{1/2}|v_1 - v_2|^{1/2} \lesssim \lambda|u_1 - u_2| \lesssim d_{\mathbf{H}}(p_1, p_2).$$

Case 2. Suppose that $C_0\lambda|u_1 - u_2| < (\kappa/16)^{1/2}|v_1 - v_2|^{1/2}$. Then,

$$d_{\mathbf{H}}(p_1, p_2)$$
$$\gtrsim \lambda|u_1 - u_2| + |g_3(u_1, v_1) - g_3(u_2, v_2) + \tfrac{1}{2}g_1(u_1, v_1)g_2(u_2, v_2) - \tfrac{1}{2}g_2(u_1, v_1)g_1(u_2, v_2)|^{1/2}.$$

Now,

$$g_3(u_1, v_1) - g_3(u_2, v_2) + \tfrac{1}{2}g_1(u_1, v_1)g_2(u_2, v_2) - \tfrac{1}{2}g_2(u_1, v_1)g_1(u_2, v_2)$$
$$= g_3(u_1, v_1) - g_3(u_1, v_2) + \tfrac{1}{2}g_2(u_2, v_2)(g_1(u_1, v_1) - g_1(u_1, v_2))$$
$$- \tfrac{1}{2}g_1(u_2, v_2)(g_2(u_1, v_1) - g_2(u_1, v_2))$$
$$+ g_3(u_1, v_2) - g_3(u_2, v_2) + \tfrac{1}{2}g_1(u_1, v_2)g_2(u_2, v_2) - \tfrac{1}{2}g_2(u_1, v_2)g_1(u_2, v_2).$$

Again assuming that ϵ is sufficiently small,

$$\left| g_3(u_1, v_1) - g_3(u_1, v_2) + \tfrac{1}{2}g_2(u_2, v_2)(g_1(u_1, v_1) - g_1(u_1, v_2)) \right.$$
$$\left. - \tfrac{1}{2}g_1(u_2, v_2)(g_2(u_1, v_1) - g_2(u_1, v_2)) \right|$$
$$= \left| \partial_v g_3(u_1, c_3) + \tfrac{1}{2}g_2(u_2, v_2)\partial_v g_1(u_1, c_1) - \tfrac{1}{2}g_1(u_2, v_2)\partial_v g_2(u_1, c_2) \right| |v_1 - v_2|$$
$$\geq \kappa/2 |v_1 - v_2|.$$

Thus, by (4.7)

$$d_{\mathbf{H}}(p_1, p_2) \gtrsim \lambda|u_1 - u_2| + (\kappa/2)^{1/2}|v_1 - v_2|^{1/2} - C_0\lambda|u_1 - u_2|$$
$$\gtrsim \lambda|u_1 - u_2| + \kappa^{1/2}|v_1 - v_2|^{1/2}$$
$$\simeq |\Psi(p_1) - \Psi(p_2)|.$$

Therefore, Ψ is bi-Lipschitz in a neighborhood of p_0. $\qquad\square$

4.3. The smooth torus. Although the set of characteristic points of a given smooth manifold is very small, it rarely is empty. By the Hairy–Ball Theorem and the classification of 2-manifolds, if a compact differentiable manifold M has no characteristic points, then it must be homeomorphic to the torus $\mathbf{S}^1 \times \mathbf{S}^1$. On the other hand, a topological torus could be $\mathbf{H}$-regular. Given numbers $0 < r < R$, define the torus

$$\mathbf{T}_{r,R} = \{(x, y, z) \colon (R - \sqrt{x^2 + y^2})^2 + z^2 = r^2\}.$$

We claim that $\mathbf{T}_{r,R}$ has no characteristic points. Recall first that both the horizontal distribution and the tangent space of $\mathbf{T}_{r,R}$ are invariant under rotations with respect to the z-axis in the following sense. If P_w (resp. $\hat{P}_w$) is the affine space containing $w \in \mathbf{T}_{r,R}$ and generated by $T_w\mathbf{T}_{r,R}$ (resp. generated by H_w) and if $\zeta \colon \mathbf{R}^3 \to \mathbf{R}^3$ is a rotation with respect to z-axis, then $\zeta(P_w) = P_{\zeta(w)}$ and $\zeta(\hat{P}_w) = \hat{P}_{\zeta(w)}$. To show the claim, suppose that a point $w \in \mathbf{T}_{r,R}$ is characteristic and write $w = (r_0, \theta_0, z_0)$ in cylindrical coordinates. By the rotation invariance of the tangent space and the horizontal distribution, every point of the circle $S = \{(r_0, \theta, z_0) \colon \theta \in [0, 2\pi)\}$ is characteristic. Therefore, S is horizontal. However, by §3.2, $(S, d_{\mathbf{H}})$ is bi-Lipschitz equivalent to $(\mathbf{S}^1, |\cdot|^{1/2})$, which is a contradiction.

Let V be a smooth unitary vector field on $\mathbf{T}_{r,R}$ such that $V(w) \in T_w\mathbf{T}_{r,R} \cap H_w$ for all $w \in \mathbf{T}_{r,R}$. By the Frobenius Theorem, V is integrable and $\mathbf{T}_{r,R}$ is foliated by the (closed) integral curves of V which are obtained by rotating one of them with respect to the z-axis. Specifically, there exists a diffeomorphism $G \colon \mathbf{S}^1 \times \mathbf{S}^1 \to \mathbf{T}_{r,R}$ such that $s \mapsto G(s, \eta)$ is a horizontal curve for each $\eta \in \mathbf{S}^1$.

We now claim that $(\mathbf{T}_{r,R}, d_{\mathbf{H}})$ bi-Lipschitz embeds in $\mathbf{R}^4$. Indeed, say τ is an identification of $\mathbf{R}^2/\mathbf{Z}^2$ with $\mathbf{S}^1 \times \mathbf{S}^1$ and define $\Psi \colon \mathbf{T}_{r,R} \to \mathbf{R}^4$ so that

$$\Psi(G(\tau(u, v))) = (u, \Phi(v)) \quad \text{for all } (u, v) \in \mathbf{R}^2/\mathbf{Z}^2.$$

According to Lemma 4.5, there is a constant $L > 1$ such that, for any $p \in \mathbf{T}_{r,R}$, there exists a neighborhood U_p of p in $\mathbf{T}_{r,R}$ wherein $\Psi|_{(U_p, d_{\mathbf{H}})}$ is L-bi-Lipschitz. By the compactness of $\mathbf{T}_{r,R}$, there exists $\epsilon > 0$ such that Ψ is bi-Lipschitz on all compact sets of $\mathbf{T}_{r,R}$ with (Euclidean) diameter at most ϵ. On the other hand, by Lemma 2.1, for all $w, w' \in \mathbf{T}_{r,R}$ with $|w - w'| \geq \epsilon$ we have

$$d_{\mathbf{H}}(w, w') \simeq_\epsilon 1 \simeq_\epsilon |\Psi(w) - \Psi(w')|.$$

Therefore, Ψ is bi-Lipschitz.

5. Some smooth surfaces with characteristic points

Here we examine the bi-Lipschitz embeddability of some special classes of smooth surfaces that contain characteristic points.

5.1. Surfaces obtained by revolution about the z-axis. Let $F\colon [0,\infty) \to \mathbf{R}$ be a $C^{1,1}$ function. Let Σ be the surface of revolution generated by F:

$$\Sigma = \{(w, F(|w|))\colon w \in \mathbf{R}^2\}.$$

Below we denote by $\mathbf{0}$ the origins $(0,0)$ and $(0,0,0)$, depending on the context.

Theorem 5.1. *Suppose that F is $C^{1,1}$ with $F'(0) = 0$, and set*

$$M := \max\left\{1, \limsup_{t\downarrow 0} \frac{|F'(t)|}{t}\right\}.$$

There exist $\epsilon > 0$, $L = L(M) > 1$, and a neighborhood $U \subset \Sigma$ of $(0,0,F(0))$ such that $(U, d_{\mathbf{H}})$ L-bi-Lipschitz embeds in $\mathbf{R}^4$.

Before proving this theorem, we observe an application of this embedding. Combined with Lemma 3.6 and the $\mathbf{H}$-regularity of $\Sigma \setminus \{(0,0,F(0))\}$, Theorem 5.1 yields the following corollary.

Corollary 5.2. *Suppose that F is $C^{1,1}$ with $F'(0) = 0$. Any compact subset of $(\Sigma, d_{\mathbf{H}})$ bi-Lipschitz embeds in some Euclidean space $\mathbf{R}^n$.*

Proof. We first show that all points of $\Sigma \setminus \{(0,0,F(0))\}$ are regular points. Indeed, suppose that $(x,y) \neq (0,0)$ is a characteristic point of Σ. Then,

$$y = -2x\frac{F'(|(x,y)|)}{|(x,y)|} \quad \text{and} \quad x = 2y\frac{F'(|(x,y)|)}{|(x,y)|}.$$

If $x = 0$ or $y = 0$, then both must be equal to 0, and this is a contradiction. However, if $x \neq 0$ and $y \neq 0$, then $y/x = x/y$ which implies that $x^2 + y^2 = 0$, and this is again a contradiction. Thus the only characteristic point of Σ is $(0,0,F(0))$.

Let S be a compact subset of Σ. By Theorem 5.1, Proposition 4.3, and the compactness of S, there exist open sets $D_1, \ldots, D_k$ such that each D_i bi-Lipschitz embeds into $\mathbf{R}^4$ and $S \subset \bigcup_{i=1}^k D_i$. By Lemma 3.6, we obtain a bi-Lipschitz embedding of $\bigcup_{i=1}^k D_i$ (and thus of S) into $\mathbf{R}^{5k-1}$. $\qquad\square$

We now prove Theorem 5.1. The idea of the proof is as follows. In Lemma 5.3, we construct a bi-Lipschitz homeomorpism of a disc centered at the origin in $\mathbf{R}^2$ such that any radial segment in the disc is mapped to a curve which is the projection of a horizontal curve in Σ. Since Σ is a surface of revolution, rotations of this horizontal curve foliate the surface. We then complete the proof of Theorem 5.1 by re-parameterizing Σ in terms of the horizontal curve (which embeds into $\mathbf{R}$) and the snowflaked unit circle (which embeds into $\mathbf{R}^3$).

Lemma 5.3. *Let $F\colon [0, +\infty) \to \mathbf{R}$ be as in Theorem 5.1. There exist $T > 0$, $L = L(M) > 1$, and an L-bi-Lipschitz map $G : B^2(0,T) \to B^2(0,T)$ with $G(0) = 0$, such that for each $s \in \mathbf{S}^1$, the curve*

$$t \mapsto (G(ts), F(|G(ts)|))$$

is horizontal. Moreover, $|G(st)| = |G(s't)|$ for any $s, s' \in \mathbf{S}^1$ and $t \in [0, T)$.

Proof. For each $t \geq 0$ consider the point $w_t = (t, 0, F(t))$. Then

$$T_{w_t}\Sigma \cap H_{w_t} = \left\{ \left(x, 2\frac{F'(t)}{t}(x - t), F(t) + F'(t)(x - t) \right) : x \in \mathbf{R} \right\}.$$

Consider now the vector field $V \colon \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}^2$ given by

$$V(ts) = s + 2(F'(t)/t)\hat{s}$$

where $t > 0$, $s, \hat{s} \in \mathbf{S}^1$ and $\hat{s}$ is the rotation of s by an angle of $\pi/2$. Note that V has been defined in such a way that, when any integral curve γ of V is lifted to a curve on Σ, this lifted curve is horizontal. That is, the curve $t \mapsto (\gamma(t), F(\gamma(t)))$ is a horizontal curve.

Fix $T > 0$ such that $|F'(t)|/t \leq 2M$ for every $t \in (0, T)$. That is,

$$|V(p)| = \sqrt{1 + \left(2\frac{F'(|p|)}{|p|} \right)^2} \in [1, 5M] \quad \text{for all } p \in B^2(\mathbf{0}, T) \setminus \{\mathbf{0}\}.$$

Then, for all $p \in B^2(\mathbf{0}, T) \setminus \{\mathbf{0}\}$ we have

(5.1) $$V(p) \cdot p = |p| \geq (5M)^{-1}|p||V(p)|.$$

From (5.1) we have that if γ is an integral curve of V, then

$$(|\gamma|^2)' = \gamma' \cdot \gamma = |\gamma| = (|\gamma|^2)^{1/2}$$

which implies that $(|\gamma|)' = 1/2$. Therefore, given $0 < t_1 \leq t_2$ in the domain of γ,

(5.2) $$|\gamma(t_2)| - |\gamma(t_1)| = \frac{1}{2}(t_2 - t_1).$$

We construct a particular integral curve in $B(\mathbf{0}, T)$ as follows. For each $i \in \mathbf{N}$ let S_i be the circle $\partial B^2(\mathbf{0}, 2^{1-i}T)$, and let A_i be the closed annulus $A(\mathbf{0}; 2^{-i}T, 2^{1-i}T)$. By (5.2), every integral curve of $V|_{A_i}$ is a C^1 curve that joins S_i with S_{i+1} defined on an interval of length $2^{1-i}T$.

For each $i \in \mathbf{N}$, let $\tilde{\gamma}^{(i)}$ be the integral curve of $V|_{A_i}$ joining S_{i+1} to the point $(2^{1-i}T, 0)$ defined on the interval $[2^{1-i}T, 2^{2-i}T]$. The vector field V is defined in such a way that rotating $\tilde{\gamma}^{(i)}$ produces all integral curves of $V|_{A_i}$. Appropriately rotating each $\tilde{\gamma}^{(i)}$, we obtain a curve γ_0 that joins $\mathbf{0}$ with the point $(T, 0)$ such that $\tilde{\gamma}|_{A_i}$ is an integral curve of $V|_{A_i}$. Note that, by a limiting argument, (5.2) also holds if one or both of the t_i is zero. Hence the domain of γ_0 is $[0, 2T]$.

For each $s \in \mathbf{S}^1$, let $\gamma_s(t) : [0, 2T] \to \mathbf{R}^2$ be the curve obtained by rotating γ_0 in such a way that $\gamma_s(2T) = Ts \in \partial B^2(\mathbf{0}, T)$. By the rotation invariance of V, the integral curves of V are exactly the curves $\{\gamma_s(t) : s \in \mathbf{S}^1\}$ which join $\mathbf{0}$ with $\partial B^2(\mathbf{0}, T)$. As discussed above, the lift of each γ_s to Σ is a horizontal curve since $\gamma_s'(t) = V(\gamma_s(t))$.

Define now $G \colon B^2(0, T) \to B^2(0, T)$ with $G(ts) = \gamma_s(2t)$. To show that G is bi-Lipschitz, fix $s_1, s_2 \in \mathbf{S}^1$ and $0 \leq t_1 \leq t_2 \leq T$. By the triangle inequality, the Mean Value Theorem, and (5.2), for some $c \in [2t_1, 2t_2]$,

$$\begin{aligned}
|G(t_1 s_1) - G(t_2 s_2)| &\leq |G(t_2 s_1) - G(t_2 s_2)| + |G(t_1 s_1) - G(t_2 s_1)| \\
&= |\gamma_0(2t_1)||s_1 - s_2| + |\gamma_{s_1}(2t_1) - \gamma_{s_1}(2t_2)| \\
&= t_1|s_1 - s_2| + |\gamma_{s_1}'(c)||t_1 - t_2| \\
&\leq t_1|s_1 - s_2| + 5M|t_1 - t_2| \simeq_M |t_1 s_1 - t_2 s_2|.
\end{aligned}$$

For the other inequality we consider two cases.

Case 1. Suppose that $10M|t_1 - t_2| \geq t_1|s_1 - s_2|$. By (5.2),

$$|G(t_1 s_1) - G(t_2 s_2)| \geq |G(t_2 s_2)| - |G(t_1 s_1)| = |\gamma_0(2t_2)| - |\gamma_0(2t_1)|$$
$$= |t_1 - t_2| \gtrsim_M |s_1 t_1 - s_2 t_2|.$$

Case 2. Suppose that $10M|t_1 - t_2| < t_1|s_1 - s_2|$. By (5.2) and the Mean Value Theorem, for some $c \in [2t_1, 2t_2]$,

$$|G(t_1 s_1) - G(t_2 s_2)| \geq |G(t_2 s_1) - G(t_2 s_2)| - |G(t_2 s_1) - G(t_1 s_1)|$$
$$= |\gamma_0(2t_2)||s_1 - s_2| - |\gamma'_{s_1}(c)||t_1 - t_2|$$
$$\geq t_1|s_1 - s_2| - 5M|t_1 - t_2|$$
$$\geq \tfrac{1}{2}t_1|s_1 - s_2| \gtrsim_M |s_1 t_1 - s_2 t_2|. \qquad \square$$

We can now prove Theorem 5.1.

Proof of Theorem 5.1. For a point $w \in \mathbf{R}^3$, we denote by $\pi(w)$ the projection of w on $\{z = 0\}$. Since $d_{\mathbf{H}}$ is invariant under vertical translations, we may assume that $F(0) = 0$. Define

$$\Sigma_0 = \{w \in \Sigma \colon \pi(w) \in B^2(\mathbf{0}, T)\}$$

and choose $\epsilon > 0$ such that $B^3(\mathbf{0}, \epsilon) \cap \Sigma \subset \Sigma_0$. Define now the homeomorphism

$$\Psi \colon \{z = 0\} \cap B^3(\mathbf{0}, T) \to \Sigma_0 \quad \text{as} \quad \Psi(st, 0) = (G(st), F(|G(st)|))$$

where $s \in \mathbf{S}^1$ and $t \in [0, T]$. We show that, if T is chosen small enough, then the map Ψ is L-bi-Lipschitz for some $L = L(M) > 1$.

Fix $s_1, s_2 \in \mathbf{S}^1$, $0 < t_1 \leq t_2 \leq T$ and points $w_1 = (s_1 t_1, 0)$ and $w_2 = (s_2 t_2, 0)$. By Lemma 3.3 it suffices to show that

$$d_{\mathbf{H}}(\Psi(w_1), \Psi(w_2)) \simeq_M (t_2 - t_1) + t_1|s_1 - s_2|^{1/2}.$$

For the rest of the proof denote $w'_1 = \Psi(w_1)$ and $w'_2 = \Psi(w_2)$. We write $G(st) = (x(st), y(st))$. Recall that $G(st) = G(s't)$ lie on the same circle centered at the origin. Therefore, since $d_{\mathbf{H}}$ and $G(st)$ are both invariant under rotations, we may assume that $x(s_1 t_1) = x(s_2 t_1)$. Then

$$d_{\mathbf{H}}(w'_1, w'_2) \simeq |G(s_1 t_1) - G(s_2 t_2)|$$

$$+ \left| F(|G(s_1 t_1)|) - F(|G(s_2 t_2)|) + \tfrac{1}{2}x(s_1 t_1)y(s_2 t_2) - \tfrac{1}{2}x(s_2 t_2)y(s_1 t_1) \right|^{1/2}$$

$$(5.3) \qquad \simeq |t_1 - t_2| + t_1|s_1 - s_2|$$

$$+ \left| F(|G(s_2 t_1)|) - F(|G(s_2 t_2)|) + \tfrac{1}{2}x(s_2 t_1)y(s_2 t_2) - \tfrac{1}{2}x(s_2 t_2)y(s_2 t_1) \right.$$

$$+ \left. \tfrac{1}{2}x(s_2 t_2)(y(s_2 t_1) - y(s_1 t_1)) \right|^{1/2}.$$

Since the points $(G(s_2 t_1), F(|G(s_2 t_1)|))$ and $(G(s_2 t_2), F(|G(s_2 t_2)|))$ lie on the horizontal curve $t \mapsto (G(s_2 t), F(|G(s_2 t)|))$, Lemma 2.2 gives a constant $C_1 = C_1(M) > 0$ such that

$$(5.4) \qquad \left| F(|G(s_2 t_1)|) - F(|G(s_2 t_2)|) + \tfrac{1}{2}x(s_2 t_1)y(s_2 t_2) - \tfrac{1}{2}x(s_2 t_2)y(s_2 t_1) \right|^{1/2}$$

$$\lesssim \|\partial_t G(s_2 t)\|_\infty |t_1 - t_2| = \|V(st)\|_\infty |t_1 - t_2| \leq C_1|t_1 - t_2|$$

Set $\epsilon = \min\{2^{-3/2}, (2C_1)^{-1}, (12L^2)^{-1}\}$ where L is the bi-Lipschitz constant of G. (Note that ϵ depends only on M.) Also, we may choose $\delta > 0$ sufficiently small so that $|x(s_2 t_1)| > |y(s_2 t_1)|$ whenever $|s_1 - s_2| < \delta$. Indeed, we assumed earlier that $x(s_1 t_1) = x(s_2 t_1)$ so that the points $G(s_1 t_1)$ and $G(s_2 t_1)$ are reflections of one another

across the x-axis. Since G is $L(M)$-bi-Lipschitz, such a δ can be chosen depending only on M. Therefore, $|x(s_2 t_1)| \geq \frac{1}{2}|G(s_2 t_1)|$ when $|s_1 - s_2| < \delta$. We consider the following cases.

Case 1. $|s_1 - s_2| \geq \delta$. By Lemma 3.3, $d_{\mathbf{H}}(w_1, w_2) \simeq_\delta t_2$. On the other hand,

$$d_{\mathbf{H}}(w_1', w_2') \geq |G(s_1 t_1) - G(s_2 t_2)| \gtrsim_M |s_1 t_1 - s_2 t_2| \simeq |t_1 - t_2| + t_1|s_1 - s_2| \simeq_\delta t_2.$$

Moreover, there exist $0 < \xi_1 < |G(s_1 t_1)|$ and $0 < \xi_2 < |G(s_2 t_2)|$ such that

$$d_{\mathbf{H}}(w_1', w_2') \leq d_{\mathbf{H}}(w_1', 0) + d_{\mathbf{H}}(w_2', 0)$$
$$= |G(s_1 t_1)| + |G(s_2 t_2)| + |F(|G(s_1 t_1)|) - F(0)|^{1/2} + |F(|G(s_2 t_2)|) - F(0)|^{1/2}$$
$$\lesssim_M (t_1 + t_2) + |F'(\xi_1)|^{1/2}|G(s_1 t_1)|^{1/2} + |F'(\xi_2)|^{1/2}|G(s_2 t_2)|^{1/2}.$$

Assuming now that T is sufficiently small we have

$$d_{\mathbf{H}}(w_1', w_2') \lesssim_M t_2 + M^{1/2}(|\xi_1|^{1/2}|G(s_1 t_1)|^{1/2} + |\xi_2|^{1/2}|G(s_2 t_2)|^{1/2})$$
$$\lesssim_M t_2 + (|G(s_1 t_1)| + |G(s_2 t_2)|) \lesssim_M t_2.$$

Therefore, $d_{\mathbf{H}}(w_1', w_2') \simeq_{M,\delta} t_2 \simeq_{M,\delta} |w_1 - w_2|$.

Case 2. $|s_1 - s_2| < \delta$ and $|t_1 - t_2| \geq \epsilon t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2}$. Note that

$$|x(s_2 t_2)(y(s_2 t_1) - y(s_1 t_1))|^{1/2} \lesssim_M |x(s_2 t_2)|^{1/2} t_1^{1/2}|s_1 - s_2|^{1/2}$$
$$\lesssim_M t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2} \lesssim_M |t_1 - t_2|.$$

Hence by (5.3), (5.4), and Lemma 3.3

$$d_{\mathbf{H}}(w_1', w_2') \lesssim_M |t_1 - t_2| + t_1|s_1 - s_2| \lesssim |t_1 - t_2| + t_1|s_1 - s_2|^{1/2} \lesssim d_{\mathbf{H}}(w_1, w_2).$$

Moreover,

$$d_{\mathbf{H}}(w_1, w_2) \lesssim |t_1 - t_2| + t_1|s_1 - s_2|^{1/2} \leq |t_1 - t_2| + t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2}$$
$$\lesssim_M |t_1 - t_2| \lesssim_M d_{\mathbf{H}}(w_1', w_2').$$

Case 3. $|s_1 - s_2| < \delta$ and $|t_1 - t_2| < \epsilon t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2}$. By (5.3), (5.4), and the choice of ϵ,

$$d_{\mathbf{H}}(w_1', w_2') \lesssim_M |t_1 - t_2| + t_1|s_1 - s_2| + |x(s_2 t_2)|^{1/2}|y(s_2 t_1) - y(s_1 t_1)|^{1/2} + |t_1 - t_2|$$
$$\lesssim |t_1 - t_2| + t_1|s_1 - s_2| + |x(s_1 t_2)|^{1/2}|y(s_2 t_1) - y(s_1 t_1)|^{1/2}$$
$$+ |x(s_2 t_2) - x(s_1 t_2)|^{1/2}|y(s_2 t_1) - y(s_1 t_1)|^{1/2}$$
$$\lesssim_M |t_1 - t_2| + t_1^{1/2} t_2^{1/2}|s_1 - s_2| + t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2} + t_1^{1/2} t_2^{1/2}|s_1 - s_2|$$
$$\lesssim |t_1 - t_2| + t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2}$$

while

$$d_{\mathbf{H}}(w_1, w_2) \gtrsim |t_1 - t_2| + t_1|s_1 - s_2|^{1/2}$$
$$= |t_1 - t_2| + t_2|s_1 - s_2|^{1/2} - (t_2 - t_1)|s_1 - s_2|^{1/2}$$
$$> |t_1 - t_2| + t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2} - \sqrt{2}\epsilon t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2}$$
$$\geq |t_1 - t_2| + t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2}.$$

For the other direction, note that, $|s_1 - s_2| < \delta$ gives $|x(s_2 t_1)| \geq \frac{1}{2}|G(s_2 t_1)|$. Thus,

$$|x(s_2 t_2)| \geq |x(s_2 t_1)| - |x(s_2 t_2) - x(s_2 t_1)| \geq \frac{1}{2}|G(s_2 t_1)| - |G(s_2 t_2) - G(s_2 t_1)|$$
$$\geq \frac{1}{2}|G(s_2 t_2)| - \frac{3}{2}|G(s_2 t_2) - G(s_2 t_1)| \geq \frac{1}{2L} t_2 - \frac{3L}{2}|t_1 - t_2| > \frac{1}{4L} t_2$$

since $\epsilon \leq 1/(12L^2)$. Therefore,

$$
\begin{aligned}
d_{\mathbf{H}}(w_1', w_2') &\gtrsim_M |t_1 - t_2| + \tfrac{1}{2}|x(s_2 t_2)|^{1/2}|y(s_2 t_1) - y(s_1 t_1)|^{1/2} - C_1|t_1 - t_2| \\
&\gtrsim_M |t_1 - t_2| + t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2} - C_1|t_1 - t_2| \\
&> |t_1 - t_2| + t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2} - C_1 \epsilon t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2} \\
&\geq |t_1 - t_2| + t_1^{1/2} t_2^{1/2}|s_1 - s_2|^{1/2} \geq |t_1 - t_2| + t_1|s_1 - s_2|^{1/2} \gtrsim d_{\mathbf{H}}(w_1, w_2)
\end{aligned}
$$

and the proof is complete. $\qquad\square$

5.2. Korányi spheres. Following the ideas in §5.1, we show directly that the Korányi sphere

$$
\mathbf{S}_K = \{p \in \mathbf{H} \colon d_K(p, 0) = 1\}
$$

admits a bi-Lipschitz embedding into $\mathbf{R}^4$. (Note that this is the sphere in the metric d_K rather than $d_{\mathbf{H}}$.) Since the metric is left invariant on $\mathbf{H}$ and commutes with dilations, this shows that any Korányi sphere admits a bi-Lipschitz embedding into $\mathbf{R}^4$.

Proposition 5.4. *There exists a bi-Lipschitz embedding of $(\mathbf{S}_K, d_{\mathbf{H}})$ into $\mathbf{R}^4$.*

Since the proof is similar to that of Theorem 5.1, we only sketch the steps and leave the details to the reader.

Proof. Given $s \in \mathbf{S}^1$ and $t \in [-1, 1]$, we define

$$
\ell_s(t) := (s(1 - |t|), t) \in \mathbf{R}^3 \quad \text{and} \quad \mathcal{C} = \{\ell_s(t) \colon t \in [-1, 1], s \in \mathbf{S}^1\}.
$$

The arc $\ell_s([-1, 1])$ is the *longitude of $\mathcal{C}$ in the direction of s* with $\ell_s(-1)$ (resp. $\ell_s(1)$) being the south pole $\mathcal{S} = (0, 0, -1)$ (resp. north pole $\mathcal{N} = (0, 0, 1)$) for all $s \in \mathbf{S}^1$. Given $\epsilon \in (0, 1)$, define the sets

$$
\mathcal{N}(\epsilon) = \{\ell_s(t) \colon s \in \mathbf{S}^1, 1 - \epsilon \leq t \leq 1\} \quad \text{and} \quad \mathcal{S}(\epsilon) = \{\ell_s(t) \colon s \in \mathbf{S}^1, -1 \leq t \leq -1 + \epsilon\}.
$$

Define also the two poles of $\mathbf{S}_K$, $\mathcal{N}^* = (0, 0, 1/4)$ and $\mathcal{S}^* = (0, 0, -1/4)$.

Working as in Lemma 5.3, we can show that there exists a bi-Lipschitz homeomorphism $G \colon \mathcal{C} \to \mathbf{S}_K$ such that $G(\mathcal{S}) = \mathcal{S}^*$, $G(\mathcal{N}) = \mathcal{N}^*$ and every longitude $\ell_s([-1, 1])$ is mapped to a horizontal curve. In fact every curve $t \mapsto G(\ell_s(t))$ is an integral curve of the vector field

$$
\begin{aligned}
&V(x, y, z) \\
&= \frac{1}{\sqrt{4(x^2 + y^2) + (x^2 + y^2)^4}} \left(-8xz - 2y(x^2 + y^2), -8yz + 2x(x^2 + y^2), (x^2 + y^2)^2\right)
\end{aligned}
$$

which lies in the intersection of $T_p \mathbf{S}_K$ and H_p at any $p = (x, y, z) \in \mathbf{S}_K$.

Define now $\Psi \colon (\mathbf{S}_K, d_{\mathbf{H}}) \to \mathbf{R}^4$ by

$$
\Psi(G(s(1 - |t|), t) = (t, (1 - |t|)\phi(s))
$$

where ϕ is the bi-Lipschitz embedding of $(\mathbf{S}^1, |\cdot|^{1/2})$ into $\mathbf{R}^3$. By §3.2.2 and Theorem 5.1, we know that Ψ is bi-Lipschitz on a neighborhood of $\mathcal{N}$ and on a neighborhood of $\mathcal{S}$. Moreover, for all $w \in \mathbf{S}_K \setminus \{\mathcal{N}^*, \mathcal{S}^*\}$, we can use the result of Lemma 4.5 to see that there exists a neighborhood of w on $\mathbf{S}_K$ on which Ψ is bi-Lipschitz with a universal constant. By the compactness of $\mathbf{S}_K$, there exist $\epsilon > 0$ and $L > 1$ such that Ψ is L-bi-Lipschitz on each $B(p, \epsilon) \cap \mathbf{S}_K$ with $p \in \mathbf{S}_K$. On the other hand, by Lemma 2.1, if $w, w' \in \mathbf{S}_K$ with $|w - w'| \geq \epsilon$, then

$$
d_{\mathbf{H}}(w, w') \simeq_\epsilon 1 \simeq_\epsilon |\Psi(w) - \Psi(w')|. \qquad\square
$$

We may follow very similar arguments to prove the following result.

Proposition 5.5. *There exists a bi-Lipschitz embedding of* $(\mathbf{S}^2, d_{\mathbf{H}})$ *into* $\mathbf{R}^4$.

As with Proposition 5.4, one needs to construct a bi-Lipschitz mapping $G\colon \mathbf{S}^2 \to \mathbf{S}^2$ such that every longitude $\ell_s([-1,1])$ is mapped to a horizontal curve. Here, the vector field V is defined as

$$V(x,y,z) = \frac{1}{\sqrt{5(x^2+y^2)}}(xz + 2y, yz - 2x, -y^2 - x^2).$$

The rest of the proof follows in the same way as above.

5.3. The surface $z = \frac{1}{2}xy$. The manifolds considered in §5.1 and in §5.2 are manifolds that contain at most two characteristic points. Now we consider the surface $z = xy/2$ which has infinitely many characteristic points, namely all points along the x-axis. Solving the ODE for dx/dy in (4.4), we obtain the horizontal curves

$$\left\{(c, t, \tfrac{1}{2}ct)\colon t \in \mathbf{R}\right\} \text{ for } c \in \mathbf{R} \quad \text{and} \quad \{(t, 0, 0)\colon t \in \mathbf{R}\}.$$

On the other hand, for any $a \in \mathbf{R}$ and $c \in \mathbf{R}$, the curve

$$\{(t, c, ct/2)\colon t \in [a, a+c]\}$$

is a snowflake; that is, it is bi-Lipschitz homeomorphic to a rescaled copy of $\Phi([0,1])$ where $\Phi\colon \mathbf{R} \to \mathbf{R}^3$ is the $\frac{1}{2}$-snowflaking map from §2. The construction of the following embedding resembles a general construction by Seo [Seo11].

Proposition 5.6. *The metric space* $(\{z = \frac{1}{2}xy\}, d_{\mathbf{H}})$ *bi-Lipschitz embeds in* $\mathbf{R}^{19}$.

Proof. For each $n, m \in \mathbf{Z}$ let

$$Q_{n,m}^{\pm} = \{(x, \pm y, \pm \tfrac{1}{2}xy)\colon m2^{-n} \le x \le (m+1)2^{-n}, 2^{-n} \le y \le 2^{1-n}\}.$$

Firstly, we show that $(Q_{0,0}^{+}, d_{\mathbf{H}})$ bi-Lipschitz embeds in $[1,2] \times [0,1]^3 \subset \mathbf{R}^4$. Indeed, consider the map

$$g\colon (Q_{0,0}^{+}, d_{\mathbf{H}}) \to \mathbf{R}^4 \quad \text{with} \quad g(x, y, \tfrac{1}{2}xy) = (y, \Phi(x)).$$

Then, for $w = (x, y, \tfrac{1}{2}xy) \in Q_{0,0}^{+}$ and $w' = (x', y', \tfrac{1}{2}x'y') \in Q_{0,0}^{+}$, we have

$$d_{\mathbf{H}}(w, w') = |x - x'| + |y - y'| + 2^{-1/2}(y + y')^{1/2}|x - x'|^{1/2}$$
$$\simeq |y - y'| + |x - x'|^{1/2} \simeq |g(w) - g(w')|.$$

Secondly, we remark that the maps $\zeta_{n,m}^{\pm}\colon (Q_{0,0}^{+}, d_{\mathbf{H}}) \to (Q_{n,m}^{\pm}, d_{\mathbf{H}})$ given by

$$\zeta_{n,m}^{\pm}(x, y, \tfrac{1}{2}xy) = (2^{-n}(x + m), \pm 2^{-n}y, \pm \tfrac{1}{2}4^{-n}(x + m)y)$$

are 2^{-n}-similarities i.e. $d_{\mathbf{H}}(w, w') = 2^{-n}d_{\mathbf{H}}(\zeta_{n,m}^{\pm}(w), \zeta_{n,m}^{\pm}(w'))$. Let also $\sigma_{n,m}^{\pm}\colon \mathbf{R}^4 \to \mathbf{R}^4$ be the similarities given by

$$\sigma_{n,m}^{\pm}(x, y, z, t) = (\pm 2^{-n}x, 2^{-n}(y + m), 2^{-n}z, 2^{-n}t).$$

Thirdly, we decompose $(\{z = \frac{1}{2}xy\}, d_{\mathbf{H}})$ into 4 pieces, we embed each piece bi-Lipschitz into $\mathbf{R}^4$, and we glue these 4 embeddings via Lemma 3.6. Define the following four families of squares

$$\mathscr{Q}_1 = \{Q_{n,m}^{\pm}\colon n \text{ is even and } m \text{ is even}\}, \quad \mathscr{Q}_2 = \{Q_{n,m}^{\pm}\colon n \text{ is even and } m \text{ is odd}\},$$
$$\mathscr{Q}_3 = \{Q_{n,m}^{\pm}\colon n \text{ is odd and } m \text{ is even}\}, \quad \mathscr{Q}_4 = \{Q_{n,m}^{\pm}\colon n \text{ is odd and } m \text{ is odd}\}.$$

Fix an $i \in \{1, 2, 3, 4\}$ and define $G_i\colon (\{y = z = 0\} \cup \bigcup \mathscr{Q}_i, d_{\mathbf{H}}) \to \mathbf{R}^4$ by

$$G_i|_{Q_{n,m}^{\pm}} = \sigma_{n,m}^{\pm} \circ g \circ (\zeta_{n,m}^{\pm})^{-1}|_{Q_{n,m}^{\pm}} \quad \text{and} \quad G_i(x, 0, 0) = (0, x, 0, 0).$$

By the first two steps, for each $Q \in \mathscr{Q}_i$, the map $G_i|_Q$ is bi-Lipschitz with bi-Lipschitz constant independent of the size of Q. It remains to show that each G_i is bi-Lipschitz. Assuming this is true, $(\{z = \frac{1}{2}xy\}, d_{\mathbf{H}})$ bi-Lipschitz embeds into $\mathbf{R}^{19}$ by Lemma 3.6.

Fix $Q, Q' \in \mathscr{Q}_i$ with $Q \neq Q'$ and fix $w = (x, y, \frac{1}{2}xy) \in Q$ and $w' = (x', y', \frac{1}{2}x'y') \in Q'$. (The case that one of them is on the x-axis follows by a limit argument). Recall that
$$d_{\mathbf{H}}(w, w') \simeq |x - x'| + |y - y'| + |y + y'|^{1/2}|x - x'|^{1/2}.$$
We claim that $d_{\mathbf{H}}(w, w') \simeq |x - x'| + |y - y'|$. To see that, consider the following two cases.

Case 1. Suppose that $Q = Q^+_{n,m}$ and $Q' = Q^+_{n,m'}$ for $m \neq m'$. Then,
$$|y + y'| \lesssim 2^{-n} \leq |m - m'|2^{-n} \lesssim |x - x'|.$$
Hence, $|y + y'|^{1/2}|x - x'|^{1/2} \lesssim |x - x'|$. The same conclusion holds if $Q = Q^-_{n,m}$ and $Q' = Q^-_{n,m'}$.

Case 2. Suppose that Case 1 does not hold. Then $Q = Q^\pm_{n,m}$ and $Q' = Q^\pm_{n',m'}$ for $n \neq n'$, and so $|y - y'| \simeq |y + y'|$. The claim follows.

With similar reasoning, we can show that, for any $Q, Q' \in \mathscr{Q}_i$ with $Q \neq Q'$ and for any $w = (x, y, \frac{1}{2}xy) \in Q$ and $w' = (x', y', \frac{1}{2}x'y') \in Q'$, we have
$$(5.5) \qquad |G_i(w) - G_i(w')| \simeq |x - x'| + |y - y'|.$$
Indeed, notice that $(g \circ (\zeta^\pm_{n,m})^{-1})(Q) \subset [1, 2] \times [0, 1]^3$ for any $Q = Q^\pm_{n,m} \in \mathscr{Q}_i$, and thus
$$G_i(Q) \subset [\pm 2^{-n}, \pm 2^{1-n}] \times [m2^{-n}, (m+1)2^{-n}] \times [0, 2^{-n}] \times [0, 2^{-n}].$$
According to the definition of $\mathscr{Q}_i$ and arguments as above, then, (5.5) holds for any $Q, Q' \in \mathscr{Q}_i$ with $Q \neq Q'$. That concludes the proof of Proposition 5.6. $\qquad\square$

6. Nonembedabbility of Heisenberg porous sets

The focus of this section is the proof of Theorem 1.4. For the rest of §6, we set $\pi\colon \mathbf{H} \to \mathbf{R}^2$ to be the 1-Lipschitz abelianization map. Given $\alpha > 0$, we define a α-cone in $\mathbf{H}$ as the set
$$(6.1) \qquad C_\alpha := \{(x, y, z)\colon |z| \leq \alpha(x^2 + y^2)\}.$$
It is easily seen that C_α is dilation and rotation invariant and $C_\alpha \subseteq C_\beta$ when $\alpha \leq \beta$. In §6.1 we define the set X and explain why there is no bi-Lipschitz embedding of X in ℓ^2, and in §6.2 we show that X is indeed porous.

6.1. Laakso graphs. Let $\{G_n\}_{n=1}^\infty$ denote the sequence of Laakso graphs defined in Section 2.2 of [Li16] and let $f_n\colon G_n \to \mathbf{H}$ denote the embedding defined starting from the bottom of p. 1625 (there it was just denoted f by abuse of notation). (Note in particular that each G_n consists only of the vertices in the graph and does not include the edges.) Recall that f_n is specified with a sequence of angles $\theta_j = \left(\sqrt{M + j}\log(M + j)\right)^{-1}$ for some M sufficiently large to be fixed. Theorem 1.2 of [Li16] showed that f_n on G_n has distortion $O((\log|G_n|)^{1/4}\sqrt{\log\log|G_n|})$.
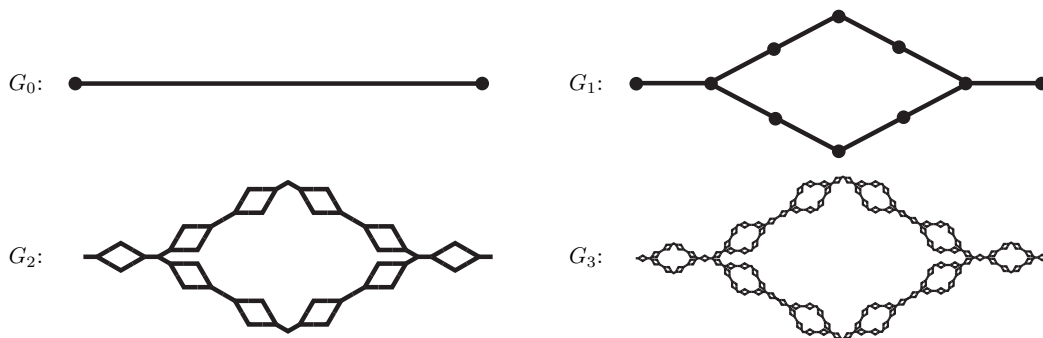
Figure 1. The construction of $\{G_n\}_{n=1}^{\infty}$.

One of the main goals of [Li16] was to use f_n to show that the image $f_n(G_n)$ cannot uniformly bi-Lipschitz embed into Markov p-convex metric spaces when $p < 4$. This is explicitly done in the proof of Corollary 1.4 of [Li16] (although one needs to remove the intermediary $\mathbf{H}(\mathbf{Z})$ that appears there). Since Hilbert space is Markov 2-convex, we get that $f_n(G_n)$ do not embed into ℓ_2 with uniform bi-Lipschitz distortion.

We now require that the diameter of each G_n is 1. Thus, the length of edges in G_n become 6^{-n}. This allows us to define a limiting compact object G_∞ also of diameter 1. It is not hard to show that G_∞ contains each G_n isometrically.

One also has that G_∞ is self-similar and is composed as a union of 10 scaled copies of G_∞. Call these level 1 copies. We use H_1 to denote any one of these copies, and they are all isometric. (Note that H_1 could refer to any level 1 copy of G_∞.) Each of these in turn are composed of 10 scaled copies of G_∞. Call these level 2 copies, and use H_2 to denote each of these copies. We can continue and identify level n copies for all $n \in \mathbf{N}$ each denoted H_n. If a copy H_i does not contain a terminal point of some H_{i-1} we say it is forked. Note that each H_i contains 8 forked and 2 non-forked copies of H_{i+1}.

Each H_i contains a unique scaled copy of G_k that is maximal in diameter. We call this $H_{i,k}$. Again, $H_{i,k}$ could refer to any one of many isometric objects.

We can define a Lipschitz map $f \colon G_\infty \to \mathbf{H}$ so that the restriction of f on each embedded copy of G_n in G_∞ is a re-scaled (and possibly translated and rotated) copy of $f_n(G_n)$. Thus, $f(G_\infty)$ is compact and cannot embed into ℓ_2 as it contains copies of $f_n(G_n)$. Define $X = f(G_\infty)$.

6.2. Proof of Theorem 1.4. From the discussion above, Theorem 1.4 immediately follows from the following proposition.

Proposition 6.1. *The set $X = f(G_\infty)$ is porous in* $\mathbf{H}$.

For the rest of §6.2, given a subset $A \subset G_\infty$, we let $\tilde{A} := \pi(f(A)) \subset \mathbf{R}^2$, and if $x \in G_\infty$, set $\tilde{x} := \pi(f(x))$. We require two lemmas from [Li16]. The first one follows from Lemma 2.6 of [Li16].

Lemma 6.2. *For M sufficiently large, if s, t are terminal points of a level n copy H_n of G_∞, then $\frac{6^{-n}}{2} \leq |\pi(f(s)) - \pi(f(t))| \leq d_{\mathbf{H}}(f(s), f(t))$.*

The second is Lemma 2.7 of [Li16].

Lemma 6.3. *For any i, the set $\tilde{H}_i$ is contained in the closed convex hull of the corresponding $\tilde{H}_{i,1}$.*

Remark 6.4. Let C and C' be convex hulls of $\tilde{H}_{i+1,1}$ for copies of H_{i+1} in H_i that are in parallel and intersect at a fork point a. Lemma 6.3 says that there exists a cone based at a of aperture at $\theta_i - \frac{\theta_{i+1}}{2} \geq \frac{\theta_i}{2}$ between C and C'.

Corollary 6.5. *For any H_n, we have $\frac{6^{-n}}{2} \leq \operatorname{diam}_{d_{\mathbf{H}}} f(H_n) \leq 6^{-n}$.*

Proof. The upper bound comes from 1-Lipschitzness of f. The lower bound comes from Lemma 6.2. □

Proposition 6.6. *There exists $M > 0$ sufficiently large so that the following holds. Let $H_i \supset H_{i+1} \supset \ldots \supset H_k$ be a nested sequence of sub-graphs so that H_{i+1} is forked in H_i. Let $S \subseteq \{i+2, \ldots, k\}$ be the indices j so that H_j is forked in H_{j-1}. Suppose there exists another forked copy $H'_{i+1} \subset H_i$ that is in parallel with H_{i+1} so that there are points $x \in H_k$ and $y \in H'_{i+1}$ for which $6^{-k} \leq |\tilde{x} - \tilde{y}| \leq 6^{-k+1}$. Then $\sum_{j \in S} \theta_j \leq 1/100$.*

Proof. We suppose for contradiction that $\sum_{j \in S} \theta_j > \frac{1}{100}$. In particular, S is nonempty so let $\ell = \min S$ and $a, b, c \in \mathbf{R}^2$ be the points of $\tilde{H}_{i,1} \subset \tilde{H}_i$ with degree at least 3. Then the distance from $\tilde{x}$ to a, b, c is at least $6^{-\ell}/2$ by Lemma 6.2. By Remark 6.4, the distance from x to any point in $\tilde{H}'_{i+1}$ (which is in parallel with $\tilde{H}_{i+1}$) is at least $6^{-\ell} \theta_i / 8$ and so $6^{-k+1} \geq 6^{-\ell} \theta_i / 8$ by the assumption of the lemma. This tells us that $k \leq \ell + 3 + \log \frac{1}{\theta_i}$ and so $|S| \leq 3 + \log \frac{1}{\theta_i} \leq \frac{1}{100} \sqrt{i + M}$ for large enough M. Thus,

$$\sum_{j \in S} \theta_j \leq \sum_{j=1}^{\sqrt{i+M}/100} \theta_{i+1+j} \leq \sum_{j=1}^{\sqrt{i+M}/100} \frac{1}{\log(M+i+j+1)\sqrt{M+i+j+1}} < \frac{1}{100},$$

which is a contradiction. □

We also choose M sufficiently large so that $100\theta_1 < \frac{1}{100}$.

Fix some $x \in X$ and choose some $k \in \mathbf{N}$. Let $y \in G_\infty$ be any point so that $f(y) = x$. For each $i \in \mathbf{N}$, there exists some H_i that contains y. We call this subset Y_i. If y is a terminal point of some H_i, then the choice of Y_i may not be unique. In this case, choose Y_i arbitrarily while ensuring that, at each step, $Y_{i+1} \subset Y_i$. We let s_i and t_i denote the source and sink of Y_i.

Let $(u, v) \in \mathbf{S}^1$ be a 90° clockwise rotation of the direction from $\tilde{s}_k$ to $\tilde{t}_k$. Let

$$x' = f(s_k) \cdot (u6^{-k+10}, v6^{-k+10}, 0)$$

and define the ball $B = B(x', 6^{-k})$. Our goal is to prove that $B \cap X = \emptyset$. It suffices to show that $B \cap (f(Y_i) \setminus f(Y_{i+1})) = \emptyset$ for every $i \in \mathbf{N}$. If $k \leq i + 5$, then it is obvious that this intersection is empty by a simple triangle inequality argument and the fact that $f(Y_i) \subseteq B(f(s_k), 6^{-k+5})$.

We may now suppose $i < k - 5$. Let $S \subseteq \{i+2, ..., k\}$ be the indices j so that Y_j is forked in Y_{j-1}. Let $P \colon \mathbf{R}^2 \to \mathbf{R}$ denote the orthogonal projection onto the linear subspace spanned by $\tilde{s}_i$ and $\tilde{t}_i$ with the orientation that $P(\tilde{s}_i) < P(\tilde{t}_i)$.

Lemma 6.7. *Suppose at least 100 of $\{Y_j\}_{j=i+2}^k$ are forked. Then*

(6.2) $$P(\pi(B)) \subseteq [P(\tilde{s}_{i+1}) + 10 \cdot 6^{-k}, P(\tilde{t}_{i+1}) - 10 \cdot 6^{-k}].$$

Proof. Let $\ell \in \{i+2, \ldots, k\}$ be the smallest integer so that Y_ℓ is forked. Thus, in Y_ℓ, there exists a non-forked $H_{\ell+1}$ separating Y_k from s_ℓ and t_ℓ (since at least one Y_j is forked for $j \in \{\ell+2, \ldots, k\}$). As ℓ was minimal in $\{i+2, \ldots, k\}$, $P(\tilde{s}_\ell) \geq P(\tilde{s}_{i+1})$ and

we get that $P(\tilde{s}_k) - P(\tilde{s}_\ell) > 6^{-\ell}/2$. As $k \geq \ell + 99$ and since the group multiplication acts in the first two coordinates simply as addition, this proves the lemma (arguing similarly for the right hand side of the interval). $\qquad\square$

Lemma 6.8. *If B intersects $f(Y_i)\backslash f(Y_{i+1})$, then Y_{i+1} must be forked and B must intersect some other $f(H_{i+1})$ where H_{i+1} is in parallel with Y_{i+1}.*

Proof. Since $B \cap f(Y_i) \neq \emptyset$, the ball B intersects some $f(H_{i+1})$ in $f(Y_i)$. We denote this H_{i+1} as Y'_{i+1}. Let θ denote the angle between the line segments connecting the terminals of $\tilde{Y}_i$ and $\tilde{Y}_k$. First suppose that $\theta \leq 1/100$. Then an elementary geometric argument in $\mathbf{R}^2$ using Lemma 6.3 shows that $\pi(B)$ cannot intersect any $\tilde{H}_{i+1}$ where H_{i+1} is in series with Y_{i+1}. Thus Y'_{i+1} must be in parallel with Y_{i+1}, and this proves the lemma in this case.

Now suppose $\theta > 1/100$. Then as $100\theta_1 \leq 1/100$, we must have that at least 100 of $\{Y_j\}_{j=i+2}^{k}$ are forked. The lemma then follows from the previous lemma. Indeed, $P(\pi(B)) \cap P(\tilde{Y}'_{i+1}) \neq \emptyset$, and so Y'_{i+1} cannot be in series with Y_{i+1}. $\qquad\square$

Recall that C_1 is the 1-cone defined in (6.1).

Lemma 6.9. $6B \subseteq f(s_k) \cdot C_1$.

Proof. We may suppose $f(s_k) = (0,0,0)$. Let $p \in B(0, 6^{-k+1})$. Then $(u6^{-k+10}, v6^{-k+10}, 0) \cdot p$ has a z-coordinate of absolute value at most $6^{-2k+2} + 6^{-k+10} \cdot 6^{-k+1} \leq 6^{-2k+12}$. On the other hand, the x and y coordinates have absolute value at least 6^{-k+9}. This easily proves the lemma (with drastic overestimation). $\qquad\square$

We now show Proposition 6.1.

Proof of Proposition 6.1. Suppose B intersects $f(Y_i)\backslash f(Y_{i+1})$. Let Y'_{i+1} denote the H_{i+1} that is in parallel with Y_{i+1} such that $B \cap f(Y'_{i+1})$ is nonempty (from Lemma 6.8). Let $p \in B \cap f(Y'_i)$. In a similar way, for $j \in \{i+2, \ldots, k\}$, let Y'_j denote the H_j that contain p with $Y'_j \subseteq Y'_{j-1}$ and $s'_k, t'_k, \tilde{s}'_k, \tilde{t}'_k$ be defined similarly. Note that the inflated ball $6B$ must intersect $\tilde{s}'_k$. Let S' be the indices j of $\{i+2, \ldots, k\}$ where Y'_j is forked in Y'_{j-1}. By Proposition 6.6, we have that both $\sum_{j \in S} \theta_j$ and $\sum_{j \in S'} \theta_j$ are less than $1/100$.

Take a geodesic path from s_k to s'_k. This is pushed into $\mathbf{H}$ via f to a piecewise affine horizontal path $p_1 p_2 \cdots p_n$ so that $p_1 = f(s_k)$ and $p_n = f(s'_k)$. As $\max\{\sum_{j \in S} \theta_j, \sum_{j \in S'} \theta_j\} \leq 1/100$ and the line from $\pi(p_1)$ to $\pi(p_n)$ is almost orthogonal to the line between the terminals of $\tilde{Y}_i$, the closed piecewise affine path $\pi(p_1) \cdots \pi(p_n)\pi(p_1)$ in $\mathbf{R}^2$ does not self intersect. It follows from Remark 6.4 that this path encloses an isosceles triangle of angle $\theta_i - \frac{\theta_{i+1}}{2}$ and base at least $\frac{|\pi(p_n) - \pi(p_1)|}{2}$ which has area

$$\frac{1}{2} \frac{|\pi(p_1) - \pi(p_n)|}{2} \frac{|\pi(p_1) - \pi(p_n)|}{2\tan\left(\theta_i - \frac{\theta_{i+1}}{2}\right)} \geq \frac{1}{16\theta_i}|\pi(p_1) - \pi(p_n)|^2.$$

Thus, we see that $p_n \notin p_1 \cdot C_{1/16\theta_{i-1}}$. As θ_i is decreasing, this means $p_n \notin p_1 \cdot C_{1/16\theta_1}$. However, from Lemma 6.9 we have that $p_n \in p_1 \cdot C_1$. This is a contradiction, which means B does not intersect $f(Y_i)\backslash f(Y_{i+1})$. This proves that $X = f(G_\infty)$ is porous in $\mathbf{H}$. $\qquad\square$

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