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107

NEGLIGIBLE SUBSETS OF THE GENERALIZED BAIRE SPACE $\omega_1^{\omega_1}$

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Aapo Halko

Contents

1. Introduction

We introduce concepts which analyze negligible subsets, like meager and strong measure zero sets, of the generalized Baire space $\mathcal{N}_1 = \omega_1^{\omega_1}$. In the topology of this space basic neighborhoods of an element $f \in \mathcal{N}_1$ are of the form

$$U(f, \alpha) = \{g \in \mathcal{N}_1 \mid g{\restriction}\alpha = f{\restriction}\alpha\}.$$

Some of the concepts, like the concept of a meager set, are direct analogies of classical concepts. Others, like the concept of a small set, are unique for the space $\mathcal{N}_1$.

Many of the results of this thesis are just the results for the Baire space $\mathcal{N} = \omega^\omega$ "lifted one cardinal up". The power of ZFC is often insufficient so we will use additional set theoretical assumptions, like CH, Kurepa's hypothesis, $\diamondsuit$, $\diamondsuit^*$, GMA and $I(\omega)$.

We use the same name for the concepts in $\mathcal{N}_1$ as for the concepts in the Baire space $\mathcal{N}$. We shall explicitly mention when we mean the concept of the Baire space or of the reals. We hope that this convention does not cause confusion.

In Telgársky [34] there is a survey of topological games such as the Banach–Mazur game, the Borel game and the point-open game. The length of those games is ω. Here we shall generalize many of those games by simply letting the game go on for δ moves for some ordinal $\delta \leq \omega_1$. As Telgársky points out many of the games are redundant in the case of Baire space. But in the study of $\mathcal{N}_1$ we need to introduce these games since the generalization of some concepts is most naturally defined by games. Prime examples are the definition of a Borel* set [11] and the definition of ω_1-perfectness [38].

Sikorski was the first to study the space $\mathcal{C}_1 = 2^{\omega_1}$ in [33]. He studied compactness properties of the space $\mathcal{C}_1$. Juhász and Weiss solved a problem of Sikorski in [15]. Shelah [31] had an application for meager subsets of $\mathcal{C}_1$. Tuuri [37] and Väänänen [38] proved the separation theorem for Π_1^1 subsets of $\mathcal{N}_1$. Landver [22] has studied the so called Baire numbers of $\mathcal{N}_1$. Väänänen [38] has studied perfect and scattered subsets of $\mathcal{N}_1$. Halko [11] studied generalized Borel sets of $\mathcal{N}_1$.

The structure of this thesis is the following: The notation used and some of the connections between topological properties of $\mathcal{N}_1$ and set theoretical assumptions are presented in Section 2. In Section 3 we define meager sets in $\mathcal{N}_1$ and study the property of Baire in $\mathcal{N}_1$. The definition and basic properties of small and strong measure zero sets are dealt with in Section 4. We introduce a generalization of the combinatorial principle $\diamondsuit$ in Section 5. In Section 6 we define several classes of negligible subsets of $\mathcal{N}_1$. Section 7 is inspired by the theorem of Galvin, Mycielski and Solovay for the strong measure zero sets of reals. In Section 8 we investigate how the generalized Martin's axiom decides the properties of negligible subsets. In Section 9 we discuss how difficult it is to define a measure for $\mathcal{N}_1$.

2. Preliminaries

Our set theoretical notation is consistent with Jech [13]. We shall work within Zermelo–Fraenkel theory with the Axiom of Choice (ZFC). We shall often use an additional assumption, which is independent of ZFC, namely the Continuum Hypothesis (CH), since it makes the environment here quite natural. We shall mention when we use CH.

If $F : A \to B$ is a function and $C \subseteq A$ then $F''C = \{F(x) \mid x \in C\}$. A κ-union is a union of κ many sets. For a set A and a cardinal λ we denote

$$[A]^\lambda = \{B \subseteq A \mid |B| = \lambda\} \text{ and } [A]^{<\lambda} = \{B \subseteq A \mid |B| < \lambda\}.$$

If α is an ordinal then $\kappa^\alpha = \{f \mid f : \alpha \to \kappa\}$ and $\kappa^{<\alpha} = \bigcup_{\beta<\alpha} \kappa^\beta$. Ordinals are, unless otherwise specified, assumed to be countable i.e. elements of ω_1. For $a \subseteq \omega_1$ the *characteristic function* $\chi_a \in \mathcal{C}_1$ is such that $\chi_a(\xi) = 1$ if and only if $\xi \in a$.

We consider the spaces $\mathcal{N}_1 = \omega_1^{\omega_1} = \{f \mid f : \omega_1 \to \omega_1\}$ and $\mathcal{C}_1 = 2^{\omega_1} = \{f \mid f : \omega_1 \to 2\}$ with a topology in which the basic open neighborhoods of $f \in \mathcal{N}_1$ are

$$U(f, \alpha) = \{g \in \mathcal{N}_1 \mid \forall \xi < \alpha[f(\xi) = g(\xi)]\}, \quad \alpha < \omega_1.$$

Let Seq be the set $\omega_1^{<\omega_1}$ and denote its elements by letters $s, t, \ldots$. If $s, t \in$ Seq, then $s \prec t$ means that s is a proper initial segment of t. The length of the sequence s is denoted by $\ell(s)$. If h is a countable function such that $\mathrm{dom}(h) \subseteq \omega_1$ and $\mathrm{ran}(h) \subseteq \omega_1$, let $[h] = \{g \in \mathcal{N}_1 \mid h \subseteq g\}$. So $U(f, \alpha) = [f{\restriction}\alpha]$. Sequences s and s' are *compatible*, denoted by $s \mid s'$, if there is a sequence t such that $s \prec t$ and $s' \prec t$; otherwise they are *incompatible*, $s \perp s'$.

A set $T \subseteq$ Seq is a *tree*, if $t{\restriction}\xi \in T$ for all $t \in T$ and $\xi < \ell(s)$. For $t \in T$ we define $\mathrm{prec}(t) = \{s \in T \mid s \prec t\}$ and $\mathrm{succ}(t) = \{t^\frown\langle\alpha\rangle \in T \mid \alpha \in \omega_1\}$. $\mathrm{Lev}_\alpha(T) = \{t \in T \mid \ell(t) = \alpha\}$. A *branch* $b \subseteq T$ is a maximal chain of T. An ω_1*-branch* is a branch of length ω_1. The set of all branches of T is denoted by $\mathcal{B}(T)$, and the set of ω_1-branches of T is denoted by $\mathcal{B}_{\omega_1}(T)$. An *Aronszajn tree* is a tree which has no ω_1-branches and $\mathrm{Lev}_\alpha(T)$ is countable for all $\alpha < \omega_1$. A *Kurepa tree* is a tree which has at least $\aleph_2$ ω_1-branches and $\mathrm{Lev}_\alpha(T)$ is at most countable for all $\alpha < \omega_1$.

The spaces $\mathcal{N}_1$ and $\mathcal{C}_1$ have the property that any countable intersection of open sets is open. Spaces of this kind are called ω_1*-additive*. The *Borel sets* of $\mathcal{N}_1$ are the members of the ω_2-algebra generated by open sets. A space $\mathcal{X}$ is T_1, if its singletons are closed.

2.1. Lemma. *If $\mathcal{X}$ is an ω_1-additive T_1-space that has a clopen basis of cardinality $\aleph_1$, then $\mathcal{X}$ is homeomorphic to a subset of 2^{ω_1}.*

Proof. Let $\langle G_\xi\rangle_{\xi<\omega_1}$ be the clopen basis of $\mathcal{X}$. Denote $G_\xi^1 = G_\xi$ and $G_\xi^0 = \mathcal{X}\setminus G_\xi$. We define $\mathcal{Y} \subseteq \mathcal{C}_1$ and a homeomorphism $F : \mathcal{X} \to \mathcal{Y}$ as follows: For each $x \in \mathcal{X}$ set $f(\xi) = i$, if and only if $x \in G_\xi^i$ and put $F(x) = f$. Let $\mathcal{Y} = \mathrm{ran}(F)$. F is one-one, since if $x \neq y$, then there is $\xi < \omega_1$ such that $x \in G_\xi$, but $y \notin G_\xi$. F is continuous, since for any $s \in 2^{<\omega_1}$, if $x \in F^{-1}([s])$ then $x \in \bigcap_{\xi<\ell(s)} G_\xi^{F(x)(\xi)} \subseteq F^{-1}([s])$. F is open since for any $\xi \in \omega_1$, $[\{(\xi, 1)\}] \cap \mathcal{Y} \subseteq F''G_\xi$. □

Thus assuming CH, $\mathcal{N}_1$ is homeomorphic to a subset of $\mathcal{C}_1$.

Let $f, g \in \mathcal{C}_1$ and $A, B \subseteq \mathcal{C}_1$. Then $f + g \in \mathcal{C}_1$ is such that $(f + g)(\xi) = f(\xi) + g(\xi) \bmod 2$ for all $\xi < \omega_1$. We denote $f + A = \{f + g \mid g \in A\}$ and $A + B = \{f + g \mid f \in A, g \in B\}$.

2.2. Definition. Let T be a tree with unique limits, i.e. if $\mathrm{prec}(t) = \mathrm{prec}(t')$ and $\ell(t) = \ell(t')$ is a limit ordinal, then $t = t'$. A *game* is a pair (T, L) where $L : T \cup \mathcal{B}(T) \to \{\mathbf{I}, \mathbf{II}\}$ is the *labelling function of the tree.* The players of the game are **I** and **II**. A *play* of the game is any $b \in \mathcal{B}(T)$. The player **I** *wins* the play b, if $L(b) = \mathbf{I}$; otherwise **II** wins. A strategy for a player Q is a function $\sigma : L^{-1}(Q) \cap T \to T$ such that $\sigma(t) \in \mathrm{succ}(t)$. The player Q has *used* his strategy σ in the play b, if $\sigma(b\restriction\xi) = b\restriction(\xi + 1)$ for every $\xi < \ell(b)$ such that $b\restriction\xi \in \mathrm{dom}(\sigma)$. A strategy σ of Q is a *winning strategy*, if Q wins every play b in which he has used σ.

We denote by $Q \uparrow G$ that player Q has winning strategy in G.

Usually we describe our games less informally, but it should be clear how to formulate the precise definition. The branches of the game tree will always have the same height, say α, and we say that the game has length α.

If $A \subseteq \mathcal{N}_1$, then the *closure* of A, $\overline{A}$, is the smallest closed set which contains A. A set P is *perfect*, if P is closed and contains no isolated points.

2.3. Lemma. *A set $A \subseteq \mathcal{N}_1$ is closed if and only if $A = \mathcal{B}_{\omega_1}(T)$ for some tree $T \subseteq \mathrm{Seq}$.*

Proof. Assume that $A = \mathcal{B}_{\omega_1}(T)$ for some tree T. Then A is closed, since if $f \notin A$ there is ξ such that $f\restriction\xi \notin T$ which means that $[f\restriction\xi] \subseteq \mathcal{N}_1\setminus A$.

Assume that A is closed. Let $T = \{f\restriction\xi \mid f \in A, \xi \in \omega_1\}$. Then obviously $A \subseteq \mathcal{B}_{\omega_1}(T)$. If $f\restriction\xi \in T$ for all $\xi < \omega_1$ then $f \in A$ because A is closed. □

A tree T is a *Jech–Kunen tree* if $|T| = \aleph_1$ and $\aleph_1 < |\mathcal{B}_{\omega_1}(T)| < 2^{\aleph_1}$. One generalization of the Cantor–Bendixson Theorem states that the cardinality of a closed subset of $\mathcal{N}_1$ is either at most $\aleph_1$ or is $2^{\aleph_1}$. Thus if T is a Jech–Kunen tree, then $\mathcal{B}_{\omega_1}(T)$ shows that this generalization of the Cantor–Bendixson Theorem fails for $\mathcal{N}_1$. The consistency of a Jech–Kunen tree was given in [12], in which Jech constructed a generic Kurepa tree T such that $|\mathcal{B}_{\omega_1}(T)| < 2^{\omega_1}$ in a model of CH and $2^{\omega_1} > \omega_2$. By assuming the consistency of an inaccessible cardinal, Kunen proved the consistency of non-existence of Jech–Kunen trees with CH (see [16, Theorem 4.8]). In Kunen's model there are also no Kurepa trees.

The differences between Kurepa trees and Jech–Kunen trees in terms of the existence have been studied recently by Jin and Shelah in [32].

The set theoretical assumption $I(\omega)$ states that there is a normal ω_2-complete ideal $\mathcal{I}$ on ω_2 such that $\mathcal{I}^+ = \{A \mid A \notin \mathcal{I}\}$ has a dense subset in which every descending sequence has a lower bound.

Väänänen [38] shows that $I(\omega)$ implies that every closed subset of $\mathcal{N}_1$ of cardinality $> \aleph_1$ is in fact ω_1-perfect (Definition 3.5), and hence of cardinality $2^{\aleph_1}$.

2.4. Definition. A set $A \subseteq \mathcal{X}$ is ω_1*-compact*, if every open cover $\mathcal{U}$ of A contains a countable subcover of A.

An ω_1-compact space is often called *Lindelöf*.

2.5. Lemma (CH). *A set $A \subseteq \mathcal{N}_1$ is ω_1-compact if and only if every sequence $\langle f_\xi \rangle_{\xi<\omega_1}$ of elements of A contains a subsequence converging to a point of A.*

Proof. Assume that there is a sequence $\langle f_\xi \rangle_{\xi<\omega_1}$ of elements of A that does not contain a subsequence converging to a point of A. For every $f \in A$ there is $\alpha_f < \omega_1$ such that $U(f, \alpha_f)$ contains at most one f_ξ. So $\mathcal{U} = \{U(f, \alpha_f) \mid f \in A\}$ has no countable subcover.

Let $\mathcal{U}$ be an open cover of A that does not contain a countable subcover. By CH we may assume that $\mathcal{U}$ has power $\aleph_1$. We enumerate $\mathcal{U} = \{U_\xi \mid \xi < \omega_1\}$. For each $\gamma < \omega_1$ choose $f_\xi \in A \setminus \bigcup_{\xi<\gamma} U_\xi$. The sequence $\langle f_\xi \rangle$ can not contain a converging sequence $f_{\xi_\delta} \to f$, since otherwise $f \in U_\gamma$ for some $\gamma < \omega_1$ and thus $f_{\xi_\delta} \notin U_\gamma$ whenever $\xi_\delta > \gamma$. $\square$

The following lemma is first proven in [15].

2.6. Lemma. *There is an ω_1-compact subset of $\mathcal{C}_1$ of cardinality $> \aleph_1$ if and only if there is a Kurepa tree with no Aronszajn subtrees.*

Proof. Assume that A is ω_1-compact and $|A| > \aleph_1$. Then A is closed, by the previous lemma, and there is a tree T such that $A = \mathcal{B}_{\omega_1}(T)$. T has more than $\aleph_1$ branches. The levels of $\mathrm{Lev}_\alpha(T)$, $\alpha < \omega_1$, are countable, since otherwise $\{[s] \mid s \in \mathrm{Lev}_\alpha(T)\}$ would be an open cover of A which has no countable subcover. So T is Kurepa. If T contained an Aronszajn subtree T', then $\{[s] \mid s \in \mathcal{B}(T')\} \cup \{[s] \mid s \in \mathrm{Lev}_\omega(T),\ s \perp T'\}$ would be an open cover of A which has no countable subcover.

Assume that there is a Kurepa tree T with no Aronszajn subtrees. Then there is one which is a binary tree with unique limits. Thus we may assume that $T \subseteq 2^{<\omega_1}$. Let $A = \mathcal{B}_{\omega_1}(T)$. Let $\mathcal{U}$ be an open cover of A. Let $T(A, \mathcal{U})$ be the set of all sequences s such that $[s] \cap A$ has no countable subcover. Obviously $T(A, \mathcal{U}) \subseteq T$ is closed under subsequences. We claim that $T(A, \mathcal{U})$ cannot contain an ω_1-branch f. Since A is closed, $f \in A$ and hence there is $U \in \mathcal{U}$ such that $f \in U$. There is $\alpha < \omega_1$ such that $U(f, \alpha) \subseteq U$ so $f \restriction \alpha \notin T(A, \mathcal{U})$, a contradiction.

Since T has no Aronszajn subtree, the only possibility is that $T(A, \mathcal{U})$ is bounded by some α. But then, by definition of $T(A, \mathcal{U})$, $[s] \cap A$ has a countable subcover for every $s \in \mathrm{Lev}_\alpha(T)$. Since $\mathrm{Lev}_\alpha(T)$ is countable, we get a countable subcover of A. $\square$

By a Theorem of Jensen assuming $V = L$ there exists a Kurepa tree with no Aronszajn subtrees. See [2, 4, 35]. Thus $V = L$ implies that there is an ω_1-compact subset of $\mathcal{C}_1$ of cardinality $> \aleph_1$.

3. Category in $\mathcal{N}_1$

We begin our investigations of negligible sets with meager sets. Many things seem to be similar to their counterparts in the theory of reals, see [28], but the structure of ω_1 makes some differences.

3.1. Definition. A set A of an ω_1-additive space $\mathcal{X}$ is *dense*, if every nonvoid open set contains a point of A. A set is *nowhere dense*, if its complement contains an open dense set. A set is said to be *meager* if it can be represented as an ω_1-union of nowhere dense sets. A set is *comeager*, if its complement is meager.

Every countable set is nowhere dense and a countable union of nowhere dense sets is nowhere dense since the topology of $\mathcal{N}_1$ is ω_1-additive. Under CH the set of *rationals* $\mathcal{Q} = \{f \in \mathcal{N}_1 \mid \exists \xi < \omega_1 \forall \delta > \xi (f(\delta) = 0)\}$ is an example of a dense set of cardinality ω_1. $\mathcal{Q}$ is meager as an ω_1-union of singletons but it is not nowhere dense since $\mathcal{Q} \cap [s] \neq \varnothing$ for every $s \in \mathrm{Seq}$. A nowhere dense set can have cardinality 2^{ω_1}. For example the closed set $P = \{f \in \mathcal{N}_1 \mid \forall \alpha (f(\alpha) \neq 0)\}$ is such a set, since for every $s \in \mathrm{Seq}$, $[s^\frown\langle 0\rangle] \cap P = \varnothing$. See also Lemma 3.7. The set of all meager sets is an ω_2-ideal.

3.2. Lemma. *A set $A \subseteq \mathcal{N}_1$ contains an open and dense set if and only if there is a function $F : \mathrm{Seq} \to \mathrm{Seq}$ such that $s \prec F(s)$ and $[F(s)] \subseteq A$ for all $s \in \mathrm{Seq}$.*

Proof. Assume that $D \subseteq A$ is open and dense. Let $s \in \mathrm{Seq}$. Since D is dense $D \cap [s] \neq \varnothing$. Since D is open there is $s' \succ s$ such that $[s'] \subseteq D$. Let $F(s) = s'$.

Assume that there is a function F that satisfies the condition of the lemma. Then $D = \bigcup_{s \in \mathrm{Seq}} [F(s)]$ is a dense and open subset of A since $[s] \cap D \supseteq [F(s)] \neq \varnothing$ for each $s \in \mathrm{Seq}$. □

For each set which contains an open and dense set let the *od-function* be a function given by Lemma 3.2. A sequence of sequences $\langle s_\xi \rangle_{\xi < \omega_1}$ is *continuous*, if $\xi < \delta$ implies $s_\xi \prec s_\delta$ for all $\xi, \delta \in \omega_1$ and $s_\gamma = \cup_{\xi < \gamma} s_\xi$ for each limit ordinal γ. The next lemma is the Baire Category Theorem for $\mathcal{N}_1$. Note that it does not need CH.

3.3. Lemma. *A countable intersection of open dense sets is open and dense; ω_1-intersection of open dense sets is dense.*

Proof. Let F_ξ be an od-function for an open dense set D_ξ for each $\xi < \omega_1$. Let $s \in \mathrm{Seq}$ be arbitrary. We can define a continuous sequence s_α $(\alpha < \omega_1)$ by induction: let $s_0 = s$ and $s_{\alpha+1} = F_\alpha(s_\alpha)$. Let $f \in \bigcap_{\alpha < \omega_1} [s_\alpha]$. Then by the definition of an od-function $f \in \bigcap_{\xi < \omega_1} D_\xi \cap [s]$; so $\bigcap_{\xi < \omega_1} D_\xi$ is dense. Also $\bigcap_{n < \omega} D_n$ is dense, and it is open as a countable intersection of open sets. □

3.4. Corollary. *A comeager set cannot be meager. In particular, $\mathcal{N}_1$ is not meager.*

Thus the ideal of meager sets is proper.

3.5. Definition. Let $A \subseteq \mathcal{N}_1$ and $x_0 \in A$. $G(A, x_0)$ is the following game of length ω_1 between players **I** and **II**: First **I** chooses $\alpha_0 < \omega_1$. Then **II** chooses $x_1 \in A$ such that $x_1 \neq x_0$ and $x_1 \restriction \alpha_0 = x_0 \restriction \alpha_0$. In general at round $\gamma < \omega_1$ player **I** chooses $\alpha_\gamma \geq \sup\{\alpha_\xi \mid \xi < \gamma\}$ and player **II** chooses $x_\gamma \in A$ such that $x_\gamma \neq x_\xi$ and $x_\gamma \restriction \alpha_\xi = x_\xi \restriction \alpha_\xi$ for all $\xi < \gamma$. Player **II** wins, if he can make all his moves. Player **I** wins otherwise.

A set A is ω_1-*perfect*, if it is closed and **II** $\uparrow G(A, x_0)$ for all $x_0 \in A$.

3.6. Lemma ([38]). *A non-empty ω_1-perfect set A has cardinality 2^{ω_1}.*

The proof of Lemma 3.3 can be strengthened to the following observation.

3.7. Lemma. *Every comeager set contains a nowhere dense ω_1-perfect subset.*

Proof. Let $A \subseteq \mathcal{N}_1$ be a comeager set. Then A has a representation $\bigcap_{\xi<\omega_1} R_\xi$ such that for each $\xi < \omega_1$ there is an od-function F_ξ for R_ξ. We construct by induction sequences $s(b) \in \mathrm{Seq}$ for each $b \in 3^{<\omega_1}$, such that for all $f \in 3^{\omega_1}$, $\langle s(f\restriction\xi)\rangle_{\xi<\omega_1}$ is continuous and for all $b, b' \in 3^{<\omega_1}$,

i) if $b \prec b'$, then $s(b) \prec s(b')$;
ii) $[s(b)] \subseteq R_\delta$, if $\ell(b) = \delta + 1$;
iii) if $\ell(b) = \ell(b')$ and $b \neq b'$, then $s(b) \perp s(b')$.

Then $P = \{\bigcap_{\xi<\omega_1}[s(g\restriction\xi)] \mid g \in 2^{\omega_1}\} \subseteq A$ will be the required ω_1-perfect set. Assume that the sequences $s(b)$, $b \in 3^{<\gamma}$, satisfying i, ii and iii, are already defined. If $\ell(b) = \gamma = \cup\gamma$, then choose $s(b) = \bigcup_{\xi<\ell(b)} s(b\restriction\xi)$. If $\gamma = \delta + 1$ and $\ell(b) = \delta$, then let $s(b\hat{\ }\langle i\rangle) = F_\delta(s(b)\hat{\ }\langle i\rangle)$ for $i < 3$. Clearly the sequences $s(b)$, $b \in 3^{<\gamma+1}$, satisfy i, ii and iii. The set P is closed and nowhere dense, since we can define an od-function F for $\mathcal{N}_1 \smallsetminus P$: Let $s \in \mathrm{Seq}$, be given. If $[s] \cap P = \varnothing$ then $F(s) = s$. If $[s] \cap P \neq \varnothing$, there is $b \in 2^{<\omega_1}$ such that $s \prec s(b)$. Then we let $F(s) = s(b\hat{\ }\langle 2\rangle)$. $\square$

3.8. Corollary. *Every comeager set $A \subseteq \mathcal{N}_1$ has cardinality $2^{\aleph_1}$.*

See [38] for more about ω_1-perfectness.

3.9. Definition. For an ideal $\mathcal{I}$ on $\mathcal{N}_1$ let

i) $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I},\ \bigcup\mathcal{A} \notin \mathcal{I}\}$,
ii) $\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I},\ \bigcup\mathcal{A} = \mathcal{N}_1\}$ and
iii) $\mathrm{unif}(\mathcal{I}) = \min\{|A| \mid A \subseteq \mathcal{N}_1,\ A \notin \mathcal{I}\}$.

We say that an ideal $\mathcal{I}$ is a κ-*ideal*, if $\mathrm{add}(\mathcal{I}) \geq \kappa$. The family of meager sets, $\mathcal{M}$, is a proper ω_2-ideal. If $2^{\omega_1} = \omega_2$ then of course $\mathrm{add}(\mathcal{M}) = \omega_2$. In Section 8 we will prove that

$$Con(ZF) \rightarrow Con(ZFC + CH + \mathrm{add}(\mathcal{M}) = 2^{\omega_1} + \omega_2 < 2^{\omega_1}).$$

The question whether every non-meager set has cardinality 2^{ω_1}, i.e. $\mathrm{unif}(\mathcal{M}) = 2^{\omega_1}$, is open.

The theory of the Banach–Mazur game on reals is studied in Oxtoby [28]. Here our game has length $\leq \omega_1$ and so we get two different games depending on what the rules are at limit ordinals.

3.10. Definition. Let $\alpha > 0$ be an ordinal. The *Banach–Mazur game of length* α for a set $A \subseteq \mathcal{N}_1$ is the following game between players **I** and **II**. The players choose sequences $s_\xi, s'_\xi \in \mathrm{Seq}$.

$$\begin{array}{lcccccc} \mathbf{I}: & s_0 & & \dots & s_\xi & & \dots \\ \mathbf{II}: & & s'_0 & \dots & & s'_\xi & \end{array}$$

where $\xi < \alpha$. Player **I** moves first on limits. The rules of the round $\delta < \alpha$ are $\bigcup_{\xi<\delta} s'_\xi \prec s_\delta$ and $s_\delta \prec s'_\delta$. Player **I** wins, if $\bigcap_{\delta<\alpha}[s'_\delta] \cap A \neq \varnothing$. We denote this game by $BM^\alpha(A)$.

The *smooth Banach–Mazur game*, smooth $BM^\alpha(A)$ or s-$BM^\alpha(A)$, is like that above but player **I** has to move $\bigcup_{\xi<\delta} s'_\xi$ at limits $\delta > 0$.

3.11. Proposition. *If* **I** $\uparrow$ *s-$BM^\alpha(A)$ then* **I** $\uparrow$ *$BM^\alpha(A)$. If* **II** $\uparrow$ *$BM^\alpha(A)$ then* **II** $\uparrow$ *s-$BM^\alpha(A)$.*

The following example shows that these implications cannot be reversed.

3.12. Example. Let $A = \{x \in \mathcal{C}_1 \mid \{\alpha \mid x(\alpha) = 0\}$ contains a cub$\}$. Then **I** $\uparrow$ $BM^{\omega_1}(A)$, but **II** $\uparrow$ s-$BM^{\omega_1}(A)$.

Proof. In $BM^{\omega_1}(A)$ the player **I** makes the move at limits. It is enough to know how he moves at limits. For each limit $\delta > 0$ let $\alpha_\delta = \min\{\alpha \in \mathrm{Lim} \mid \alpha > \sup\{\ell(s'_\xi) \mid \xi < \delta\}\}$. Then **I** chooses $s_\delta \in \mathrm{Seq}$ such that $\ell(s_\delta) = \alpha_\delta + 1$, $s_\delta \succ s_\xi$, for all $\xi < \delta$, and $s_\delta(\alpha_\delta) = 0$. Since $\{\alpha_\delta \mid \delta < \omega_1\}$ is cub, player **I** wins the play no matter what the other moves are.

In s-$BM^{\omega_1}(A)$ the player **II** makes the move at limits $\delta > 0$. As above let $s'_\delta \in \mathrm{Seq}$ such that $\ell(s'_\delta) = \alpha_\delta + 1$, $s'_\delta \succ s_\xi$, for all $\xi < \delta$, and $s'_\delta(\alpha_\delta) = 1$. Since $\{\alpha_\delta \mid \delta < \omega_1\}$ is cub, player **II** wins the play no matter what the other moves are. □

3.13. Lemma. **II** $\uparrow$ *(s-)$BM^\alpha(A)$, where $\alpha < \omega_1$, if and only if* **II** $\uparrow$ *$BM^1(A)$.*

Proof. Assume that τ is a winning strategy for **II** in (s-)$BM^\alpha(A)$. Let s be **I**'s move in $BM^1(A)$. To get a winning strategy for **II** in $BM^1(A)$ (i.e. to find $s' \succ s$ such that $[s'] \cap A = \varnothing$) player **I** plays a game $BM^\alpha(A)$ where he starts with $s_0 = s$ and **II** uses strategy τ. So let $s'_0 = \tau(s_0)$, and choose s_ξ, $0 < \xi < \alpha$, such that $\bigcup_{\xi<\delta} s'_\xi \prec s_\delta$ where $s'_\delta = \tau(\langle s_\xi \rangle_{\xi<\delta})$. Now we can let $s' = \bigcup_{\xi<\alpha} s'_\xi$.

Let τ be **II**'s winning strategy in $BM^1(A)$. In (s-)$BM^\alpha(A)$ player **II** starts with τ and then he can choose his moves arbitrarily. □

Since a winning strategy for **II** in $BM^1(A)$ is an od-function for $\mathcal{N}_1 \setminus A$ we get the following proposition.

3.14. Proposition. *A set $A \subseteq \mathcal{N}_1$ is nowhere dense if and only if* $\mathbf{II} \uparrow BM^1(A)$.

The proof of the following theorem is an elaboration of [17, Proposition 27.3, p. 373] which is proved for the Baire space. The forward directions were originally noted by Mazur, and Banach showed the converse. Oxtoby generalized it to arbitrary topological spaces. See [17] for the references.

3.15. Theorem (CH). *i) A is meager if and only if* $\mathbf{II} \uparrow BM^{\omega_1}(A)$.
ii) $[s]\setminus A$ is meager for some $s \in$ Seq if and only if $\mathbf{I} \uparrow$ *s-*$BM^{\omega_1}(A)$.

Proof. i) Let $A = \bigcup_{\xi<\omega_1} R_\xi$, where R_ξ, $\xi < \omega_1$, are nowhere dense. Let F_ξ be an od-function for $\mathcal{N}_1 \setminus R_\xi$. If s_ξ is the move of player **I** at round ξ, let $s'_\xi = F_\xi(s_\xi)$. Now $f = \bigcup_{\xi<\omega_1} s_\xi \notin A$.

Suppose now that **II** has a winning strategy τ. For each partial play according to τ of the form $p = \langle s_\xi, s'_\xi\rangle_{\xi<\gamma}$, let $p_* = \bigcup_{\xi<\gamma} s'_\xi$ and set

$$D_p = \{x \in \mathcal{N}_1 \mid p_* \subseteq x \to \exists t \in \mathrm{Seq}\setminus\{\langle\rangle\}(\tau(p\hat{\ }\langle p_*\hat{\ }t\rangle) \subseteq x)\}.$$

Then each D_p is open and dense: for if $u \in$ Seq, either $u \not\supseteq p_*$ so that $[u] \subseteq D_p$, or else there is a $t \in \mathrm{Seq}\setminus\{\langle\rangle\}$ such that $p_*\hat{\ }t = u$ and so $[\tau(p\hat{\ }\langle p_*\hat{\ }t\rangle)] \subseteq [u] \cap D_p$. There are ω_1 partial plays p. Moreover, for any $x \in \bigcap_p D_p$ we will recursively define a play $\langle s_\xi, s'_\xi\rangle$ according to τ such that $x = \bigcup_{\xi<\omega_1} s'_\xi$. Assume that $p = \langle s_\xi, s'_\xi\rangle_{\xi<\gamma}$ is defined. Then let $s_\gamma = p_*\hat{\ }t$ where $t\setminus\{\langle\rangle\}$ is such that $\tau(p\hat{\ }\langle p_*\hat{\ }t\rangle) \subseteq x$; such a t exists since $x \in D_p$. Then $x \notin A$ because τ is a winning strategy for **II**. Consequently, $A \subseteq \bigcup_p(\mathcal{N}_1\setminus D_p)$, an ω_1-union of nowhere dense sets.

ii) Assume that $[s]\setminus A$ is meager for some $s \in$ Seq. Let $[s] \cap A \supseteq \bigcap_{\xi<\omega_1} D_\xi$ where D_ξ are open and dense in $[s]$. Let **I**'s first move s_0 be s. At $\delta + 1$th move **I** chooses $s_{\delta+1}$ such that $s'_\delta \prec s_{\delta+1}$ and $[s_{\delta+1}] \subseteq D_\delta$. At limit γ player **I** has to move $\bigcup_{\xi<\gamma} s'_\xi$. This is a winning strategy for **I** in s-$BM^{\omega_1}(A)$.

If τ is **I**'s winning strategy in s-$BM^{\omega_1}(A)$ then let $s = \tau(\langle\rangle)$. Then τ is **II**'s winning strategy for $BM^{\omega_1}([s]\setminus A)$. By i) $[s]\setminus A$ is meager. □

3.16. Definition. A set A is *weakly meager*, if $\mathbf{II} \uparrow$ s-$BM^{\omega_1}(A)$.

By Proposition 3.11 and Theorem 3.15 every meager set is weakly meager. The set A in Example 3.12 is weakly meager but not meager.

3.17. Theorem. *The set of all weakly meager sets is a proper ω_2-ideal.*

Proof. It is clear that if $A \subseteq B$ then $\mathbf{II} \uparrow$ s-$BM^{\omega_1}(B)$ implies $\mathbf{II} \uparrow$ s-$BM^{\omega_1}(A)$.

Assume that σ_δ is **II**'s winning strategy in s-$BM^{\omega_1}(A_\delta)$. Let $\pi : \omega_1 \times \omega_1 \to \omega_1$ be a bijection such that $\xi < \xi'$ implies $\pi(\delta,\xi) < \pi(\delta,\xi')$ for all δ. We define a strategy σ for **II** in s-$BM^{\omega_1}(\bigcup_{\delta<\omega_1} A_\delta)$ by

$$\sigma(\langle s_\xi\rangle_{\xi\leq\pi(\xi,\delta)}) = \sigma_\delta(\langle s_{\pi(\xi',\delta)}\rangle_{\xi'\leq\xi}).$$

If $x \in \bigcap_{\xi<\omega_1}[s'_\xi]$ where s'_ξ are the moves given by σ then $x \notin A_\delta$ since $\langle s'_{\pi(\xi,\delta)}\rangle_{\xi<\omega_1}$ is a play according to σ_δ. Thus σ is a winning strategy for **II**.

We will prove that if A is weakly meager then the complement of A is a dense set of cardinality $2^{\aleph_1}$. Hence $\mathcal{N}_1$ can not be weakly meager. Let σ be a winning strategy for **II** in s-$BM^{\omega_1}(A)$. Fix $g \in 2^{\omega_1}$ and $s \in$ Seq. Let $s_0 = s$, $s_{\xi+1} = s'_\xi{}^\frown\langle g(\xi)\rangle$ and $s'_\xi = \sigma(\langle s_\delta\rangle_{\delta<\xi})$. Now $x \in \bigcap_{\xi<\omega_1}[s_\xi]$ is in $[s]\setminus A$, and for each different g we get a different such an x. □

3.18. Definition. A set A has *the property of Baire*, if there is an open set G such that $A \triangle G$ is meager. We denote by PB the class of sets with the property of Baire.

3.19. Theorem. *There is a set which does not have the property of Baire.*

Proof. Using the axiom of choice let $\mathcal{F}$ be a uniform ultrafilter on ω_1 extending the filter $\{a \subseteq \omega_1 \mid |\omega_1\setminus a| < \omega_1\}$. Let $F = \{\chi_a \in \mathcal{C}_1 \mid a \in \mathcal{F}\}$. Then $\mathcal{F}_s = \{a \in \mathcal{F} \mid s \subseteq \chi_a\}$ is an ultrafilter on $\omega_1\setminus\ell(s)$ for every $s \in$ Seq. Let $F_s = \{\chi_a \in \mathcal{C}_1 \mid a \in \mathcal{F}_s\}$. We prove that F does not have the property of Baire. We use here the following facts

i) For every $f \in \mathcal{C}_1$, a set M is meager if and only if $f + M$ is;
ii) Let $\overline{1} = \chi_{\omega_1}$. For every $a \subseteq \omega_1$, $\chi_a + \overline{1} = \chi_{\omega_1\setminus a}$. Thus $F + \overline{1} = \mathcal{C}_1\setminus F$, since $\mathcal{F}$ is an ultrafilter.

Suppose, towards a contradiction, that there is an open G such that $F \triangle G$ is meager.

Case 1. G is empty. Then F and $F + \overline{1} = \mathcal{C}_1\setminus F$ are both meager which contradicts the Baire Category Theorem (Corollary 3.4).

Case 2. $[s] \subseteq G$ for some $s \in$ Seq. Then $(F \triangle G) \cap [s] = [s]\setminus F = [s]\setminus F_s$ should be meager. Let $f \in \mathcal{C}_1$ be such that $f(\xi) = 0$, if $\xi < \ell(s)$, and $f(\xi) = 1$ otherwise. Then $f + [s]\setminus F = [s] \cap F = F_s$ is also meager. This contradicts Corollary 3.4. □

The next lemma implies that the Borel sets have the property of Baire. The proof is just like the one for the Baire space, e.g. in [27].

3.20. Lemma. *The family of sets with the property of Baire is closed under complement and* $\bigcup_{\omega_1}$.

Proof. If P is closed then let P^* be the open set $\{x \in P \mid \exists\alpha(U(x,\alpha) \subseteq P)\}$. Now $P\setminus P^*$ is nowhere dense since its complement $(\mathcal{N}_1\setminus P) \cup P^*$ is open and dense: if $x \in P\setminus P^*$ then for each α there is some $y \in U(x,\alpha)\setminus P$, hence there is $\beta > \alpha$ such that $U(y,\beta) \subseteq \mathcal{N}_1\setminus P$ since P is closed. So $P \triangle P^*$ is meager.

If there is closed set P such that $Q \triangle P$ is meager then Q has the property of Baire, because

$$Q \triangle P^* = (Q\setminus P^*) \cup (P^*\setminus Q) \subseteq (Q\setminus P) \cup (P\setminus P^*) \cup (P\setminus Q)$$

is meager.

Let $'$ denote complement. Now if Q has the property of Baire there is an open P such that $Q \triangle P$ is meager. Since $Q' \triangle P' = Q \triangle P$ we conclude that Q' has the property of Baire.

Assume that the sets P_ξ, $\xi < \omega_1$, have the property of Baire. There are open sets Q_ξ, $\xi < \omega_1$, such that the sets $P_\xi \triangle Q_\xi$ are meager. Then

$$(\bigcup_{\xi<\omega_1} P_\xi)\triangle(\bigcup_{\xi<\omega_1} Q_\xi) \subseteq \bigcup_{\xi<\omega_1}(P_\xi \triangle Q_\xi)$$

is meager. Since $\bigcup_{\xi<\omega_1} Q_\xi$ is open this shows that $\bigcup_{\xi<\omega_1} P_\xi$ has the property of Baire. □

3.21. Lemma. *Assume that A has the property of Baire. Then A is meager if and only if for every $s \in$ Seq, $[s]\smallsetminus A$ is not meager.*

Proof. If there is $s \in$ Seq such that A is meager and $[s]\smallsetminus A$ is meager then we get that $[s]$ is meager, which contradicts the Baire Category Theorem.

If A is not meager, then there is a non-empty open B such that $A \triangle B$ is meager. Choose $[s] \subseteq B$. Now $[s]\smallsetminus A \subseteq B\smallsetminus A$ is meager. □

3.22. Corollary. *If A has the property of Baire then $BM^{\omega_1}(A)$ is determined.*

Proof. We use Theorem 3.15. If A is meager then $\mathbf{II} \uparrow BM^{\omega_1}(A)$. Otherwise $[s]\smallsetminus A$ is meager for some $s \in$ Seq, hence $\mathbf{I} \uparrow$ s-$BM^{\omega_1}(A)$ which implies $\mathbf{I} \uparrow BM^{\omega_1}(A)$. □

3.23. Definition. Let $A_s \subseteq \mathcal{N}_1$ for all $s \in$ Seq. *The Suslin operation of the system* (A_s) is the set

$$\bigcup_{f\in\mathcal{N}_1}\bigcap_{\xi\in\omega_1} A_{f\restriction\xi}.$$

A system (A_s) is *regular*, if

i) $A_s \subseteq A_{s'}$ for all $s, s' \in$ Seq such that $s' \prec s$ and

ii) $A_s = \bigcap_{\xi<\ell(s)} A_{s\restriction\xi}$, for all $s \in$ Seq such that $\ell(s)$ is a limit ordinal.

If $(A_s)_{s\in\mathrm{Seq}}$, is any system, then the system (B_s) where $B_{\langle\rangle} = A_{\langle\rangle}$ and $B_s = \bigcap_{\xi<\ell(s)} A_{s\restriction\xi}$, for $s \in$ Seq, is regular and

$$\bigcup_{f\in\mathcal{N}_1}\bigcap_{\xi\in\omega_1} A_{f\restriction\xi} = \bigcup_{f\in\mathcal{N}_1}\bigcap_{\xi\in\omega_1} B_{f\restriction\xi}.$$

If (A_s) is a system, let for each $x \in \mathcal{N}_1$, $T(x) = \{s \in \mathrm{Seq} \mid x \in A_s\}$. Then $x \in \bigcup_{f\in\mathcal{N}_1}\bigcap_{\xi\in\omega_1} A_{f\restriction\xi}$ iff there is an ω_1-branch in $T(x)$. If (A_s) is regular then for all x, $T(x)$ is a tree which has maximal elements if $x \notin \bigcup_{f\in\mathcal{N}_1}\bigcap_{\xi\in\omega_1} A_{f\restriction\xi}$.

3.24. Conjecture (CH). *The family of sets with the property of Baire is closed under Suslin operation.*

The proof of Conjecture 3.24 may need an additional hypothesis such as $I(\omega)$. At least $2^\omega < 2^{\omega_1}$ must be assumed as the following example shows.

3.25. Example. Assume that $2^\omega = 2^{\omega_1}$. Then the property of Baire is not closed under the Suslin operation. Theorem 3.19 gives a set $A \subseteq \mathcal{C}_1$ which does not have the property of Baire. Let $A = \{f_s \mid s \in 2^\omega\}$. Set $A_s = \{f_{s\restriction\omega}\}$ for each $s \in \mathrm{Seq}$ such that $\ell(s) \geq \omega$ and $A_s = \mathcal{C}_1$ otherwise. Then $A = \bigcup_{g\in\mathcal{N}_1}\bigcap_{\xi<\omega_1} A_{g\restriction\xi}$ is a set obtained by Suslin operation from singletons.

4. Strong measure zero sets

E. Borel [1] introduced strong measure zero sets of reals. He conjectured that they all are countable. But using CH one can construct a Lusin set which is a strong measure zero set of cardinality ω_1. R. Laver [23] was able to prove the consistency of the Borel conjecture.

Here we examine analogically the strong measure zero sets of $\mathcal{N}_1$.

4.1. Definition. Let A be subset of $\mathcal{N}_1$.

i) Assume $X \subseteq \omega_1$. We say that A is *X-small*, if for each $\alpha \in X$ there is $f_\alpha \in \mathcal{X}$ such that $A \subseteq \bigcup_{\alpha\in X} U(f_\alpha, \alpha)$. We say that A is *σ-X-small*, if for each $\alpha \in X$ there are $f_\alpha^n \in \mathcal{X}$, $n \in \omega$, such that $A \subseteq \bigcup_{\alpha\in X}\bigcup_{n\in\omega} U(f_\alpha^n, \alpha)$.

ii) A set A is *small*, if A is σ-$\{\alpha\}$-small for every $\alpha < \omega_1$.

iii) Let $Z \subseteq \mathcal{P}(\omega_1)$. A set A is *Z-null*, if A is X-small for every $X \in Z$.

iv) A set A has *strong measure zero*, if A is X-small for all $X \subseteq \omega_1$ such that $|X| = \omega_1$ i.e. A is $[\omega_1]^{\omega_1}$-null.

v) Let $\langle\alpha_\xi\rangle_{\xi<\omega_1}$ be a sequence. A set A is *$\langle\alpha_\xi\rangle$-small*, if there are f_ξ, $\xi < \omega_1$, such that $A \subseteq \bigcup_{\xi\in\omega_1}[f_\xi\restriction\alpha_\xi]$.

For $X, Y \in [\omega_1]^{\omega_1}$ we denote $X < Y$ if there is a bijection $F : X \to Y$ such that $\alpha < F(\alpha)$ for all $\alpha \in X$. Similarly $X \leq Y$. For each $\alpha \in \omega_1$, let $X+\alpha = \{\xi+\alpha \mid \xi \in X\}$. Let S_X be the collection of all X-small sets.

4.2. Proposition. *If $X < Y$, then $S_Y \subsetneq S_X$.*

Proof. It is clear that $S_Y \subseteq S_X$. If $X < Y$, then $X < X+1 \leq Y$. Let $A = \bigcup_{\xi\in X} U(f_\xi, \xi)$, where $f_\xi\restriction\xi \perp f_\delta\restriction\delta$, when $\xi \neq \delta$. We will prove that $A \notin S_{X+1}$. Let $B = \bigcup_{\xi\in X} U(g_\xi, \xi+1) \in S_{X+1}$. We will show that $A \neq B$ by defining $f \in A\setminus B$. Let $f\restriction\xi_0 = f_0\restriction\xi_0$ where $\xi_0 = \min X$. For each $\xi \in X$, choose $f(\xi) = g_\xi(\xi)+1$. Otherwise define $f(\xi)$ arbitrarily. Now $f \in A\setminus B$. $\square$

4.3. Lemma. *i) A is small if and only if A is $[\omega_1]^\omega$-null.*

ii) If A is small, then it has strong measure zero.

Proof. i) Assume that A is small and $|X| = \omega$. Let $\alpha = \sup X$. Since A is σ-$\{\alpha\}$-small there are f_ξ, $\xi \in X$, such that $A \subseteq \bigcup_{\xi\in X} U(f_\xi, \alpha)$. But $\bigcup_{\xi\in X} U(f_\xi, \alpha) \subseteq \bigcup_{\xi\in X} U(f_\xi, \xi)$ and so A is X-small.

Assume that A is X-small for all $X \in [\omega_1]^\omega$. If $\alpha \in \omega_1$ is given then there are f_n, $n \in \omega$, such that $A \subseteq \bigcup_{n\in\omega} U(f_n, \alpha+n)$. So $A \subseteq \bigcup_{n\in\omega} U(f_n, \alpha)$ and hence A is σ-$\{\alpha\}$-small.

ii) Let A be small and let $X \in [\omega_1]^{\omega_1}$ be arbitrary. Let $\alpha = \sup Y$ where Y is (any) infinite subset of X. Choose f_ξ, $\xi \in Y$, such that $A \subseteq \bigcup_{\xi\in Y} U(f_\xi, \alpha)$. Now $A \subseteq \bigcup_{\xi\in X} U(f_\xi, \xi)$, where f_ξ, $\xi \in X \setminus Y$, are arbitrary. □

4.4. Lemma. *The following are equivalent for $A \subseteq \mathcal{N}_1$.*

i) A has strong measure zero,

ii) A is $\langle\alpha_\xi\rangle$-small for every sequence $\langle\alpha_\xi\rangle$,

iii) A is σ-X-small for every uncountable $X \subseteq \omega_1$.

Proof. i) implies ii). Let $\langle\alpha_\xi\rangle$ be any sequence. Let $X = \{\alpha_\xi \mid \xi < \omega_1\}$. If A is X-small then A is $\langle\alpha_\xi\rangle$-small.

ii) implies iii). Assume that A is $\langle\alpha_\xi\rangle$-small for all $\langle\alpha_\xi\rangle$. Let $X \in [\omega_1]^{\omega_1}$ be given. Enumerate $X = \{\alpha_\xi \mid \xi < \omega_1\}$. If A is $\langle\alpha_\xi\rangle$-small then A is X-small and hence σ-X-small.

iii) implies i). Assume that A is σ-X-small for every uncountable $X \subseteq \omega_1$. Let $X \in [\omega_1]^{\omega_1}$ be given. We split $X = \bigcup_{\alpha<\omega_1} X_\alpha$ where the sets X_α, $\alpha < \omega_1$, are countable and disjoint. Let $\beta_\alpha = \sup X_\alpha$. Since A is σ-$\{\beta_\alpha \mid \alpha < \omega_1\}$-small, there are f^n_α so that

$$A \subseteq \bigcup_{\alpha<\omega_1} \Big(\bigcup_{n\in\omega} U(f^n_\alpha, \beta_\alpha)\Big).$$

Choose g_ξ, $\xi \in X$, such that $\{g_\xi \mid \xi \in X_\alpha\} = \{f^n_\alpha \mid n \in \omega\}$. Now $A \subseteq \bigcup_{\xi\in X} U(g_\xi, \xi)$. □

4.5. Lemma. *The class of the strong measure zero sets is an $\aleph_2$-complete ideal.*

Proof. Assume that $A = \bigcup_{\xi<\omega_1} A_\xi$ where the sets A_ξ have strong measure zero. Let $X \in [\omega_1]^{\omega_1}$ and split $X = \bigcup_{\xi<\omega_1} X_\xi$ where $|X_\xi| = \omega_1$. Choose f_ξ, $\xi \in X$, such that $A_\delta \subseteq \bigcup_{\xi\in X_\delta} U(f_\xi, \xi)$ for all δ. Now $A \subseteq \bigcup_{\xi\in X} U(f_\xi, \xi)$. Thus A has strong measure zero. □

Assume that $Z \subseteq [\omega_1]^{\omega_1}$ satisfies the following: For each $X \in Z$ there are disjoint $X_\alpha \in Z$, $\alpha < \omega_1$, such that $X \leq \bigcup_{\alpha<\omega_1} X_\alpha$. Then we see as in the previous lemma that the class of Z-null sets is closed under ω_1-unions.

The next proposition shows the close connection of $\mathcal{N}_1$ being X-small and the diamond principles. We remaind that for $X \subseteq \omega_1$ the principle $\diamondsuit(X)$ states that there are sets $A_\alpha \subseteq \alpha$, $\alpha \in X$, such that for all $A \subseteq \omega_1$ the set $\{\alpha \in X \mid A \cap \alpha = A_\alpha\}$ is stationary. We shall return to this in Section 5.

4.6. Proposition. *i) Every countable set is small. Every set of cardinality ω_1 has strong measure zero.*

ii) If $A \subseteq \mathcal{N}_1$ is X-small where X is non-stationary and $0 \notin X$, then $\mathcal{N}_1 \setminus A$ has cardinality $2^{\aleph_1}$. Hence $\mathcal{N}_1$ does not have strong measure zero.

iii) Assume $\diamondsuit$. $\mathcal{N}_1$ is X-small for every cub set X.

iv) Assume $\diamondsuit(X)$. Then $\mathcal{N}_1$ is X-small. In particular, assuming $V = L$, $\mathcal{N}_1$ is X-small for every stationary set X.

Proof. i) Obvious.

ii) We may assume that $|X| = \omega_1$. Since X is non-stationary there is a cub set C such that $X \cap C = \varnothing$. For each $\alpha \in X$ let $F(\alpha) = \max(C \cap \alpha)$. Now there are f_α such that

$$A \subseteq \bigcup_{\alpha \in X} U(f_\alpha, \alpha) \subseteq \bigcup_{\alpha \in X} U(f_\alpha, F(\alpha) + 1) = B.$$

Since $\{\alpha \in X \mid F(\alpha) = \beta\}$ is countable for each β, we can define for each $g \in 2^{\omega_1}$

$$f_g(\beta) = \sup\{f_\alpha(\beta) \mid F(\alpha) = \beta\} + 1 + g(\beta)$$

for all $\beta \in \omega_1$. Now $f_g \neq f_{g'}$, if $g \neq g'$, and $f_g \notin B$ for every $g \in 2^{\omega_1}$.

iii) $\diamondsuit$ implies that there are sequences h_α, $\alpha < \omega_1$, such that for every $f \in \mathcal{N}_1$ the set $\{\alpha < \omega_1 \mid f\restriction\alpha = h_\alpha\}$ is stationary (see [13, (22.20)] or Lemma 5.3). So for every cub set X, $\{\alpha \in X \mid f\restriction\alpha = h_\alpha\}$ is stationary; hence $\mathcal{N}_1 = \bigcup_{\alpha \in X} [h_\alpha]$.

iv) $V = L$ implies $\diamondsuit(E)$ for every stationary E. Thus for each stationary E there are sequences h_α, $\alpha \in \omega_1$, such that for every $f \in \mathcal{N}_1$ the set $\{\alpha \in E \mid f\restriction\alpha = h_\alpha\}$ is stationary. □

4.7. Example. i) There is a small set of cardinality ω_1. Let $f \in \mathcal{N}_1$. Let

$$f^\delta(\xi) = \begin{cases} f(\xi), & \text{if } \xi < \delta \\ \delta & \text{otherwise.} \end{cases}$$

Now $A = \{f^\delta \mid \delta < \omega_1\}$ is small: Let α be given. Then $A \subseteq U(f, \alpha) \cup \bigcup_{\delta<\alpha} U(f^\delta, \alpha)$.

ii) Let $\overline{\delta}$ be the constant function δ. The set $\{\overline{\delta} \mid \delta < \omega_1\}$ is a set of cardinality ω_1, which is not small.

iii) Let $\mathcal{Q} = \{f \in \mathcal{N}_1 \mid \exists\delta\forall\xi > \delta[f(\xi) = f(\delta)]\}$. Then $\mathcal{Q}$ is dense, $|\mathcal{Q}| = 2^\omega$ and player **II** has a winning strategy in $BM^{\omega_1}(\mathcal{Q})$. He just makes sure that the result is not ultimately constant. If CH then $\mathcal{Q}$ has strong measure zero by 4.6.i).

4.8. Proposition. *There is a Kurepa tree if and only if there is a small set of cardinality $> \omega_1$.*

Proof. Let $T \subseteq \text{Seq}$ be a Kurepa tree i.e. levels of T are countable and the set of ω_1-branches, $\mathcal{B}_{\omega_1}(T)$, is of cardinality $\geq \omega_2$. Then $\mathcal{B}_{\omega_1}(T)$ is closed and small. On the other hand let A be a small set of cardinality $\geq \omega_2$. For each α choose $s_n^\alpha \in \text{Seq}$, $n \in \omega$, such that

$$A \subseteq \bigcup_{n \in \omega} [s_n^\alpha]$$

where $\ell(s_n^\alpha) = \alpha$ and $A \cap [s_n^\alpha] \neq \varnothing$. Now $T = \{s_n^\alpha \mid n \in \omega, \alpha \in \omega_1\}$ is a Kurepa tree. □

Let *the weak generalized Borel conjecture* (wGBC) be the statement that every small set has cardinality $\leq \omega_1$. If we Lévy collapse an inaccessible cardinal to $\aleph_2$ we get a model in which there are no Kurepa trees, hence wGBC is consistent with ZF. On the other hand $V = L$ implies that there is a Kurepa tree, so $\neg$wGBC is consistent with ZF.

The *generalized Borel conjecture* is the following statement.

Every strong measure zero set has power at most $\aleph_1$.

We call it GBC for short. By Lemma 4.3.ii) GBC implies wGBC. So $V = L$ implies the consistency of $\neg$GBC. We will see later that for the consistency of $\neg$GBC it is enough to assume $2^{\aleph_1} = \aleph_2$, see Lemma 6.4. The consistency of GBC is an open problem.

4.9. Proposition (CH). *If $D \subseteq [\omega_1]^{\omega_1}$ is such that $|D| = \omega_1$ then there is a D-null set Z and a meager set M such that $\mathcal{N}_1 = Z \cup M$.*

Proof. Let $\mathcal{Q} = \{q_\alpha \mid \alpha < \omega_1\}$. Let us enumerate $D = \{X_\alpha \mid \alpha < \omega_1\}$ and $X_\alpha = \{x_{\alpha\beta} \mid \beta < \omega_1\}$. Let $Z_\alpha = \bigcup_{\beta \in \omega_1} U(q_\beta, x_{\alpha\beta})$ and $Z = \bigcap_{\alpha<\omega_1} Z_\alpha$. It is clear that each Z_α is open and dense. So $M = \mathcal{N}_1 \setminus Z$ is meager. It is also clear that Z is D-null. □

4.10. Remark. Since M in Proposition 4.9 is meager, Z has cardinality 2^{ω_1} by Corollary 3.8. It is easy to see that **I** has a winning strategy in $BM^{\omega_1}(Z)$.

4.11. Proposition. *If A is small, then A is nowhere dense.*

Proof. We show **II** $\uparrow BM^1(A)$. If **I** plays s_0, then since $A \subseteq \bigcup_{n\in\omega} U(f_n, \ell(s_0) + 1)$ for some $f_n \in \mathcal{N}_1$, player **II** can choose $s_0' \in \omega_1^{\ell(s_0)+1}$ such that $s_0 \prec s_0'$ and $[s_0'] \cap A = \varnothing$. □

The converse of Proposition 4.11 does not hold. In fact:

4.12. Proposition (CH). *There is a nowhere dense set which does not have strong measure zero.*

Proof. Let $\mathcal{Q} = \{q_\alpha \mid \alpha < \omega_1\}$. Let $D = \bigcup_{\alpha<\omega_1} U(q_\alpha, \alpha + 1)$. Now D is open and dense, so $A = \mathcal{N}_1 \setminus D$ is nowhere dense. If A had strong measure zero, there would be some f_ξ, $\xi < \omega_1$, such that $A \subseteq \bigcup_{\xi<\omega_1} U(f_\xi, \xi + 1)$. Choosing $f \in \mathcal{N}_1$ such that for each $\xi < \omega_1$ $f(\xi) \neq q_\xi(\xi)$ and $f(\xi) \neq f_\xi(\xi)$ we would get $f \in A \setminus \bigcup_{\xi<\omega_1} U(f_\xi, \xi + 1)$ which is absurd. □

The idea for the following theorem is from Richard Laver. Let $P = \mathrm{Seq}$, and for $p, q \in P$, $p \leq q$ if and only if $q \subseteq p$.

4.13. Theorem. *Assume that $M \models ZFC+CH$. Let $P = \mathrm{Seq}^M$ and let G be P-generic over M. Then $\mathcal{N}_1 \cap M$ has strong measure zero in $M[G]$.*

Proof. The partial ordering P is countably closed and has $\aleph_2$-c.c., so forcing with it preserves all cardinals. Let $f \in \mathcal{N}_1 \cap M[G]$ be arbitrary. Let $\underset{\sim}{f}$ be a name for f. Working in M, let $C_\xi = \{p \in P \mid \ell(p) \geq \xi, p \Vdash \underset{\sim}{f}(\xi) = \gamma \text{ for some } \gamma\}$. Let $A_\xi \subseteq C_\xi$ be a maximal antichain such that $\ell(p) \geq \xi$ for all $p \in A_\xi$. For each $p \in A_\xi$, choose a bijection

$$g_\xi^p : \mathrm{succ}(p) \to \omega_1^\gamma,$$

where γ is such that $p \Vdash \underset{\sim}{f}(\xi) = \gamma$ and $\mathrm{succ}(p) = \{p\hat{\ }\langle\alpha\rangle \mid \alpha \in \omega_1\}$. Working in $M[G]$, let

$$I_\xi = [g_\xi^p(p')],$$

where p is the unique element of $G \cap A_\xi$ and p' is the unique element of $G \cap \mathrm{succ}(p)$. Now $\mathcal{N}_1 \cap M \subseteq \bigcup_{\xi \in \omega_1} I_\xi$, since for each $h \in \mathcal{N}_1 \cap M$ the set

$$D_h = \{p \mid \exists\xi\exists q, q'(q, q' \geq p, q \in A_\xi, q' \in \mathrm{succ}(q), h \in [g_\xi^q(q')])\}.$$

is dense in P. Hence, since $D_h \cap G \neq \varnothing$, $h \in \bigcup_{\xi \in \omega_1} I_\xi$. □

It is sometimes easier to manipulate sequences then sets. When we speak about sequences $\langle\alpha_\xi\rangle$, we assume that $\alpha_\xi \leq \alpha_\delta$, if $\xi \leq \delta$.

4.14. Definition. Assume that $A \subseteq \mathcal{N}_1$ and $\langle\alpha_\xi\rangle$ is an increasing sequence.

i) A is $\langle\alpha_\xi\rangle$-*subsmall*, if A is $\langle\beta_\xi\rangle$-small for some $\langle\beta_\xi\rangle$ such that $\beta_\xi \geq \alpha_{\xi+n}$ for all $n \in \omega$.

ii) A is σ-$\langle\alpha_\xi\rangle$-small, if there is $\{f_\xi^n \in \mathcal{N}_1 \mid \xi < \omega_1,\ n \in \omega\}$, such that

$$A \subseteq \bigcup_{\xi < \omega_1} \bigcup_{n \in \omega} U(f_\xi^n, \alpha_\xi).$$

Notice that A is $\langle\alpha_\xi\rangle$-subsmall, if and only if A is $\langle\sup\{\alpha_{\xi+n} \mid n \in \omega\}\rangle$-small. We can get some closure properties for these concepts.

4.15. Lemma. *Assume that $A \subseteq \mathcal{N}_1$ and $\langle\alpha_\xi\rangle$ is an increasing sequence.*

i) If A is $\langle\alpha_\xi\rangle$-small, then A is σ-$\langle\alpha_\xi\rangle$-small.

ii) If A is $\langle\alpha_\xi\rangle$-subsmall, then A is $\langle\alpha_\xi\rangle$-small.

iii) If A is σ-$\langle\alpha_{\xi+\omega}\rangle$-small, then A is $\langle\alpha_\xi\rangle$-subsmall.

iv) If A_n, $n \in \omega$, are $\langle\alpha_\xi\rangle$-subsmall, then $\bigcup_{n\in\omega} A_n$ is $\langle\alpha_\xi\rangle$-subsmall.

v) If A_n, $n \in \omega$, are σ-$\langle\alpha_\xi\rangle$-small, then $\bigcup_{n\in\omega} A_n$ is σ-$\langle\alpha_\xi\rangle$-small.

Proof. i), ii) and v) are trivial.

iii) Assume that $A \subseteq \bigcup_{\xi<\omega_1} [s_{\xi+\omega}]$, where $\ell(s_{\xi+\omega}) = \alpha_{\xi+\omega}$. For each limit ordinal ν set $s'_{\nu+n} = s_{\nu+\omega}$. Now $\ell(s'_{\nu+n}) = \ell(s_{\nu+\omega}) = \alpha_{\nu+\omega} \geq \alpha_{\nu+k}$ for all $k \in \omega$. It is clear that $A \subseteq \bigcup_{\xi<\omega_1} [s'_\xi]$.

iv) Assume that $A_n \subseteq \bigcup_{\xi<\omega_1}[s^n_\xi]$, where $\ell(s^n_\xi) \geq \alpha_{\xi+n}$ for all $n \in \omega$. Let $\rho : \omega \times \omega \to \omega$ be one-one and onto such that $\rho(n,m) \geq n$ for all $n \in \omega$. For each limit ordinal ν set

$$s^\omega_{\nu+\rho(n,m)} = s^m_{\nu+n}.$$

Now $\ell(s^\omega_{\nu+\rho(n,m)}) = \ell(s^m_{\nu+n}) \geq \alpha_{\nu+k}$ for all $k \geq n$, hence the same holds for all $k \geq \rho(n,m)$. Now it is clear that

$$\bigcup_{n\in\omega} A_n \subseteq \bigcup_{\xi<\omega_1}[s^\omega_\xi]$$

and so $\bigcup_{n\in\omega} A_n$ is $\langle\alpha_\xi\rangle$-subsmall. □

5. Diamonds

Here we shall study further the connection between X-smallness of $\mathcal{N}_1$ and the diamond principles seen in Proposition 4.6. We will introduce a generalization of the $\diamondsuit$-principle.

5.1. Definition. i) For each $E \subseteq \omega_1$, $A \subseteq \mathcal{N}_1$ and $\mathcal{I} \subseteq \mathcal{P}(\omega_1)$ the principle $\diamondsuit(E,A,\mathcal{I})$ states that

$$\exists\langle s_\alpha\rangle_{\alpha<\omega_1} \in \mathrm{Seq}^{\omega_1}\forall f \in A[\{\alpha \in E \mid f\restriction\alpha = s_\alpha\} \notin \mathcal{I}].$$

ii) For each $E \subseteq \omega_1$, $A \subseteq \mathcal{N}_1$ and $\mathcal{I} \subseteq \mathcal{P}(\omega_1)$ the principle $\diamondsuit^-(E,A,\mathcal{I})$ states that

$$\exists\langle S_\alpha\rangle_{\alpha<\omega_1} \in ([\mathrm{Seq}]^\omega)^{\omega_1}\forall f \in A[\{\alpha \in E \mid f\restriction\alpha \in S_\alpha\} \notin \mathcal{I}].$$

Thus $\diamondsuit(E,A,\mathcal{I})$ says that there is a sequence $\langle s_\xi\rangle$ which captures each element $f \in A$ E_f-many times where $E_f \subseteq E$ is some set not in $\mathcal{I}$. Here we are mainly interested in $\diamondsuit(E,A,\{\varnothing\})$ since it is equivalent to A being E-small. Let CUB be the cub-filter on ω_1 and NS be the ideal of non-stationary sets of ω_1. Then $\diamondsuit(\omega_1,\mathcal{N}_1,NS)$ is the usual $\diamondsuit$-principle and for $E \subseteq \omega_1$, $\diamondsuit(E,\mathcal{N}_1,NS)$ is usually denoted by $\diamondsuit(E)$, see Lemma 5.3. $\diamondsuit(\omega_1,\mathcal{N}_1,\{\varnothing\})$ is equivalent with $\diamondsuit$, see [3]. Also $\diamondsuit(\omega_1,\mathcal{N}_1,NS)$ is equivalent with $\diamondsuit^-(\omega_1,\mathcal{N}_1,NS)$, see [5].

The following proposition follows directly from the definition.

5.2. Proposition. *i) If $A \subseteq B$ then $\diamondsuit(E,B,\mathcal{I})$ implies $\diamondsuit(E,A,\mathcal{I})$.*
ii) If $E \subseteq E'$ and $\mathcal{I}$ is closed under subsets then $\diamondsuit(E,A,\mathcal{I})$ implies $\diamondsuit(E',A,\mathcal{I})$.
iii) If $\mathcal{I}' \subseteq \mathcal{I}$ then $\diamondsuit(E,A,\mathcal{I})$ implies $\diamondsuit(E,A,\mathcal{I}')$.
iv) If $\mathcal{I}$ is closed under subsets and $E \in \mathcal{I}$ then $\neg\diamondsuit(E,A,\mathcal{I})$.
The same implications hold for $\diamondsuit^-$.

5.3. Lemma. *For each stationary $E \subseteq \omega_1$, the following are equivalent* *i)* $\diamondsuit(E)$

ii) $\diamondsuit(E, \mathcal{C}_1, NS)$

iii) $\diamondsuit(E, \mathcal{N}_1, NS)$.

Proof. i) implies iii). Assume $\diamondsuit(E)$ and let $\langle \diamondsuit_\alpha \rangle$ be a $\diamondsuit(E)$-sequence. Let $\pi : \omega_1 \times \omega_1 \to \omega_1$ be a bijection. There is a cub set C such that if $\alpha \in C$ then α is a limit ordinal and $\pi\restriction\alpha$ is a bijection $\alpha \times \alpha \to \alpha$. For $\alpha \in C$ let $s_\alpha = \{\pi^{-1}(\xi) \mid \xi \in \diamondsuit_\alpha\}$ if it is a function. Otherwise set $s_\alpha = \bar{0}$. Let $f \in \mathcal{N}_1$ and $X = \{\pi(\langle \xi, \delta \rangle) \mid \langle \xi, \delta \rangle \in f\} = \pi''f$. Now $\{\alpha \in E \mid X \cap \alpha = \diamondsuit_\alpha\}$ is stationary, so $\{\alpha \in C \cap E \mid X \cap \alpha = \diamondsuit_\alpha\} = \{\alpha \in C \cap E \mid f\restriction\alpha = s_\alpha\}$ is stationary.

ii) implies i). Assume that $\langle s_\alpha \rangle$ is a $\diamondsuit(E, \mathcal{C}_1, NS)$-sequence. Let $\diamondsuit_\alpha = \{\xi < \alpha \mid s_\alpha(\xi) = 1\}$ for each $\alpha < \omega_1$. Now $\langle \diamondsuit_\alpha \rangle$ is a $\diamondsuit(E)$-sequence because if $A \subseteq \omega_1$ then by letting $f = \chi_A$ we see that $\{\alpha \in E \mid f\restriction\alpha = s_\alpha\} = \{\alpha \in E \mid A \cap \alpha = \diamondsuit_\alpha\}$ is stationary.

iii) implies ii). Assume that $\langle s_\alpha \rangle$ is a $\diamondsuit(E, \mathcal{N}_1, NS)$-sequence. Then $\{\alpha \in E \mid f\restriction\alpha = s_\alpha\}$ is stationary for all $f \in \mathcal{N}_1$. So $\{\alpha \in E \mid f\restriction\alpha = s_\alpha\}$ is stationary for all $f \in \mathcal{C}_1$ i.e. $\diamondsuit(E, \mathcal{C}_1, NS)$ holds. □

5.4. Definition. For $A \subseteq \mathcal{N}_1$, let $\mu(A) = \{E \subseteq \omega_1 \mid \neg\diamondsuit(E, A, NS)\}$ and $\mu'(A) = \{E \subseteq \omega_1 \mid \neg\diamondsuit(E, A, \{\varnothing\})\}$.

Thus $\mu'(A) = \{E \subseteq \omega_1 \mid A \text{ not } E\text{-small}\}$. Note that $\mu'(A) \subseteq \mu(A)$ because $\diamondsuit(E, A, NS)$ implies $\diamondsuit(E, A, \{\varnothing\})$.

5.5. Proposition. *i) If $\diamondsuit$ holds, then $E \notin \mu(\mathcal{N}_1)$ for all cub set E.*

ii) If $\diamondsuit(E)$, then $E \notin \mu(\mathcal{N}_1)$.

iii) If E is non-stationary, then $E \in \mu'(\mathcal{N}_1)$.

iv) If $V = L$ then $\mu(\mathcal{N}_1) = NS$.

Proof. i) If $\diamondsuit$ holds, then $\mathcal{N}_1$ is E-small for every cub set E by Proposition 4.6.iii. ii) $\mathcal{N}_1$ is E-small by Lemma 4.6.iii. iii) See Lemma 4.6.ii. iv) If $V = L$ then $\diamondsuit(E)$ holds for every stationary E, see [14]. The claim follows then from ii) and iii). □

It is an open problem whether $\mu(\mathcal{N}_1) = \mu'(\mathcal{N}_1)$ i.e. whether there is a stationary E such that $\diamondsuit(E, \mathcal{N}_1, \{\varnothing\})$, but not $\diamondsuit(E, \mathcal{N}_1, NS)$. The following proposition shows that μ and μ' have some of the properties of a measure.

5.6. Proposition. *i) $\mu(\varnothing) = \mu'(\varnothing) = \varnothing$.*

ii) If $A \subseteq B$, then $\mu(A) \subseteq \mu(B)$ and $\mu'(A) \subseteq \mu'(B)$.

iii) If A is small, then $\mu'(A) \subseteq \{E \mid |E| < \omega\}$.

iv) If A has strong measure zero, then $\mu'(A) \subseteq \{E \mid |E| \leq \omega\}$.

v) If A is Z-null, then $\mu'(A) \cap Z = \varnothing$.

Proof. i) $\diamondsuit(E, \varnothing, NS)$ holds trivially for every $E \subseteq \omega_1$.

ii) If $A \subseteq B$ then $\diamondsuit(E, B, NS)$ implies trivially $\diamondsuit(E, A, NS)$.

iii) If A is small and $E \subseteq \omega_1$, $|E| \geq \omega$, then there are $s_\alpha \in \alpha^\alpha$, $\alpha \in E$, such that $A \subseteq \bigcup_{\alpha \in E}[s_\alpha]$ i.e. $\diamondsuit(E, A, \{\varnothing\})$. See Lemma 4.3.i).

iv) If A has strong measure zero and $E \subseteq \omega_1$, $|E| = \omega_1$, then there are $s_\alpha \in \omega_1^\alpha$, $\alpha \in E$, such that $A \subseteq \bigcup_{\alpha \in E}[s_\alpha]$ i.e. $\diamondsuit(E, A, \{\varnothing\})$.

v) If $X \in Z$ then $\diamondsuit(X, A, \{\varnothing\})$ i.e. $X \notin \mu'(A)$. □

The idea for the following theorem is from [6, Theorem 1.10, p. 144].

5.7. Theorem. *$\mu(\mathcal{N}_1)$ is an ω_1-ideal.*

Proof. Let $I = \{E \subseteq \omega_1 \mid \neg\diamondsuit(E, \mathcal{N}_1, NS)\}$. Let $E_n \in I$, $n \in \omega$, and $E = \bigcup_{n \in \omega} E_n$. Let $\theta : \omega_1 \times \omega \to \omega_1$ be a bijection. There is a cub set C of limit ordinals such that if $\alpha \in C$ then $\theta''(\alpha \times \omega) = \alpha$. If $\alpha \in C$ then set for each $s \in \omega_1^\alpha$, $n \in \omega$ and $\xi < \alpha$

$$s^{(n)}(\xi) = s(\theta(\xi, n)).$$

To prove $\neg\diamondsuit(E, \mathcal{N}_1, NS)$, let $\langle s_\alpha \rangle_{\alpha \in E}$ be arbitrary. We choose elements $f_n \in \mathcal{N}_1$ and cub sets C_n such that for each $n \in \omega$

$$C_n \cap C \cap \{\alpha \in E_n \mid f_n \restriction \alpha = s_\alpha^{(n)}\} = \varnothing.$$

Let $f(\theta(\xi, n)) = f_n(\xi)$ and $D = C \cap \bigcap_{n \in \omega} C_n$. We claim that $D \cap \{\alpha \in E \mid f \restriction \alpha = s_\alpha\} = \varnothing$. If $\alpha \in C \cap E$, let n be such that $\alpha \in E_n$. If $f \restriction \alpha = s_\alpha$, then for all $\xi < \alpha$, $f(\theta(\xi, n)) = s_\alpha(\theta(\xi, n))$, which means $f_n \restriction \alpha = s_\alpha^{(n)}$. Therefore $\alpha \notin C_n$. □

We do not know for which sets $A \subseteq \mathcal{N}_1$, the set $\mu(A)$ is an ω_1-ideal.

6. Variants

There has recently been a lot of research into negligible subsets of reals. For example, the so called point-open game has been studied in [30] and [29]. We shall define the classes of negligible subsets of $\mathcal{N}_1$ by generalizing the corresponding definitions for the reals. We shall get similar hierarchy results for the classes of $\mathcal{N}_1$ as for the classes of reals.

6.1. Definition. i) Let $A \subseteq \mathcal{N}_1$. The *point-open game G_A* is the following game between players **I** and **II**. The game G_A has length ω_1. On the ξth round player **I** chooses an element $f_\xi \in \mathcal{N}_1$ and then player **II** chooses an ordinal $\varepsilon_\xi \in \omega_1$. Player **I** wins, if

$$A \subseteq \bigcup_{\xi \in \omega_1} U(f_\xi, \varepsilon_\xi).$$

Otherwise player **II** wins.

ii) Let $A \subseteq \mathcal{N}_1$. The game G_A^* is the following game between players **I** and **II**. The game G_A^* has length ω_1. On the ξth round player **II** chooses an open cover $\mathcal{J}_\xi$ of A and then player **I** chooses an element $J_\xi \in \mathcal{J}_\xi$. Player **I** wins, if

$$A \subseteq \bigcup_{\xi \in \omega_1} J_\xi.$$

otherwise player **II** wins.

iii) A set A of cardinality 2^{ω_1} is called a *Lusin set*, if $|A \cap R| \leq \omega_1$ for every nowhere dense set R.

iv) A set A has *Rothberger's property*, if for all $\{\varepsilon_\xi^f \in \omega_1 \mid \xi \in \omega_1, f \in A\}$ there is $\langle f_\xi \rangle_{\xi \in \omega_1}$ such that

$$A \subseteq \bigcup_{\xi \in \omega_1} U(f_\xi, \varepsilon_\xi^{f_\xi}).$$

v) Let $A, Y \subseteq \mathcal{N}_1$. A set A is *concentrated around* Y, if $|A \setminus G| < \omega_1$ for every open set $G \supseteq Y$. A set is *concentrated*, if it is concentrated around some Y, with $|Y| = \omega_1$.

vi) A set A has property *CM*, if there is f such that for all $g \in A$, $|\{\xi < \omega_1 \mid f(\xi) = g(\xi)\}| = \omega_1$.

In the study of reals strong measure zero is called property C and Rothberger's property is property C'', see [26].

We say that games G and G' are *equivalent*, if $Q \uparrow G$ if and only if $Q \uparrow G'$ for $Q = \mathbf{I}$ and $Q = \mathbf{II}$. The proof of the following lemma is an easy generalization of the proof for the Baire space, [8].

6.2. Lemma. *The games G_A and G_A^* are equivalent.*

Proof. Suppose τ is **I**'s winning strategy in G_A. A winning strategy for **I** in G_A^* is the following. There are auxiliary moves $\varepsilon_\xi \in \omega_1$ and f_ξ that **I** uses to define his winning strategy. If $\mathcal{J}_\xi$, $J_\xi \in \mathcal{J}_\xi$, f_ξ and ε_ξ have been played for $\xi < \gamma$ and **II** has chosen a cover $\mathcal{J}_\gamma$ then **I** chooses $J_\gamma \in \mathcal{J}_\gamma$ such that $f_\gamma \in J_\gamma$ where f_γ is a move given by $\tau(\langle f_\xi, \varepsilon_\xi \rangle_{\xi<\gamma})$. Let ε_γ be such that $U(f_\gamma, \varepsilon_\gamma) \subseteq J_\gamma$. Clearly $A \subseteq \bigcup_\xi J_\xi$, so **I** wins G_A^*.

Suppose that τ is **I**'s winning strategy in G_A^*. We define **I**'s winning strategy in G_A as follows: Let $f_0 \in A$ be such that $\{U(f_0, \alpha) \mid \alpha < \omega_1\} \subseteq \{\tau(\mathcal{J}) \mid \mathcal{J}$ an open cover of $A\}$. There must be such a point, since otherwise for each $f \in A$ there is $\alpha_f < \omega_1$ such that $U(f, \alpha_f)$ is not of the form $\tau(\mathcal{J})$ for some open cover $\mathcal{J}_0$ of A; put $\mathcal{J} = \{U(f, \alpha_f) \mid f \in A\}$ for a contradiction. Let **II** choose any open neighborhood $U_0 = U(f_0, \alpha_0)$. Then $U_0 = \tau(\mathcal{J}_0)$ for some open cover of A. Assume that f_ξ, U_ξ and $\mathcal{J}_\xi$ for $\xi < \gamma$ have been played. Then **I** chooses $f_\gamma \in A$ such that $\{U(f_\gamma, \alpha) \mid \alpha < \omega_1\} \subseteq \{\tau(\langle \mathcal{J}_\xi \rangle_{\xi<\gamma}{}^\frown\langle \mathcal{J} \rangle) \mid \mathcal{J}$ an open cover of $A\}$. Let **II** choose any open neighborhood $U_\gamma = U(f_\gamma, \alpha_\gamma)$. Then $U_\gamma = \tau(\mathcal{J}_\gamma)$ for some open cover of A. Now **I** wins the play $\langle f_\xi, U_\xi \rangle_{\xi<\omega_1}$ since $\langle \mathcal{J}_\xi, U_\xi \rangle_{\xi<\omega_1}$ is a play of G_A^* in which **I** uses the winning strategy τ.

Suppose that τ is a winning strategy for **II** in G_A. A winning strategy for **II** in G_A^* is the following. Let $\mathcal{J}_\xi$ and J_ξ for $\xi < \gamma$ be the previous moves. Player **II** uses auxiliary moves $f_\xi \in J_\xi$ and $\varepsilon_\xi \in \omega_1$ such that $U(f_\xi, \varepsilon_\xi) \subseteq J_\xi$ to define his winning strategy. On the round γ player **II** chooses $\mathcal{J}_\gamma = \{U(f, \tau(\langle f_\xi, \varepsilon_\xi \rangle_{\xi<\gamma})) \mid f \in A\}$. If **I** chooses $J_\gamma \in \mathcal{J}_\gamma$, $J_\gamma = U(f, \tau(\langle f_\xi, \varepsilon_\xi \rangle_{\xi<\gamma}))$, we let $f_\gamma = f$ and $\varepsilon_\gamma = \tau(\langle f_\xi, \varepsilon_\xi \rangle_{\xi<\gamma})$. Now $A \not\subseteq \bigcup_{\xi<\omega_1} U(f_\xi, \varepsilon_\xi)$ implies $A \not\subseteq \bigcup_{\xi<\omega_1} J_\xi$.

Suppose that τ is **II**'s winning strategy in G_A^*. We describe a winning strategy of **II** in G_A. If f_0 is **I**'s first move then **II** chooses α_0 such that $U(f_0, \alpha_0) \subseteq J \in \mathcal{J}_0 = \tau()$. Assume f_ξ, α_ξ and $\mathcal{J}_\xi$ for $\xi < \gamma$ have been defined. Let $J_\xi = U(f_\xi, \alpha_\xi)$. Then if **I** plays f_γ, **II** chooses α_γ such that $U(f_\gamma, \alpha_\gamma) \subseteq J \in \mathcal{J}_\gamma = \tau(\langle J_\xi \rangle_{\xi<\gamma})$. Player **II** wins the game G_A. $\square$

6.3. Lemma. *i) A Lusin set is concentrated around every dense set.*

ii) If A is concentrated, then it has Rothberger's property.

*iii) If not **II** $\uparrow$ G_A where G_A is the point-open game, then A has Rothberger's property.*

iv) If $A \subseteq \mathcal{N}_1$ has Rothberger's property then A has strong measure zero.

Proof. i) Assume that A is a Lusin set and Y is dense. Then every open $G \supseteq Y$ is dense. We have $|A \setminus G| \leq \omega_1$, since A is Lusin, so A is concentrated around Y.

ii) Let A be concentrated around $Y = \{y_\xi \mid \xi < \omega_1\}$. Let $\langle \varepsilon_\xi^f \rangle$ be a sequence. Let $G = \bigcup_{\xi<\omega_1} U(y_\xi, \varepsilon_{2\xi}^{y_\xi})$. Now $A \setminus G = \{z_\xi \mid \xi < \omega_1\}$, so

$$A \subseteq \bigcup_{\xi<\omega_1} U(y_\xi, \varepsilon_{2\xi}^{y_\xi}) \cup \bigcup_{\xi<\omega_1} U(z_\xi, \varepsilon_{2\xi+1}^{z_\xi}).$$

iii) Assume that A does not have Rothberger's property. Then there is a sequence $\langle \varepsilon_\xi^f \rangle$ such that for each $\langle f_\xi \rangle$ we have $A \not\subseteq \bigcup_{\xi\in\omega_1} U(f_\xi, \varepsilon_\xi^{f_\xi})$. But then player **II** has a winning strategy by choosing $\mathcal{J}_\xi = \{U(f, \varepsilon_\xi^f) \mid f \in A\}$ on the ξth round in the point-open game G_A^*. Here we use lemma 6.2.

iv) Let $\langle \varepsilon_\xi \rangle$ be arbitrary and let $\varepsilon_\xi^f = \varepsilon_\xi$. $\square$

If CH holds then every Lusin set is concentrated by Lemma 6.3.i) since then $\mathcal{Q}$ is a dense set of cardinality $\aleph_1$.

The following theorem shows that the existence of these negligible sets is consistent with ZFC.

6.4. Theorem. *Assume $2^{\aleph_1} = \aleph_2$. Then there is a Lusin set.*

Proof. Let $\{R_\xi \mid \xi < \omega_2\}$ be the set of all closed nowhere dense sets. By the Baire Category Theorem 3.3 $\bigcup_{\delta<\xi} R_\delta \neq \mathcal{N}_1$ for all $\xi < \omega_2$. Let $E = \{e_\xi \mid \xi < \omega_2\}$, where $e_\xi \notin \{e_\delta \mid \delta < \xi\} \cup \bigcup_{\delta<\xi} R_\delta$. E is Lusin because if R is nowhere dense then $E \cap R \subseteq E \cap \overline{R} \subseteq \{e_\xi \mid \xi < \gamma\}$ where γ is such that $\overline{R} = R_\gamma$. $\square$

6.5. Lemma. *$A \subseteq \mathcal{N}_1$ has Rothberger's property if and only if for every sequence of open covers $\langle \mathcal{J}_\xi \rangle_{\xi<\omega_1}$ of A there are $J_\xi \in \mathcal{J}_\xi$ such that $A \subseteq \bigcup_{\xi<\omega_1} J_\xi$.*

Proof. Assume that $A \subseteq \mathcal{X}$ has Rothberger's property. Let $\langle \mathcal{J}_\xi \rangle_{\xi<\omega_1}$ be a sequence of open covers of A. For each $f \in A$ and $\xi < \omega_1$ choose $J_f \in \mathcal{J}_\xi$ such that $f \in J_f$ and then pick $\varepsilon_\xi^f \in \omega_1$ such that $[f \restriction \varepsilon_\xi^f] \subseteq J_f$. By Rothberger's property there are $f_\xi \in A$ such that $A \subseteq \bigcup_{\xi<\omega_1} [f_\xi \restriction \varepsilon_\xi^{f_\xi}] \subseteq \bigcup_{\xi<\omega_1} J_{f_\xi}$.

For the other direction, if $\langle \varepsilon_\xi^f \rangle$ is given, put $\mathcal{J}_\xi = \{[f \restriction \varepsilon_\xi^f] \mid f \in A\}$ for each $\xi < \omega_1$. $\square$

6.6. Lemma. *$A \subseteq \mathcal{N}_1$ has Rothberger's property if and only if $F''A$ has property CM for every continuous function $F : A \to \mathcal{N}_1$.*

Proof. Assume that A has Rothberger's property and F is continuous. We show that $F''A$ has Rothberger's property. Let $\langle \varepsilon_\alpha^{F(f)} \rangle_{f \in A, \alpha<\omega_1}$ be given. Since F is continuous we can choose δ_α^f such that $F''[f \restriction \delta_\alpha^f] \subseteq [F(f) \restriction \varepsilon_\alpha^{F(f)}]$. By Rothberger's property there are $f_\alpha \in A$, $\alpha < \omega_1$, such that

$$A \subseteq \bigcup_{\alpha<\omega_1} [f_\alpha \restriction \delta_\alpha^{f_\alpha}].$$

Hence $F''A \subseteq \bigcup_{\alpha<\omega_1} [F(f_\alpha) \restriction \varepsilon_\alpha^{F(f_\alpha)}]$.

Next we prove that every set A with Rothberger's property has the CM property. Let $\pi : \omega_1 \times \omega_1 \to \omega_1$ be a bijection. Fix $\delta < \omega_1$ for a moment and let $\mathcal{J}_\xi^\delta = \{[(\pi(\xi,\delta), \alpha)] \mid \alpha < \omega_1\}$. Using lemma 6.5 for $\langle \mathcal{J}_\xi^\delta \rangle_{\xi<\omega_1}$ for each $\delta < \omega_1$ we find f such that $A \subseteq \bigcap_{\delta<\omega_1} \bigcup_{\xi<\omega_1} J_\xi^\delta$ where $J_\xi^\delta = [(\pi(\xi,\delta), f(\pi(\xi,\delta)))] \in \mathcal{J}_\xi^\delta$. Thus f has the required property.

Assume that $F''A$ has property CM for every continuous function $F : A \to \mathcal{N}_1$. Let $\mathcal{J}_\xi = \{J_\delta^\xi \mid \delta < \omega_1\}$ where the elements of $\mathcal{J}_\xi$ are clopen and disjoint. Define $F : A \to \mathcal{N}_1$ by $F(x)(\xi) = \delta$ if $x \in J_\delta^\xi$. Then F is continuous. Thus there is f such that for all $g \in F''A$, $|\{\delta \mid f(\delta) = g(\delta)\}| = \omega_1$. Then $A \subseteq \bigcup_{\xi<\omega_1} J_\xi^{f(\xi)}$. $\square$

6.7. Lemma (CH). **I** $\uparrow G_A$ *if and only if* $|A| \leq \aleph_1$.

Proof. If $A = \{f_\xi \mid \xi < \omega_1\}$ then **I**'s winning strategy is to play f_ξ on the ξth round.

Assume that $|A| > \aleph_1$. Let τ be a strategy for **I**. We will prove that it is not a winning strategy. Let $S = \tau''\mathrm{Seq}$. Since, by CH, $|S| \leq \aleph_1$, there is $f \in A \setminus S$. Now there is a play $\langle f_\xi, [f_\xi \restriction \alpha_\xi] \rangle$ of G_A where **I** uses τ and **II** chooses ordinals α_ξ such that $f \notin [f_\xi \restriction \alpha_\xi]$. So the play is winning for **II**. $\square$

Pawlikowski [29] showed in the Baire space that an uncountable subset of reals A has Rothberger's property if and only if **II** $\not\uparrow G_A$.

To show that these classes of negligible sets are not the same we need some additional set theoretical assumptions, because under GBC all these classes are just the collection of subsets of cardinality at most ω_1. Here is a lemma which separates classes of strong measure zero sets and concentrated sets. It is a generalization of the theorem of Besicovich. A reference to it and many of the results concerning negligible subset of the Baire space can be found in [26].

6.8. Lemma (CH). *If* $2^{\omega_1} = \omega_2$ *then there is a set which has strong measure zero, but is not concentrated.*

Proof. We construct a sequence $\langle P_\xi \rangle_{\xi<\omega_2}$ of disjoint ω_1-perfect nowhere dense sets such that every meager set M intersects only ω_1 many of them. Let R_ξ, $\xi < \omega_2$, be an enumeration of all closed nowhere dense sets. Assume that P_ξ, $\xi < \delta < \omega_2$, have been constructed. By Lemma 3.7 there is an ω_1-perfect nowhere dense set $P_\delta \subseteq \mathcal{N}_1 \setminus \bigcup_{\xi<\delta}(P_\xi \cup R_\xi)$. Let $E_\xi \subseteq P_\xi$ be a Lusin set (relativized to P_ξ). Then $E = \cup\{E_\xi \mid \xi < \omega_2\}$ has strong measure zero: Given a sequence $\langle \alpha_\xi \rangle_{\xi<\omega_1}$ let $G = \bigcup_{\xi<\omega_1} U(q_\xi, \alpha_{2\xi})$. By the property of the sequence $\langle P_\xi \rangle$ there is $\delta < \omega_2$ such that $P_\xi \subseteq G$ for all $\xi \geq \delta$. Since $\bigcup_{\xi<\delta} E_\xi$ has strong measure zero, it is $\langle \alpha_{2\xi+1} \rangle$-small.

E is not concentrated, since if D has power ω_1, there is $\xi < \omega_2$ such that $D \cap P_\xi = \varnothing$. Hence $G = \mathcal{N}_1 \setminus P_\xi$ is open and $D \subseteq G$, but $E \setminus G$ has power ω_2 because it contains E_ξ. □

We recall that $\mathrm{cov}(\mathcal{M})$ is the smallest cardinal κ such that $\mathcal{N}_1$ can be covered by κ many meager sets. The following theorem is a generalization of the corresponding theorem of Galvin [8] for the point-open game for reals.

6.9. Theorem (CH). *If* $A \subseteq \mathcal{C}_1$ *is such that* $|A| < \mathrm{cov}(\mathcal{M})$ *then* **II** *does not have a winning strategy in point-open game* G_A.

Proof. Let $\mathcal{F} = \{F : \omega_1 \to \mathrm{Seq} \mid \bigcup_{\xi<\omega_1}[F_\alpha(\xi)] = \mathcal{C}_1\}$. $\mathcal{F}$ is nonvoid by CH. We consider the game G_A^*. We may assume that at the αth move, **II** chooses $F_\alpha \in \mathcal{F}$. Then **I** chooses $i_\alpha \in \omega_1$. Player **I** wins the play $\langle F_\alpha, i_\alpha \rangle_{\alpha<\omega_1}$ if $A \subseteq \bigcup_{\alpha<\omega_1}[F_\alpha(i_\alpha)]$. Let σ : $\mathrm{Seq} \to \mathcal{F}$ be any strategy for **II** in G_A^*. Given any $g \in \mathcal{N}_1$, let $D_g = \bigcup_{\alpha<\omega_1}[\sigma(g\restriction\alpha)(g(\alpha))]$. For each $x \in A$ the set $F_x = \{g \in \mathcal{N}_1 \mid x \in D_g\}$ is open and dense: Given any $s \in \mathrm{Seq}$ choose $\xi \in \omega_1$ such that $x \in [\sigma(s)(\xi)]$. Then $[s^\frown\langle\xi\rangle] \subseteq F_x$. Since $|A| < \mathrm{cov}(\mathcal{M})$, it follows that $\bigcap_{x\in A} F_x \neq \varnothing$. Choose $g \in \bigcap_{x\in A} F_x$; then $D_g \subseteq A$, and so σ is not a winning strategy. □

7. Stationary strong measure zero sets

In an unpublished paper Galvin, Mycielski and Solovay proved a nice characterization for strong measure zero sets of reals. The theorem was announced in [9]. Their theorem says that a set of reals A has strong measure zero if and only if for every open dense set D there is $x \in 2^\omega$ such that $x + A \subseteq D$. The proof uses a variant of the point-open game. We will generalize the theorem for $\mathcal{C}_1$, Theorem 7.8. The classical proof uses compactness of 2^ω, but 2^{ω_1} is not ω_1-compact by Lemma 2.6. Instead of compactness we will use the $\diamondsuit^*$-principle. We thank Stevo Todorçevic for his valuable help with this.

Let $\mathrm{NCub} = \{E \subseteq \omega_1 \mid E$ does not contain a cub set$\}$.

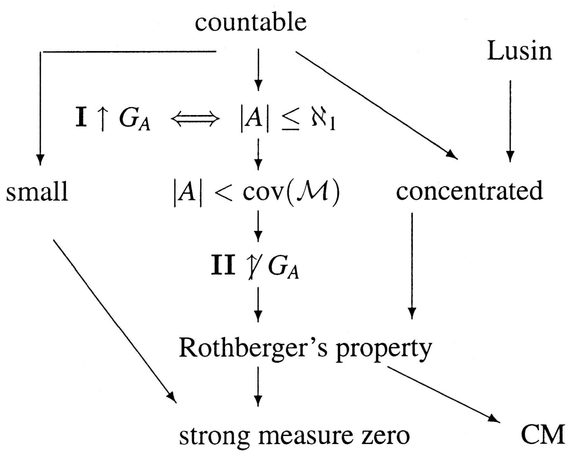

Figure 1: Properties of a negligible set A.

7.1. Definition. $\diamondsuit^*$ is the principle $\diamondsuit^-(\omega_1, \mathcal{N}_1, \mathrm{NCub})$. Let $\langle \diamondsuit_\alpha \rangle_{\alpha<\omega_1}$ be a $\diamondsuit^*$-sequence where $\diamondsuit_\alpha = \{f_{\alpha n} \mid n \in \omega\}$.

If $V = L$ then $\diamondsuit^*$. See [5].

A set is Π^0_2 if it is an ω_1-intersection of open sets.

7.2. Lemma. *Let $\langle s_\alpha \rangle_{\alpha\in\omega_1}$ be a sequence of functions such that* $\mathrm{dom}(s_\alpha) \in [\omega_1 \setminus \alpha]^\omega$ *and* $\mathrm{ran}(s_\alpha) \subseteq \omega_1$. *Then*

i) $\{x \in \mathcal{N}_1 \mid \{\alpha \mid s_\alpha \subseteq x\} \neq \varnothing\}$ *is a dense open set,*

ii) $\{x \in \mathcal{N}_1 \mid \{\alpha \mid s_\alpha \subseteq x\}$ *has cardinality* $\aleph_1\}$ *is a dense* Π^0_2 *set,*

iii) $\{x \in \mathcal{N}_1 \mid \{\alpha \mid s_\alpha \subseteq x\}$ *is stationary*$\}$ *is a dense set.*

Proof. Let D be any of the sets above. D is non-empty since we can define $x \in D$ as follows. We define inductively $\alpha_0 = 0$, $\alpha_\delta = \bigcup_{\xi<\delta}[\sup(\mathrm{dom}(s_{\alpha_\xi}))]$, $x\restriction\mathrm{dom}(s_{\alpha_\delta}) = s_{\alpha_\delta}$ and $x(\xi) = 0$ otherwise. Then $\{\alpha \mid s_\alpha \subseteq x\} \supseteq \{\alpha_\delta \mid \delta \in \omega_1\} \in \mathrm{Cub}$ and so $x \in D$. Now it is easy to see that D is dense, since $s\hat{}(x\restriction(\omega_1 \setminus \ell(s))) \in D$ for every $s \in \mathrm{Seq}$. The set $\{x \mid |\{\alpha \mid s_\alpha \subseteq x\}| = \aleph_1\}$ is

$$\bigcap_{\delta<\omega_1} \bigcup_{\delta<\xi<\omega_1} [s_\xi]$$

so it is Π^0_2. □

The following lemma gives us a tool to handle open and dense sets. It also gives us a way to avoid the use of compactness. In the proof we will use an abbreviation $[\alpha, \beta) = \{\xi \mid \alpha \leq \xi < \beta\}$.

7.3. Lemma. *Assume that $\langle \Diamond_\alpha \rangle$ is a $\Diamond^*$-sequence, where $\Diamond_\alpha = \{f_{\alpha n} \in \alpha^\alpha \mid n \in \omega\}$. Assume that $D \subseteq \mathcal{N}_1$ is open and dense. There exists a sequence $\langle s_\alpha \rangle_{\alpha<\omega_1}$ of countable functions such that* $\mathrm{dom}(s_\alpha) \in [\omega_1 \setminus \alpha]^\omega$ *and* $\mathrm{ran}(s_\alpha) \subseteq \omega_1$ *with the property*

$$D' = \{x \in \mathcal{N}_1 \mid \{\alpha \mid s_\alpha \subseteq x\} \textit{ is stationary}\} \subseteq D.$$

Proof. For each α we let $\beta_0^\alpha = \alpha$ and we choose, by induction on $n \in \omega$, an ordinal β_{n+1}^α and a function $s_\alpha^n : [\beta_n^\alpha, \beta_{n+1}^\alpha) \to \omega_1$, such that

$$[f_{\alpha n} \cup s_\alpha^0 \cup \ldots \cup s_\alpha^n] \subseteq D.$$

This is possible, since D is open and dense. Let $s_\alpha = \bigcup_{n\in\omega} s_\alpha^n$. Now assume that $x \in D'$. Then $\{\alpha \mid s_\alpha \subseteq x\}$ is stationary and $\{\alpha \mid x\restriction\alpha \in \Diamond_\alpha\}$ is cub by $\Diamond^*$. Thus there is $\alpha \in \omega_1$ and $n \in \omega$ such that $x\restriction\alpha = f_{\alpha n}$ and $s_\alpha \subseteq x$. By the choice of s_α we have $x \in [f_{\alpha n} \cup s_\alpha^0 \cup \ldots \cup s_\alpha^n] \subseteq D$. $\square$

7.4. Proposition. *If A has strong measure zero and $\langle \alpha_\xi \rangle_{\xi<\omega_1}$ is a sequence then there are f_ξ, $\xi < \omega_1$, such that*

$$A \subseteq \bigcap_{\delta\in\omega_1} \bigcup_{\xi>\delta} U(f_\xi, \alpha_\xi).$$

Proof. Split $\omega_1 = \bigcup_{\delta<\omega_1} X_\alpha$ where $X_\delta \in [\omega_1]^{\omega_1}$, $\delta < \omega_1$, are disjoint. Since A has strong measure zero there are f_ξ such that $A \subseteq \bigcup_{\xi\in X_\delta} U(f_\xi, \alpha_\xi)$ for all $\delta \in \omega_1$. Then $\{\xi \mid f \in U(f_\xi, \alpha_\xi)\}$ has cardinality ω_1 for all $f \in A$, and the claim follows. $\square$

In the previous proposition $\{\xi \mid f \in U(f_\xi, \alpha_\xi)\}$ has cardinality ω_1 for all $f \in A$. Although we can not prove the Galvin-Mycielski-Solovay theorem for strong measure zero sets, we can get a similar theorem for a stronger notion, where we require that this set is stationary.

7.5. Definition. A set $A \subseteq \mathcal{N}_1$ has property *stationary C* or *stationary strong measure zero*, if for every $\langle \varepsilon_\alpha \rangle_{\alpha<\omega_1} \in \omega_1^{\omega_1}$ there is $\langle I_\alpha \rangle \in \prod_{\alpha<\omega_1} \omega_1^{\varepsilon_\alpha}$ such that

$$A \subseteq \bigcap_{C \text{ cub}} \bigcup_{\alpha\in C} [I_\alpha].$$

A set $A \subseteq \mathcal{N}_1$ has property *stationary σ-C*, if for every $\langle \varepsilon_\alpha \rangle_{\alpha<\omega_1} \in \omega_1^{\omega_1}$ there is $\langle \mathcal{I}_\alpha \rangle \in \prod_{\alpha<\omega_1} [\omega_1^{\varepsilon_\alpha}]^\omega$ such that

$$A \subseteq \bigcap_{C \text{ cub}} \bigcup_{\alpha\in C} \bigcup_{I\in\mathcal{I}_\alpha} [I].$$

7.6. Lemma. *i) Stationary C implies stationary σ-C and stationary σ-C implies strong measure zero.*

ii) The sets with property stationary σ-C is an $\aleph_2$-ideal.

Proof. i) Implications from stationary C to stationary σ-C and from stationary σ-C to σ-C are clear. σ-C implies strong measure zero by Lemma 4.4.

ii) Assume that the sets A_β, $\beta < \omega_1$, have property stationary σ-C. Let $\langle \varepsilon_\alpha \rangle$ be a an arbitrary sequence. Choose $\langle \mathcal{I}_\alpha^\beta \rangle$ such that

$$A_\beta \subseteq \bigcap_{C \text{ cub}} \bigcup_{\alpha \in C} \bigcup_{I \in \mathcal{I}_\alpha^\beta} [I]$$

Let $\mathcal{I}_\alpha = \bigcup_{\beta<\alpha} \mathcal{I}_\alpha^\beta$. Let $x \in A_\beta$ for some $\beta \in \omega_1$. If C is any cub set, there is some $\alpha \in C$ such that $\alpha \geq \beta$, $I \in \mathcal{I}_\alpha^\beta \subseteq \mathcal{I}_\alpha$ and $x \in [I]$. Thus $\bigcup_{\beta<\omega_1} A_\beta$ has property stationary σ-C. $\square$

There are many open problems with these concepts. We do not know whether every small set has stationary strong measure zero. Of course, if A is small then it is stationary σ-C. We do not know if stationary C is equivalent to stationary σ-C.

7.7. Lemma. *Assume $\diamondsuit^*$. There are $\mathcal{M}_\alpha$, $\alpha \in \omega_1$, where $\mathcal{M}_\alpha$ is countable set of countable models such that for each model (ω_1, R) the set*

$$\{\alpha \in \omega_1 \mid (\alpha, R\restriction\alpha) \in \mathcal{M}_\alpha\}$$

contains a cub set.

Proof. Let $\langle \diamondsuit_{\alpha n} \subseteq \alpha \mid n \in \omega,\ \alpha \in \omega_1 \rangle$ be a $\diamondsuit^*$-sequence. Thus for each $X \subseteq \omega_1$

$$\{\alpha \mid \exists n (X \cap \alpha = \diamondsuit_{\alpha n})\}$$

contains a cub set. Let k be the arity of R. Let $\pi : \omega_1^k \to \omega_1$ be a bijection. There is a cub set C of limit ordinals such that for all $\alpha \in C$, $\pi\restriction\alpha$ is a bijection from α^k onto α. Let $M_{\alpha n} = (\alpha, \pi^{-1}\diamondsuit_{\alpha n})$ if it is a model; otherwise let $M_{\alpha n} = \varnothing$. Let $\mathcal{M}_\alpha = \{M_{\alpha n} \mid n \in \omega\}$. Let $M = (\omega_1, R)$ be any model, and $X = \pi''R = \{\pi(\bar{x}) \mid \bar{x} \in R\}$. Then $\{\alpha \mid \exists n (X \cap \alpha = \diamondsuit_{\alpha n})\}$ contains a cub set D. If $\alpha \in C \cap D$ then there is n such that $R\restriction\alpha = \pi^{-1}\diamondsuit_{\alpha n}$. $\square$

Now we can prove the main theorem of this section.

7.8. Theorem. *Assume $\diamondsuit^*$. Let $X \subseteq 2^{\omega_1}$. Then i) implies ii) and ii) implies iii), where*

i) X has stationary strong measure zero,

ii) for every open dense set D there is $x \in 2^{\omega_1}$ such that $x + X \subseteq D$,

iii) X has strong measure zero.

Proof. Since $\diamondsuit^*$ implies CH the set $\mathcal{Q}$ of all ultimately constant functions has cardinality $\aleph_1$. So let $\mathcal{Q} = \{q_\alpha \mid \alpha < \omega_1\}$. Assume that X has the property ii). Then X has strong measure zero since if $\langle \varepsilon_\alpha \rangle_{\alpha<\omega_1}$ is arbitrary then $\bigcup_{\alpha<\omega_1} U(q_\alpha, \varepsilon_\alpha)$ is open and dense. Thus, there is $x \in 2^{\omega_1}$ such that $x+X \subseteq \bigcup_{\alpha<\omega_1} U(q_\alpha, \varepsilon_\alpha)$ i.e. $X \subseteq \bigcup_{\alpha<\omega_1} U(x+q_\alpha, \varepsilon_\alpha)$.

For the harder implication we assume that X has stationary strong measure zero. Let D be a dense open set.

We will define two games which are variants of the point-open game. Let Γ_1 be a game $(\varepsilon_\alpha, I_\alpha)_{\alpha\in\omega_1}$ where on round α player **I** chooses $\varepsilon_\alpha \in \omega_1$ and then player **II** chooses $I_\alpha \in 2^{\varepsilon_\alpha}$. Player **I** wins the game Γ_1 if there is $x \in 2^{\omega_1}$ such that

$$x + \bigcap_{C \text{ cub}} \bigcup_{\alpha\in C} [I_\alpha] \subseteq D.$$

Let Γ_2 be a game $(\varepsilon_\alpha, I_\alpha)_{\alpha\in\omega_1}$ where the player **II** wins if

$$X \subseteq \bigcap_{C \text{ cub}} \bigcup_{\alpha\in C} [I_\alpha].$$

We will prove the following two claims.

1. Claim. **I** *has a winning strategy in* Γ_1.

2. Claim. **I** *does not have a winning strategy in* Γ_2.

Thus there is a play $(\varepsilon_\alpha, I_\alpha)_{\alpha\in\omega_1}$ which is a winning play for **I** in Γ_1 but at the same time is a winning play for **II** in Γ_2. So we have that

$$x + X \subseteq x + \bigcap_{C \text{ cub}} \bigcup_{\alpha\in C} [I_\alpha] \subseteq D.$$

Proof of Claim 1. Let $\langle s_\alpha \rangle_{\alpha<\omega_1}$ be a sequence for D given by Lemma 7.3. We may assume that $\min \operatorname{dom}(s_\alpha) > \sup \operatorname{dom}(s_\beta)$ for all $\alpha > \beta$. Here is **I**'s plan. There are auxiliary moves $T_\xi \in \mathrm{Seq}$ which **I** uses to define his winning strategy. His first move is $\varepsilon_0 = \sup \operatorname{dom}(s_0) + 1$. After **II** has chosen his I_0 of length ε_0, player **I** will choose $T_0 \in 2^{\varepsilon_0}$ such that $s_0 \subseteq I_0 + T_0$. At the αth move **I** will play $\varepsilon_\alpha = \sup \operatorname{dom}(s_\alpha) + 1$. After **II** has chosen his I_α of length ε_α, **I** will then choose $T_\alpha \in 2^{\varepsilon_\alpha}$ such that $\bigcup_{\xi<\alpha} T_\xi \subseteq T_\alpha$ and $s_\alpha \subseteq I_\alpha + T_\alpha$. Let $x \in \bigcap_{\alpha<\omega_1} [T_\alpha]$. If $f \in x + \bigcap_{C \text{ cub}} \bigcup_{\alpha\in C} [I_\alpha]$ then $\{\alpha \mid s_\alpha \subseteq f\}$ is stationary i.e. $f \in D$.

Proof of Claim 2. Let $\langle \mathcal{M}_\alpha \rangle_{\alpha<\omega_1}$ where $\mathcal{M}_\alpha = \{M_{\alpha n} \mid n \in \omega\}$ is a $\diamondsuit^*$-sequence of models from Lemma 7.7. Assume that σ is a strategy for **I**. We will define a play in which **I** uses σ and **II** wins the game Γ_2. Let

$$\delta_\alpha = \sup\{\sigma(\langle I_\xi \rangle_{\xi<\alpha}) \mid \exists n(\langle I_\xi \rangle_{\xi<\alpha} \in M_{\alpha n})\}.$$

Since X has stationary strong measure zero we can choose sequences I_α, $\alpha < \omega_1$, such that $\ell(I_\alpha) = \delta_\alpha$ and

(1)
$$X \subseteq \bigcap_{C \text{ cub}} \bigcup_{\alpha \in C} [I_\alpha].$$

Let $\varepsilon_\alpha = \sigma(\langle I_\xi \rangle_{\xi<\alpha})$ for each $\alpha < \omega_1$. By $\diamondsuit^*$ there is a cub set C such that if $\alpha \in C$ then there is $n \in \omega$ such that $\langle I_\xi \rangle_{\xi<\alpha} \in M_{\alpha n}$. Thus $\varepsilon_\alpha \leq \delta_\alpha$ for every $\alpha \in C$. If C' is any cub set then $X \subseteq \bigcup_{\alpha \in C \cap C'} [I_\alpha]$ by (1). $\square$

7.9. Conjecture. *A set $X \subseteq 2^{\omega_1}$ has strong measure zero if and only if X has stationary strong measure zero.*

8. GMA and negligible sets

Martin's axiom (MA) decides many properties of negligible subsets of the Baire space [7]. For example MA implies that the union of $\lambda < 2^{\aleph_0}$ meager sets is meager. (See e.g. [18].) In [31] Shelah introduced a version of generalized Martin's axiom (GMA) which he used to prove similar results for higher cardinals. More information about other versions of GMA is in [39]. Here, we will look for the applications of GMA to the structure of negligible susets of $\mathcal{N}_1$.

Let $(P, \leq)$, $|P| < 2^{\aleph_1}$, be a partial order. Elements of P are called *conditions*. As usual we denote $p \mid q$ iff there is r such that $r \leq p$ and $r \leq q$. The greatest lower bound of conditions of p and q is denoted by $p \wedge q$. The greatest lower bound of conditions of p_n, $n \in \omega$, is denoted by $\inf_{n\in\omega} p_n$. $D \subseteq P$ is *dense*, if for all $p \in P$ there is $q \in D$ such that $q \leq p$. $G \subseteq P$ is a *filter*, if

i) $p \wedge q \in G$ for all $p, q \in G$,
ii) if $p \in G$ and $p \leq q$ then $q \in G$.

If $\mathcal{D}$ is a family of subsets of P then we say that $G \subseteq P$ is *$\mathcal{D}$-generic*, if $G \cap D \neq \varnothing$ for all $D \in \mathcal{D}$.

Shelah's *weak generalized Martin's axiom for* $\aleph_2$ (GMA) is the following:

Assume that a partial order $(P, \leq)$ satisfies
i) if $p \mid q$ where $p, q \in P$ then $p \wedge q \in P$.
ii) if $p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots$, $n \in \omega$, then $\inf_{n\in\omega} p_n \in P$.
iii) if $p_\xi \in P$, for all $\xi < \omega_2$, then there is a cub set $C \subseteq \omega_2$ and a regressive function $f : \omega_2 \to \omega_2$ such that

$$\forall \alpha, \beta \in C[\mathrm{cf}(\alpha), \mathrm{cf}(\beta) > \omega,\ f(\alpha) = f(\beta) \Rightarrow p_\alpha \mid p_\beta].$$

Then if $\mathcal{D} = \{D_\xi \mid \xi < \lambda\}$ where $\lambda < 2^{\aleph_1}$ is a family of dense subsets of P, there is a $\mathcal{D}$-generic filter $G \subseteq P$.

Notice that iii) is a strengthening of $\aleph_2$-c.c.

Shelah proves the consistency of CH, $2^{\aleph_1} > \aleph_2$ and GMA relative to consistency of ZFC. As an application he proves Theorem 8.2 below. We use an additional assumption to ensure that $|P| < 2^{\omega_1}$. First a simple lemma.

8.1. Lemma. *Assume that P has cardinality ω_1. Then P satisfies the following: If $p_\xi \in P$, $\xi < \omega_2$, then there is a cub set $C \subseteq \omega_2$ and a regressive function $f : \omega_2 \to \omega_2$ such that for all $\alpha, \beta \in C$ if $\mathrm{cf}(\alpha), \mathrm{cf}(\beta) > \omega$ and $f(\alpha) = f(\beta)$ then $F(p_\alpha) = F(p_\beta)$. In particular P satisfies GMA iii).*

Proof. Let $\langle p_\alpha \rangle_{\alpha<\omega_2}$ be a sequence of conditions. For each α let $g(\alpha) = \min\{\beta \mid p_\beta = p_\alpha\}$ and

$$f(\alpha) = \begin{cases} g(\alpha), & \text{if } g(\alpha) < \alpha \\ 0, & \text{otherwise.} \end{cases}$$

Clearly f is regressive. Now $\gamma = \sup\{\alpha \mid g(\alpha) = \alpha\} < \omega_2$, since otherwise there would be ω_2 elements in P. Now $C = \omega_2 \setminus \gamma$ is the required cub set. □

8.2. Theorem ([31]). *Assume that $\lambda^\omega < 2^{\aleph_1}$ for each $\lambda < 2^{\aleph_1}$. Assume CH, $2^{\aleph_1} > \aleph_2$ and GMA. Then the union of $\lambda < 2^{\aleph_1}$ meager subsets of $\mathcal{N}_1$ is meager.*

Proof. Let B_ξ, $\xi < \lambda$, be nowhere dense sets. Let P be the set of all countable sequences of pairs

$$p = \langle (U_\xi, E_\xi) \rangle_{\xi<\gamma},\ \gamma < \omega_1,$$

such that

i) each U_ξ is the union of countably many basic neighborhoods $[s]$;
ii) each E_ξ is countable subset of λ; and
iii) for each ξ, U_ξ is disjoint from $\bigcup_{\alpha\in E_\xi} B_\alpha$.

A condition $p' = \langle (U'_\xi, E'_\xi) \rangle_{\xi<\gamma'}$ is stronger than $p = \langle (U_\xi, E_\xi) \rangle_{\xi<\gamma}$, if $\gamma' \geq \gamma$ and for each $\xi < \gamma$, $U'_\xi \supseteq U_\xi$ and $E'_\xi \supseteq E_\xi$.

Clearly P satisfies GMA i) and ii). To show that P satisfies condition iii of GMA let $Q = \{\langle U_\xi \rangle_{\xi<\gamma} \mid \langle U_\xi, E_\xi \rangle_{\xi<\gamma} \in P\}$ and define a function $F : P \to Q$ by $F(\langle U_\xi, E_\xi \rangle_{\xi<\gamma}) = \langle U_\xi \rangle_{\xi<\gamma}$. Note that conditions $p = \langle (U_\xi, E_\xi) \rangle_{\xi<\gamma}$ and $q = \langle (U'_\xi, E'_\xi) \rangle_{\xi<\gamma'}$ are compatible, if $\gamma = \gamma'$ and for all $\xi < \gamma$, $U_\xi = U'_\xi$, i.e. $F(p) = F(q)$. By CH, the set Q has cardinality $|(\mathrm{Seq}^{<\omega_1})^{<\omega_1}| = \omega_1$. So applying Lemma 8.1 if $p_\xi \in P$, $\xi < \omega_2$, then there is a cub set $C \subseteq \omega_2$ and a regressive function $f : \omega_2 \to \omega_2$ such that for all $\alpha, \beta \in C$ if $\mathrm{cf}(\alpha), \mathrm{cf}(\beta) > \omega$ and $f(\alpha) = f(\beta)$ then $F(p_\alpha) = F(p_\beta)$, and hence $p_\alpha \mid p_\beta$. Hence P satisfies GMA iii).

For each $\alpha < \lambda$ and all $\xi < \omega_1$, $s \in \mathrm{Seq}$, let

$$\begin{aligned} D_\alpha &= \{p \mid p = \langle (U_\xi, E_\xi) \rangle_{\xi<\gamma},\ \alpha \in E_\delta \text{ for some } \delta < \gamma\} \\ E_{\xi s} &= \{p \mid p = \langle (U_\zeta, E_\zeta) \rangle_{\zeta<\gamma},\ U_\xi \cap [s] \neq \varnothing\}. \end{aligned}$$

Since each B_ξ, $\xi < \lambda$, is nowhere dense, it is clear that for all ξ and s, every condition p can be extended to a condition $p' \in E_{\xi s}$, and hence each $E_{\xi s}$ is dense in P. (Just choose $[s'] \subseteq [s] \setminus \bigcup_{\delta \in E_\xi} B_\delta$ and put $U'_\xi = U_\xi \cup [s']$ and $E'_\xi = E_\xi$.) Each D_α is also dense in P, since if p is a condition, then $p' = p^\frown\langle([s], \{\alpha\})\rangle \in D_\alpha$, where $[s] \cap B_\alpha = \varnothing$. By GMA, there exists a filter $G \subseteq P$ such that $G \cap D_\alpha \neq \varnothing$ for all $\alpha < \lambda$, and $G \cap E_{\xi s} \neq \varnothing$ for all ξ, s. For each $\xi < \omega_1$ we let

$$H_\xi = \bigcup\{U_\xi \mid (\exists p \in G) p = \langle \ldots, (U_\xi, E_\xi), \ldots \rangle\}$$

Since $E_{\xi s}$ is dense for all s, H_ξ is a dense open set.

Now if $\alpha < \lambda$, then because D_α is a dense, there exists $\xi < \omega_1$ such that H_ξ is disjoint from B_α, and hence B_α is disjoint from $\bigcap_{\xi < \omega_1} H_\xi$. Therefore, $\bigcup_{\alpha \in \lambda} B_\alpha$ is disjoint from $\bigcap_{\xi < \omega_1} H_\xi$. □

Carlson showed that MA implies 2^ω-additivity of the ideal of strong measure zero sets of the Baire space, see [7]. It is an open problem whether this result generalizes to $2^{\aleph_1}$-additiveness of the ideal of strong measure zero sets of $\mathcal{N}_1$ under *CH* and *GMA*. However, we can prove the following theorem.

8.3. Theorem. *Assume CH and GMA. If $A \subseteq \mathcal{N}_1$ has cardinality $< 2^{\aleph_1}$ then A has strong measure zero.*

Proof. Let $Q = \{q_\xi \mid \xi < \omega_1\}$. Let $\langle \alpha_\xi \rangle$ be an arbitrary increasing sequence. Let $P = (\mathrm{Seq}, \supseteq)$. Clearly P satisfies conditions i), ii) and iii) of GMA. For each $f \in A$, let

$$D_f = \{s \in \mathrm{Seq} \mid \exists \xi [f \in U(q_{s(\xi)}, \alpha_\xi)]\}.$$

D_f is dense, because if $s \in \mathrm{Seq}$, $\ell(s) = \xi$, then $s' = s^\frown\langle\delta\rangle \in D_f$ where $\delta \in \omega_1$ is such that $f \in U(q_\delta, \alpha_\xi)$. For each $\gamma < \omega_1$, let E_γ be the dense set $\{s \mid \gamma \in \mathrm{dom}(s)\}$. Let $\mathcal{D} = \{D_f \mid f \in A\} \cup \{E_\gamma \mid \gamma < \omega_1\}$. Since $|\mathcal{D}| < 2^{\aleph_1}$ by GMA there is a $\mathcal{D}$-generic G. Let $h = \cup G$. Now for each $f \in A, f \in \bigcup_{\xi < \omega_1} U(q_{h(\xi)}, \alpha_\xi)$, since there is $\delta < \omega_1$ such that $h \restriction \delta \in D_f$. □

8.4. Definition. A set $E \subseteq \mathcal{N}_1$ of cardinality $2^{\aleph_1}$ is *a generalized Lusin set* or *a GLusin set*, if $|E \cap R| < 2^{\omega_1}$ for every nowhere dense set R.

8.5. Theorem. *Assume CH, $2^{\aleph_1} > \aleph_2$ and GMA. There is a GLusin set.*

Proof. Let $\{R_\xi \mid \xi < 2^{\aleph_1}\}$ be the set of all closed nowhere dense sets. Under *CH*, $2^{\aleph_1} > \aleph_2$ and *GMA* we can apply Theorem 8.2. Then $\bigcup_{\delta < \xi} R_\delta \neq \mathcal{N}_1$ for every $\xi < 2^{\aleph_1}$. So we can define $E = \{e_\xi \mid \xi < 2^{\aleph_1}\}$ where $e_\xi \notin \{e_\delta \mid \delta < \xi\} \cup \bigcup_{\delta < \xi} R_\delta$. E is GLusin because if R is nowhere dense then $E \cap R \subseteq E \cap \overline{R} \subseteq \{e_\xi \mid \xi < \gamma\}$ where γ is such that $\overline{R} = R_\gamma$. □

We lift the results in Goldstern, Judah and Shelah [10] one cardinal up:

8.6. Theorem. *If $2^{\aleph_1}$ is a regular cardinal and there exists a generalized Lusin set, then*

i) every subset of $\mathcal{N}_1$ of cardinality less than $2^{\aleph_1}$ has strong measure zero,

ii) there exists a strong measure zero set of cardinality $2^{\aleph_1}$, and

iii) every generalized Lusin set has strong measure zero.

Proof. i) Let A be a generalized Lusin set and $X \subseteq \mathcal{C}_1$, $|X| < 2^{\aleph_1}$, and M a meager set. In view of the first part of Theorem 7.8 it suffices to prove that there is an $f \in \mathcal{C}_1$ such that $(f+X) \cap M = \varnothing$. For each $f \in \mathcal{C}_1$, $-f+M$ is meager and therefore $|A \cap (-f+M)| < 2^{\aleph_1}$. Since $2^{\aleph_1}$ is regular, $A \setminus \bigcup_{f \in X}(-f+M)$ is not empty. If f belongs to this set we have that $(f+X) \cap M = \varnothing$.

ii) follows from iii).

iii) Let $\langle \alpha_\xi \rangle$ be arbitrary. The set $G = \bigcup_{\xi<\omega_1} U(q_\xi, \alpha_{2\xi})$ is comeager, so $|A \setminus G| < 2^{\aleph_1}$. By i) there are f_ξ, $\xi < \omega_1$, such that $A \setminus G \subseteq \bigcup_{\xi<\omega_1} U(f_\xi, \alpha_{2\xi+1})$. Therefore

$$A \subseteq \bigcup_{\xi<\omega_1} U(q_\xi, \alpha_{2\xi}) \cup \bigcup_{\xi<\omega_1} U(f_\xi, \alpha_{2\xi+1}).$$

□

8.7. Corollary. *Assume CH, $2^{\aleph_1} > \aleph_2$ is regular and GMA. There is a strong measure zero set of size $2^{\aleph_1}$.*

We believe that

8.8. Conjecture. Con(ZF) *implies* Con$(ZFC + CH + GBC)$.

Since GBC implies wGBC, the existence of a Kurepa-tree implies $\neg$GBC (Proposition 4.8), hence the consistency strength of GBC is at least as strong as the existence of an inaccessible cardinal. Thus the consistency of *GBC* would be a strong refutation of Kurepa's hypothesis. Of course $2^{\omega_1} > \omega_2$ and $\neg$GMA must hold in the model for GBC.

9. Measure in $\mathcal{N}_1$

We show that difficulties arise when we try to define a measure in $\mathcal{N}_1$.

9.1. Definition. Let $D \subseteq [\omega_1]^{\omega_1}$. A set A is *D-measurable*, if for every $X \in D$ there are a closed F and an open G such that $F \subseteq A \subseteq G$ and $G \setminus F$ is X-small. A is *measurable*, if it is $[\omega_1]^{\omega_1}$-measurable.

We do not know whether all open and closed sets are measurable.

Another possibility to define measurability is the following.

9.2. Definition. A set A is *D-measurable′*, if there is a Borel set B such that $A \triangle B$ is D-null. A is *measurable′*, if it is $[\omega_1]^{\omega_1}$-measurable′.

It is easy to prove that if A_n, $n \in \omega$, are measurable so are $\mathcal{N}_1 \smallsetminus A_0$ and $\bigcup_{n\in\omega} A_n$. We do not know how to prove that $\bigcup_{\alpha\in\omega_1} A_\alpha$ is measurable whenever the A_α, $\alpha < \omega_1$, are. A family $\langle A_\alpha \rangle_{\alpha<\omega_1}$ is *locally countable*, if for every $f \in \mathcal{N}_1$ there is $\xi < \omega_1$ such that $\{\alpha \mid U(f,\xi) \cap A_\alpha \neq \varnothing\}$ is countable.

9.3. Lemma. *If $\langle F_\delta \rangle_{\delta<\omega_1}$ is a locally countable family of closed sets, then $\bigcup_{\delta<\omega_1} F_\delta$ is closed.*

Proof. Let $f \notin \bigcup_{\delta<\omega_1} F_\delta$. Since $\langle F_\delta \rangle$ is a locally countable family, there is ξ such that $I = \{\delta \mid U(f,\xi) \cap F_\delta \neq \varnothing\}$ is countable. Let $\xi_0 = \xi \cup \sup\{\min\{\beta \mid U(f,\beta) \cap F_\delta = \varnothing\} \mid \delta \in I\}$. Now $U(f,\xi_0) \subseteq \mathcal{N}_1 \smallsetminus \bigcup_{\delta<\omega_1} F_\delta$. □

9.4. Lemma. *If A_δ, $\delta < \omega_1$, are measurable and the family $\langle A_\delta \rangle_{\delta<\omega_1}$ is locally countable then $\bigcup_{\delta\in\omega_1} A_\delta$ is measurable.*

Proof. Let $X \in [\omega_1]^{\omega_1}$ be arbitrary. We split $X = \bigcup_{\delta<\omega_1} X_\delta$. Choose open set G_δ and closed set F_δ such that $F_\delta \subseteq A_\delta \subseteq G_\delta$ and $G_\delta \smallsetminus F_\delta$ is X_δ-small. Let $G = \bigcup_{\delta<\omega_1} G_\delta$ and $F = \bigcup_{\delta<\omega_1} F_\delta$. By Lemma 9.3 F is closed. It suffices to prove that $G \smallsetminus F$ is X-small. This holds since

$$G \smallsetminus F \subseteq \bigcup_{\delta<\omega_1} (G_\delta \smallsetminus F_\delta)$$

and $\bigcup_{\delta<\omega_1} (G_\delta \smallsetminus F_\delta)$ is X-small. □

9.5. Problem. Are the definitions of measurability equivalent? Are Borel sets measurable?

The following observation is due to Ville Hakulinen. Let μ be a "measure" in $\mathcal{N}_1$ which satisfies the following conditions.

i) Clopen sets $A = \{f \mid f(0) < \omega, f(1) < \omega_1\}$, $B_\alpha = \{f \mid f(0) < \omega, f(1) < \alpha\}$, $C_n = \{f \mid f(0) < n, f(1) < \omega_1\}$ are measurable for all $\alpha \in \omega_1$ and $n \in \omega$.

ii) $\mathrm{ran}(\mu)$ is a linear order.

iii) If $\alpha < \beta$ then $\mu(B_\alpha) < \mu(B_\beta)$ and if $n < m$ then $\mu(C_n) < \mu(C_m)$.

Then $A = \bigcup_{n\in\omega} C_n = \bigcup_{\alpha<\omega_1} B_\alpha$, but it can not be that $\mu(A) = \sup_{n\in\omega} \mu(C_n)$ and $\mu(A) = \sup_{\alpha<\omega_1} \mu(B_\alpha)$, because the former is ω-cofinal and the latter is ω_1-cofinal. Hence μ cannot be both ω-additive and ω_1-additive.

A family $\mathcal{F}$ is *almost disjoint*, if for all $A, B \in \mathcal{F}$, $A \cap B$ is countable.

9.6. Lemma (CH). *There is a family of cardinality $2^{\aleph_1}$ of almost disjoint subsets of ω_1.*

Proof. By CH it is sufficient to produce $2^{\aleph_1}$ almost disjoint subsets of Seq. For each $f \in 2^{\omega_1}$ let $S(f) = \{f{\restriction}\xi \mid \xi < \omega_1\}$. Now $\{S(f) \mid f \in 2^{\omega_1}\}$ is the required family. □

The notion of strong measure zero is not perfectly suitable for the concept of measure zero:

9.7. Proposition (CH). *The family*

$$\{A \mid A \text{ closed and does not have strong measure zero}\}$$

does not have 2^{ω_1}*-c.c.*

Proof. If $X \subseteq \omega_1$ let $A(X) = \{f \in \mathcal{N}_1 \mid f(\xi) = 0 \text{ for all } \xi \notin X\}$. The set $A(X)$ is closed, since $A(X) = \bigcap_{\xi \notin X}\{f \mid f(\xi) = 0\}$. Let $\{X_\xi \subseteq \omega_1 \mid \xi < 2^{\aleph_1}\}$ be an almost disjoint family given by Lemma 9.6. Let $A_\xi = A(X_\xi)$. If $\xi \neq \xi'$ then $|A_\xi \cap A_{\xi'}| \leq \omega_1$: if $f \in A_\xi \cap A_{\xi'}$ then $f(\xi) = 0$ for all $\xi \notin X_\xi \cap X_{\xi'}$; thus $f \in \mathcal{Q}$.

Assume that $X \subseteq \omega_1$ and $|X| = \omega_1$. Then the set $A(X)$ does not have strong measure zero, since it is not $(X+1)$-small. Let $B = \bigcup_{\xi \in X} U(f_\xi, \xi + 1)$. We define an element $f \in A(X) \smallsetminus B$ by letting $f(\xi) \neq f_\xi(\xi)$ for each $\xi \in X$ and $f(\xi) = 0$ for each $\xi \notin X$. □

9.8. Theorem. *Assume either* $I(\omega)$*, or CH and* $2^{\aleph_1} = \aleph_2$*. Then there is a non-measurable set.*

Proof. We will construct a set $B \subseteq \mathcal{N}_1$ such that $B \cap C \neq \varnothing$ and $(\mathcal{N}_1 \smallsetminus B) \cap C \neq \varnothing$ for every closed set C of cardinality $> \omega_1$. The set B is called a *Bernstein set*. If the set B were measurable there would be a closed set F and an open set G such that $F \subseteq B \subseteq G$ and $G \smallsetminus F$ is $\langle \alpha + 1\rangle_{\alpha<\omega_1}$-small. By the property of B the sets F and $\mathcal{N}_1 \smallsetminus G$ have cardinality $\leq \omega_1$, therefore $\mathcal{N}_1 \smallsetminus (G \smallsetminus F) \subseteq (\mathcal{N}_1 \smallsetminus G) \cup F$ has cardinality $\leq \omega_1$. This contradicts Lemma 4.6.ii.

The construction of B. Let $\mathcal{F}$ be the set of closed sets of cardinality $> \omega_1$. $\mathcal{F}$ has cardinality $2^{\aleph_1}$, since $\mathbf{\Pi}^1_1 \restriction \mathcal{N}_1 = \{\bigcap_{\xi<\omega_1}(\mathcal{N}_1 \smallsetminus [f(\xi)]) \mid f : \omega_1 \to \text{Seq}\}$ and for example $\{[s] \cup \{f\} \mid f \in \mathcal{N}_1, s \in \text{Seq}\} \subseteq \mathcal{F}$. Let $\mathcal{F} = \{F_\xi \mid \xi < 2^{\aleph_1}\}$. Wellorder $\mathcal{N}_1$. If we assume $I(\omega)$, then by Väänänen [38] every closed set of cardinality $> \omega_1$ has cardinality $2^{\aleph_1}$. If we assume $\omega_2 = 2^{\omega_1}$, then trivially every closed set of cardinality $> \omega_1$ has cardinality 2^{ω_1}. Now we are ready to define the set B by transfinite induction. Assume that sets B_ξ and B'_ξ, $\xi < \gamma$, have been defined. Let f_γ and g_γ be two smallest elements in $F_\gamma \smallsetminus (\bigcup_{\xi<\gamma} B_\xi \cup \bigcup_{\xi<\gamma} B'_\xi)$. Put $B_\gamma = \{f_\gamma\} \cup \bigcup_{\xi<\gamma} B_\xi$ and $B'_\gamma = \{g_\gamma\} \cup \bigcup_{\xi<\gamma} B'_\xi$. Now $B = \bigcup_{\xi<2^{\aleph_1}} B_\xi$ has the properties of a Bernstein set. □

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