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ON HYPERBOLIC TYPE METRICS

RIKU KLÉN

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SUOMALAINEN TIEDEAKATEMIA

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Turku, February 2009

Riku Klén

Contents

1. Introduction

The notion of the distance between two points is one of the key concepts of mathematics in diverse areas such as geometry, linear algebra, topology, function theory, functional analysis and applied mathematics. The study of distances led to the notion of a metric space introduced by M. Fréchet in his thesis in 1906. A clue of the multitude of metrics and their many applications may be obtained by inspecting the dictionary of distances [8].

In geometric function theory, which is the area of mathematics this thesis belongs to, metrics are used in numerous ways. The basic function theoretic entities, such as conformal mappings and analytic functions, have properties best expressed in terms of the hyperbolic or chordal metric rather than the usual Euclidean metric. For instance, Nevanlinna's principle of the hyperbolic metric says that analytic functions are distance decreasing with respect to the hyperbolic metric [22, p. 50].

The discovery of the hyperbolic geometry by J. Bolyai and N. Lobachevsky two centuries ago was a great sensation. They solved, in particular, the two millenniums old question about the parallel postulate, which fails in the hyperbolic geometry. There are worlds very different from what an Euclidean observer sees. For instance, it may happen that seeing the space locally is not enough to produce the correct global picture. In geometric function theory these ideas were developed among others by F. Klein, H. Poincaré, H.A. Schwarz and C. Carathéodory. An important research theme is to investigate the characteristic features of metric spaces and to compare different geometries to each other as well as to classify low-dimensional manifolds (*i.e.* locally Euclidean spaces) in terms of the metrics they carry.

The key metric of this thesis, the quasihyperbolic metric, was introduced and studied by F.W. Gehring and his students in the 1970's [9, 10]. Thereafter it has become an important tool in many problems on topics such as geometric function theory and theory of mappings. The quasihyperbolic metric has been recently studied in several PhD theses [11, 13, 18, 19] and research articles [12, 21, 25, 26, 27].

Conformal invariants and conformally invariant metrics have an important role in geometric function theory. One of the most important conformally invariant metrics is the hyperbolic metric of the unit ball B^n or the half-space $\mathbb{H}^n$ used extensively both in the planar case $n = 2$ as well as in the higher dimensions $n \geq 3$ [3].

In the planar case, one can use the Riemann mapping theorem to extend this definition to the case of simply connected plane domains. In fact, this definition can be extended even to the case of all plane domains with at least three boundary points with respect to the extended plane $\overline{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$ by use of so called universal covering maps [14, p. 126]. Neither one of these methods is applicable to the higher dimensional case.

The quasihyperbolic metric can be defined for every domain $G \subset \mathbb{R}^n$ with at least two boundary points with respect to $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. It was introduced by F.W. Gehring and B.P. Palka [10] in 1976. It turns out that for the case of planar simply connected domains the quasihyperbolic metric is comparable to the hyperbolic metric [4, (8.4)] whereas for the case of plane domains with isolated boundary points this is not the case [14, p. 138, 253].

For a domain $G \subsetneq \mathbb{R}^n$, $n \geq 2$ we define the *quasihyperbolic length* of a rectifiable arc $\gamma \subset G$ by

$$\ell_k(\gamma) = \int_\gamma \frac{|dz|}{d(z, \partial G)},$$

where $d(z, \partial G)$ is the Euclidean distance between z and ∂G, and the *quasihyperbolic metric* by

$$k_G(x, y) = \inf_\gamma \ell_k(\gamma), \tag{1.1}$$

where the infimum is taken over all rectifiable curves in G joining x and y.

During the past three decades the quasihyperbolic metric has found many applications in geometric function theory [17, 24, 26, 30]. Even so, the geometry defined by the quasihyperbolic metric has been studied very little and some very basic questions remain open. In [31] M. Vuorinen presented a list of questions of this type. Certain of these questions have already been discussed in [19]. The purpose of this thesis is to shed light on some of these geometric properties.

One of the open questions in [31] was to determine the modulus of continuity of the identity mapping between two metric spaces. We will consider this problem by comparing the quasihyperbolic metric, the distance ratio metric and the spherical metric, where the two latter metrics are defined in Section 2. We will start the comparison with a simple domain, punctured space. These results immediately generalize for a general subdomain of $\mathbb{R}^n$. However, the results are sharp only in the domain $\mathbb{R}^n \setminus \{0\}$.

The trigonometry defined by the hyperbolic metric in the half-plane and in the unit disk is classical [3] while the trigonometry defined by the quasihyperbolic metric has not been studied even in simple domains, apart from the half-plane $\mathbb{H}^2$. In Section 4 we will explore the quasihyperbolic trigonometry in punctured plane, which is one of the few domains where the explicit formula for the quasihyperbolic distance is known. One of our main results in this section is the Rule of Cosines for the quasihyperbolic metric. Some of these results also follow easily from their

Euclidean counterparts if we use the exponential mapping, see [20, p. 38]. However, the key idea of this section is to represent the proofs in a way that generalization is possible. This section raises the natural question of finding suitable counterparts of the results for domains other than the punctured plane.

The quasihyperbolic metric has recently been studied [18, 25, 21] in convex domains. However, there is little knowledge about the quasihyperbolic metric in non-convex domains apart from the punctured space. In Section 5 we will estimate the quasihyperbolic metric in two simple non-convex domains. We will estimate the quasihyperbolic length of a closed simple curve in twice punctured plane and find an estimation for the quasihyperbolic metric in the Euclidean annulus.

Finally, in Section 6 we will consider the geometry of metric balls defined by the quasihyperbolic, distance ratio and spherical metrics. We will point out that the shape of these metric balls depends on the radius as well as the shape of the domain. We will consider connectivity and local convexity properties, like convexity and starlikeness, of the metric balls.

2. Notation

In this section we introduce some definitions and notation. We begin with the concept of metric spaces and define several particular metrics.

Let X be a nonempty set. A function $m\colon X \times X \to [0,\infty)$ is a *metric* on X if for all $x, y, z \in X$

(1) $m(x,y) = m(y,x) \geq 0$,
(2) $m(x,y) = 0$ if and only if $x = y$,
(3) $m(x,y) \leq m(x,z) + m(z,y)$ (triangle inequality).

A *metric space* (X,m) consists of a nonempty set X and a metric m on X. If $f : [0,\infty) \to [0,\infty)$ is an increasing function such that $f(t)/t$ is decreasing, then $(X, f \circ m)$ is a metric space whenever (X,m) is [1, p. 146]. In particular, the function $f(t) = t^\alpha$, for $\alpha \in (0,1)$, satisfies this condition.

Let (X,m) be a metric space and γ a curve in X. If

$$m(x,y) + m(y,z) = m(x,z)$$

for all $x, z \in \gamma$ and $y \in \gamma'$, where γ' is the subcurve of γ joining x and z, then γ is a *geodesic segment* or shortly a *geodesic*. We denote a geodesic between x and y by $J_m[x,y]$.

A metric space (X,m) is a *geodesic metric space* if there exists a geodesic segment joining any two points $x, y \in X$.

In each metric space (X,m) we define a *metric ball* or *m-ball* with center $x \in X$ and radius $r > 0$ by

$$B_m(x,r) = \{y \in X \colon m(x,y) < r\}. \tag{2.1}$$

In the dimension $n = 2$ we call $B_m(x, r)$ a *metric disk* or *m-disk*. We denote the *m-diameter* of a nonempty set $A \subset X$ by

$$\operatorname{diam}_m(A) = \sup_{x,y \in A} m(x, y).$$

We use notation $\mathbb{R}^n$ for n-dimensional Euclidean space, $B^n(x, r)$ and $S^{n-1}(x, r)$ for Euclidean balls and spheres, respectively, with radius $r > 0$ and center $x \in \mathbb{R}^n$. We abbreviate $B^n(r) = B^n(0, r)$, $B^n = B^n(1)$, $S^{n-1}(r) = S^{n-1}(0, r)$ and $S^{n-1} = S^{n-1}(1)$. We often identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. We denote by $\measuredangle(x, y, z) \in [0, \pi]$ the angle between line segments $[x, y]$ and $[y, z]$ at point y.

A domain $G \subset \mathbb{R}^n$ is *starlike with respect to* $x \in G$ if for all $y \in G$ the line segment $[x, y]$ is contained in G and G is *strictly starlike with respect to* x if each ray from the point x meets ∂G at exactly one point. If G is starlike with respect to x for all $x \in G$ then it is *convex*. A domain G is *strictly convex* if for all points $x, y \in \partial G$ the open line segment (x, y) is contained in G.

The *distance ratio metric* or *j-metric* in a proper subdomain G of the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is defined by

$$j_G(x, y) = \log\left(1 + \frac{|x - y|}{\min\{d(x), d(y)\}}\right),$$

where $d(x)$ is the Euclidean distance between x and ∂G. If the domain G is understood from the context we use the notation j instead of j_G. The distance ratio metric was first introduced by F.W. Gehring and B.G. Osgood [9] and in the above form by M. Vuorinen [29]. The metric space (G, j_G) is not geodesic for any domain G [15, Theorem 2.10].

Let $G, G' \subset \mathbb{R}^n$ be domains such that $G \subset G'$ and let $x, y \in G$. The quasihyperbolic metric defined by (1.1) and the distance ratio metric are monotone with respect to the domain, *i.e.*

$$k_G(x, y) \geq k_{G'}(x, y) \quad \text{and} \quad j_G(x, y) \geq j_{G'}(x, y).$$

By definition of the quasihyperbolic and the distance ratio metrics it is evident that the shape of the domain G, or more precisely the boundary ∂G, defines the distances $k_G(x, y)$ and $j_G(x, y)$ for $x, y \in G$. This fact is studied in item 5.17. F.W. Gehring and B.P. Palka showed [10, Lemma 2.1 (2.2)] that

$$j_G(x, y) \leq k_G(x, y) \tag{2.2}$$

for all domains $G \subsetneq \mathbb{R}^n$ and $x, y \in G$. On the other hand, M. Vuorinen has shown [30, Lemma 3.7 (2)] that if $|x - y| < s d(x)$, $s \in (0, 1)$, then

$$k_G(x, y) \leq \frac{1}{1 - s} j_G(x, y).$$

It is easy to see that both of the metrics k_G and j_G are invariant under similarities and Euclidean isometries [29, p. 34].

The explicit formula for the quasihyperbolic metric is known only in a very few domains. One of such domains is the punctured space $\mathbb{R}^n \setminus \{0\}$. G.J. Martin and B.G. Osgood showed [20, page 38] that for $x, y \in \mathbb{R}^n \setminus \{0\}$ and $n \geq 2$

$$k_{\mathbb{R}^n \setminus \{0\}}(x,y) = \sqrt{\alpha^2 + \log^2 \frac{|x|}{|y|}}, \tag{2.3}$$

where $\alpha = \measuredangle(x,0,y) \in [0,\pi]$. If the domain G is understood from the context we use the notation k instead of k_G. F.W. Gehring and B.G. Osgood proved that for any domain $G \subsetneq \mathbb{R}^n$ the metric space (G, k_G) is a geodesic metric space [9, Lemma 1]. Note that the formula (2.3) is invariant under inversions $x \mapsto r^2 x/|x|^2$, $x \in \mathbb{R}^n \setminus \{0\}$.

The *spherical metric* in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ is defined by

$$q(x,y) = \begin{cases} \dfrac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & x \neq \infty \neq y, \\ \dfrac{1}{\sqrt{1+|x|^2}}, & y = \infty. \end{cases}$$

The metric space $(\overline{\mathbb{R}^n}, q)$ is not geodesic, which can be seen for example by choosing $x = e_1$ and $z = -e_1$. Now $q(x,z) = 1$ and $q(x,y) = q(y,z) = 1/\sqrt{2}$ for all $y = (0, y_2, \ldots, y_n) \in \mathbb{R}^n$. For the radii $r \in (0, 1/\sqrt{2}]$ we have

$$\operatorname{diam}_q(B_q(x,r)) = \operatorname{diam}_q(\overline{B_q(x,r)}) = \operatorname{diam}_q(\partial B_q(x,r)) = 2r\sqrt{1-r^2}$$

whereas for $r \in (1/\sqrt{2}, 1)$

$$1 = \operatorname{diam}_q(B_q(x,r)) = \operatorname{diam}_q(\overline{B_q(x,r)}) > \operatorname{diam}_q(\partial B_q(x,r)) = 2r\sqrt{1-r^2}.$$

Therefore, in a metric space the inequality $\operatorname{diam}_m(\overline{B_m(x,r)}) < 2r$ may hold. It is also possible that $\operatorname{diam}_m(B_m(x,r)) < \operatorname{diam}_m(\overline{B_m(x,r)})$ as the example $X = \mathbb{Z}$, m is the Euclidean metric and $r = 1$ shows.

2.4. **Open problem.** For the quasihyperbolic metric we may ask whether there exists a radius $r_0 > 0$ such that

$$k(\partial B_k(x,r)) = 2r$$

for all $r \in (0, r_0]$ and $x \in G$, where $G \subsetneq \mathbb{R}^n$ is a domain.

For convex domains Open problem 2.4 has been solved [21, Theorem 3.18].

Let (X_1, m_1) and (X_2, m_2) be metric spaces and $f \colon X_1 \to X_2$ be a function. Suppose that there exists a continuous function $\omega_f \colon [0, r_1) \to [0, r_2)$, $r_1, r_2 > 0$, with $\omega_f(0) = 0$ such that

$$m_2(f(x), f(y)) \leq \omega_f(m_1(x,y)).$$

This function ω_f, if it exists, is called the *modulus of continuity*. The existence of the modulus of continuity is equivalent to the uniform continuity of f.

3. Comparison of metrics

In [31] several general topics were listed which are largely open for further investigation in the setup of metric spaces. One of these topics dealt with the study of the uniform continuity of mappings between metric spaces. A specific example is to study whether the identity mapping id is uniformly continuous as a mapping

$$id\colon (G,m_1) \to (G,m_2), \tag{3.1}$$

where m_1 and m_2 are metrics on G. We can see that for any domain $G \subsetneq \mathbb{R}^n$ the identity mapping

$$id\colon (G,k_G) \to (G,j_G)$$

is uniformly continuous and the modulus of continuity is also identity in view of (2.2).

In this section we study whether the identity mapping is uniformly continuous as a mapping in (3.1) for $m_i \in \{q, k_G, j_G\}$ and for different domains G. We consider first the quasihyperbolic and distance ratio metrics in $\mathbb{R}^n \setminus \{0\}$, then in $\mathbb{R}^n \setminus \{z\}$ for $z \in \mathbb{R}^n$ and finally for a general domain $G \subset \mathbb{R}^n$. The result for $\mathbb{R}^n \setminus \{0\}$ is formulated in the following theorem.

3.2. **Theorem.** *For all $x, y \in \mathbb{R}^n \setminus \{0\}$*

(i) $2q(x,y) \le k(x,y) \le \dfrac{\pi}{\log 3} j(x,y)$,

(ii) $q(x,y)\log 3 \le j(x,y) \le k(x,y)$.

The constant in the first inequality of (i) is the best possible and the second inequality of (i) holds with equality for $x = -y$. The first inequality of (ii) holds with equality for $x = -y$, $|x| = 1$, and the second inequality of (ii) holds with equality for $\measuredangle(x, 0, y) = 0$.

3.3. **Punctured space.** Next we compare the quasihyperbolic, the spherical and the distance ratio metrics in the punctured space $\mathbb{R}^n \setminus \{0\}$ and in a general punctured space $\mathbb{R}^n \setminus \{z\}$, $z \in \mathbb{R}^n$. By [18, Theorem 1.6], $k_G(x,y) \le \pi j_G(x,y)/\log 3$ for $G = \mathbb{R}^n \setminus \{0\}$ and $x, y \in G$. Combining this with (2.2) gives us

$$j(x,y) \le k(x,y) \le \frac{\pi}{\log 3} j(x,y) \tag{3.4}$$

for $x, y \in G = \mathbb{R}^n \setminus \{0\}$. By [30, 3.32 (1)]

$$q(x,y) \le \frac{e^{k_G(x,y)} - 1}{2}$$

for $x, y \in G = \mathbb{R}^n \setminus \{0\}$.

The results of this section were motivated by the following open problem posed by M. Vuorinen [31, 8.2] in 2006:

3.5. **Open problem.** Does there exist a constant c such that

$$q(x,y) \le c k(x,y)$$

for $x, y \in \mathbb{R}^n \setminus \{0\}$?

Besides 3.5 we may ask the same question for the distance ratio metric and furthermore, we may ask whether there exist functions ω_1 and ω_2 such that $k(x,y) \le \omega_1(q(x,y))$ and $j(x,y) \le \omega_2(q(x,y))$ for all $x,y \in \mathbb{R}^n \setminus \{0\}$.

It turns out that such functions ω_1 and ω_2 do not exist as the following proposition shows.

3.6. **Proposition.** *Let $G = \mathbb{R}^n \setminus \{z\}$, $z \in \mathbb{R}^n$, and $m \in \{k_G, j_G\}$. Then there does not exist a function ω such that*

$$m(x,y) \le \omega(q(x,y))$$

for all $x,y \in G$. Moreover, the mapping $id\colon (\mathbb{R}^n \setminus \{0\}, q) \to (\mathbb{R}^n \setminus \{0\}, m)$ is not uniformly continuous.

Proof. By the symmetry of G we may assume $z = te_1$ for $t \ge 0$. Let us assume, on the contrary, that there exist a function $\omega\colon [0,r_1) \to [0,r_2)$, $r_1, r_2 > 0$, such that $k(x,y) \le \omega(q(x,y))$ for all $x,y \in G$. Let us fix $c \in [0,1]$ and denote

$$x_i = e_1\left(t + \frac{1}{i}\right) \quad \text{and} \quad y_i = e_1 \frac{c\sqrt{1-c^2}(1+2it+i^2(1+t^2)) - i - i^2 t}{c^2(1+2it+i^2(1+t^2)) - i^2}.$$

Now for all $i \ge 1$ we have $q(x_i,y_i) = c$, $|x_i| \to t$ as $i \to \infty$ and

$$|y_i| = \frac{c\sqrt{1-c^2}(1/i^2+2t/i+(1+t^2)) - 1/i - t}{c^2(1/i^2+2t/i+(1+t^2)) - 1} \to \frac{c\sqrt{1-c^2}(1+t^2) - t}{c^2(1+t^2)-1}$$

as $i \to \infty$. We will show that $f(t) < 0$, $t \ge 0$, for the function

$$f(t) = \frac{c\sqrt{1-c^2}(1+t^2) - t}{c^2(1+t^2)-1} - t = \frac{c(1+t^2)(\sqrt{1-c^2} - ct)}{c^2(1+t^2)-1}.$$

Clearly $\sqrt{1-c^2} - ct < (>)0$ and $c^2(1+t^2) - 1 > (<)0$ for $t > (<)\sqrt{1-c^2}/c$ and thus $f(t) < 0$ for $t \ne \sqrt{1-c^2}/c$. By the l'Hospital Rule

$$\lim_{t \to \sqrt{1-c^2}/c} f(t) = \lim_{t \to \sqrt{1-c^2}/c} \frac{c(2\sqrt{1-c^2}t - c - 3ct^2)}{2c^2 t} = -\frac{1}{2c\sqrt{1-c^2}}$$

we have $f(t) < 0$ for $t \ge 0$.

Because $d(x_i) \to 0$ as $i \to \infty$ and $(c\sqrt{1-c^2}(1+t^2) - t)/(c^2(1+t^2)-1) < t$, we have

$$\infty > \omega(c) \ge k(x_i,y_i) \ge j(x_i,y_i) = \log\left(1 + \frac{|x_i - y_i|}{\min\{d(x_i), d(y_i)\}}\right) \to \infty$$

as $i \to \infty$. This contradiction completes the proof for the case $m = k$. For $m = j$ the proof is similar. □

Before answering the question 3.5 we introduce a useful lemma.

3.7. **Lemma.** *Let $x,y > 0$.*

(i) *For* $\alpha \in [0,\pi]$ *and* $x \neq y$

$$\frac{x^2+y^2-2xy\cos\alpha}{\alpha^2+(\log x-\log y)^2} \le \frac{(x-y)^2}{(\log x-\log y)^2}.$$

(ii) *For* $x \neq y$ *we define*

$$g(x,y) = \frac{|x-y|}{\sqrt{1+x^2}\sqrt{1+y^2}|\log x-\log y|}$$

and

$$g(x,x) = \lim_{y\to x} g(x,y) = \frac{x}{1+x^2}.$$

Then $g(x,y) \le 1/2$.

Proof. (i) By [7, (2)] we have $xy(\log x-\log y)^2 \le (x-y)^2$. Since $1-\cos\alpha = 2\sin^2(\alpha/2) \le \alpha^2/2$, it follows that

$$(1-\cos\alpha)xy(\log x-\log y)^2 \le \frac{\alpha^2}{2}(x-y)^2,$$

which is equivalent to the claim.

(ii) If $x \neq y$, then by [7, (2)]

$$\frac{|x-y|}{\sqrt{1+x^2}\sqrt{1+y^2}|\log x-\log y|} \le \frac{x+y}{2\sqrt{1+x^2}\sqrt{1+y^2}} = \frac{q(x,-y)}{2} \le \frac{1}{2}.$$

By the arithmetic-geometric mean inequality $g(x,x) \le 1/2$. □

Let us define, for a domain $G \subsetneq \mathbb{R}^n$ and the metric $m \in \{k_G, j_G\}$, the following constant

$$c_{G,m} = \sup_{\substack{x,y\in G\\ x\neq y}} \frac{q(x,y)}{m(x,y)}.$$

Clearly we have $q(x,y) \le c_{G,m} m(x,y)$ for all $x,y \in G$. We solve now Open problem 3.5 for the quasihyperbolic metric.

3.8. **Theorem.** *For* $G = \mathbb{R}^n \setminus \{0\}$ *and* $x,y \in G$ *we have*

$$\frac{q(x,y)}{k_G(x,y)} \le \frac{1}{2}.$$

Moreover, $c_{G,k} = 1/2$.

Proof. Let us denote $\alpha = \measuredangle(x,0,y)$ and assume $|x| \neq |y|$. By the definition of $q(x,y)$ and $k(x,y)$ and Lemma 3.7

$$\begin{aligned}\frac{q(x,y)}{k(x,y)} &= \frac{\sqrt{|x|^2+|y|^2-2|x||y|\cos\alpha}}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}\sqrt{\alpha^2+\log^2(|y|/|x|)}}\\ &\le \frac{||x|-|y||}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}|\log(|y|/|x|)|}\\ &\le \frac{1}{2}.\end{aligned}$$

If $|x| = |y|$, then

$$\frac{q(x,y)}{k(x,y)} = \frac{2|x|\sin(\alpha/2)}{(1+|x|^2)\alpha} \le \frac{|x|}{1+|x|^2} \le \frac{1}{2}.$$

Let us finally show that the constant $1/2$ is the best possible. By choosing $|x| = 1 = |y|$ we have

$$\frac{q(x,y)}{k_G(x,y)} = \frac{\sin(\alpha/2)}{\alpha} \to \frac{1}{2}$$

as $\alpha \to 0$ and the assertion follows. □

Next we find a solution for 3.5 in the case of the distance ratio metric.

3.9. **Theorem.** *For $G = \mathbb{R}^n \setminus \{0\}$ and $x, y \in G$ we have*

$$\frac{q(x,y)}{j_G(x,y)} \le \frac{1}{\log 3}$$

with equality for x and y such that $x = -y$ and $|x| = 1$. In particular, $c_{G,j} = 1/\log 3$.

Proof. We may assume $|x| \le |y|$ and denote $\alpha = \measuredangle(x,0,y)$. If $\alpha = 0$, then by the definition of $q(x,y)$ and $j(x,y)$ and Lemma 3.7 *(ii)*

$$\frac{q(x,y)}{j(x,y)} = \frac{|y|-|x|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}\log(|y|/|x|)} \le \frac{1}{2} < \frac{1}{\log 3}$$

and the assertion follows.

Assume $\alpha > 0$. Let us consider the function

$$f(a) = \frac{c}{\sqrt{1+|x|^2}\sqrt{1+a^2}\log(1+c/|x|)}$$

for $a > |x|$. Clearly $f(a)$ is decreasing in a. Therefore, by definition $q(x,y)/j(x,y)$ is a decreasing function in $|y|$ for fixed $|x-y|$.

If $|x-y| \geq 2|x|$, then the quantity $q(x,y)/j(x,y)$ is maximized when $|x-y| = |x|+|y|$ and thus

$$\begin{aligned} \frac{q(x,y)}{j(x,y)} &= \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}\log\left(1+\frac{|x-y|}{|x|}\right)} \\ &\leq \frac{|x|+|y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}\log\left(2+\frac{|y|}{|x|}\right)} \\ &\leq \frac{|x|+|y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\frac{1}{\log 3} \leq \frac{1}{\log 3}, \end{aligned}$$

where the last inequality follows from the fact that $(|x|+|y|) \leq (\sqrt{1+|x|^2}\sqrt{1+|y|^2})$ is equivalent to $0 \leq (1-|x||y|)^2$.

If $|x-y| < 2|x|$, then the quantity $q(x,y)/j(x,y)$ is maximized when $|x| = |y|$ and thus

$$\begin{aligned} \frac{q(x,y)}{j(x,y)} &= \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}\log\left(1+\frac{|x-y|}{|x|}\right)} \\ &\leq \frac{2|x|\sin(\alpha/2)}{(1+|x|^2)\log(1+2\sin(\alpha/2))} \\ &\leq \frac{2|x|}{(1+|x|^2)}\frac{1}{\log 3} \leq \frac{1}{\log 3}. \end{aligned}$$

where the second inequality follows from the fact that for $a \in [0,1]$ the function $a/\log(1+2a)$ is decreasing.

By choosing $|x| = 1$ and $y = -x$ we have $q(x,y)/j_G(x,y) = 1/\log 3$ and the assertion follows. □

Proof of Theorem 3.2. The assertion follows from equation (3.4) and Theorems 3.8 and 3.9. □

We compare next the quasihyperbolic, the spherical and the distance ratio metrics in a general punctured space $\mathbb{R}^n \setminus \{z\}$, $z \in \mathbb{R}^n$. By definition $k_{\mathbb{R}^n\setminus\{0\}}(x,y) = k_{\mathbb{R}^n\setminus\{z\}}(x-z,y-z)$ and $j_{\mathbb{R}^n\setminus\{0\}}(x,y) = j_{\mathbb{R}^n\setminus\{z\}}(x-z,y-z)$ for all $x,y \in \mathbb{R}^n \setminus \{0\}$. Therefore, we are interested only in the relation between the spherical metric and the metric $m \in \{k_{\mathbb{R}^n\setminus\{z\}}, j_{\mathbb{R}^n\setminus\{z\}}\}$.

3.10. **Lemma.** *Let $z \in \mathbb{R}^n$, $t = |z|$ and $G = \mathbb{R}^n \setminus \{z\}$. Then*

$$c_{G,k} \leq \frac{1}{2} + \frac{1}{4}t(t+\sqrt{4+t^2})$$

and equality holds for $t = 0$.

Proof. By [30, Lemma 1.54 (4)]

$$q(x,y) \leq bq(x-z,y-z)$$

for $x, y, z \in \mathbb{R}^n$ and $b = 1 + (|z|(|z| + \sqrt{4 + |z|^2}))/2$. Therefore, by Theorem 3.8

$$q(x,y) \le bq(x-z, y-x) \le \frac{b}{2} k_{\mathbb{R}^n \setminus \{z\}}(x-z, y-z) = \frac{b}{2} k_{\mathbb{R}^n \setminus \{0\}}(x,y)$$

and equality holds for $t = 0$. Therefore the assertion follows. □

In Lemma 3.10 we obtained an upper bound for $c_{G,k}$ in the case of $\mathbb{R}^n \setminus \{z\}$, $z \in \mathbb{R}^n$. The following theorem gives a lower bound for $c_{G,k}$.

3.11. **Theorem.** *Let $z \in \mathbb{R}^n$, $t = |z|$ and $G = \mathbb{R}^n \setminus \{z\}$. Then*

$$c_{G,k} \ge \frac{t + \sqrt{1+t^2}}{2}$$

and equality holds for $t = 0$.

Proof. By the symmetry of G we may assume $z = te_1$ and $n = 2$. We choose $x = d + hi$ and $y = d - hi$ for $d = t - \sqrt{1+t^2}$ and $h > 0$. Now

$$\frac{q(x,y)}{k_G(x,y)} = \frac{h}{(1+h^2+d^2)\arctan(h/(t-d))}$$

implying

$$\begin{aligned} \lim_{h\to 0} \frac{q(x,y)}{k_G(x,y)} &= \lim_{h\to 0} \frac{1}{(1+h^2+d^2)} \frac{h}{\arctan(h/(t-d))} \\ &= \frac{1}{1+d^2} \lim_{h\to 0} (t-d)\left(1 + \frac{h^2}{(t-d)^2}\right) \\ &= \frac{t-d}{1+d^2} = \frac{\sqrt{1+t^2}}{1+(t-\sqrt{1+t^2})^2} = \frac{t+\sqrt{1+t^2}}{2} \end{aligned}$$

and the assertion follows. Equality in the claim holds for $t = 0$ by Theorem 3.8. □

Let us introduce the counterpart of Lemma 3.10 for the distance ratio metric.

3.12. **Lemma.** *Let $z \in \mathbb{R}^n$, $t = |z|$ and $G = \mathbb{R}^n \setminus \{z\}$. Then*

$$c_{G,j} \le \frac{1}{\log 3} + \frac{1}{2\log 3} t(t + \sqrt{4+t^2})$$

and equality holds for $t = 0$.

Proof. By [30, Lemma 1.54 (4)]

$$q(x,y) \le bq(x-z, y-z)$$

for $x, y, z \in \mathbb{R}^n$ and $b = 1 + (|z|(|z| + \sqrt{4 + |z|^2}))/2$. Therefore by Theorem 3.9

$$q(x,y) \le bq(x-z, y-x) \le \frac{b}{\log 3} j_{\mathbb{R}^n \setminus \{z\}}(x-z, y-z) = \frac{b}{\log 3} j_{\mathbb{R}^n \setminus \{0\}}(x,y)$$

and equality holds for $t = 0$. Therefore the assertion follows. □

Similarly as for the quasihyperbolic metric, we find a lower bound for $c_{G,j}$ in the case $G = \mathbb{R}^n \setminus \{z\}$, $z \in \mathbb{R}^n$.

3.13. Lemma. *Let $z \in \mathbb{R}^n$, $t = |z|$ and $G = \mathbb{R}^n \setminus \{z\}$. Then*

$$c_{G,j} \geq \frac{1+t^2-t\sqrt{1+t^2}}{\sqrt{1+t^2}\left(1+t^2-t\sqrt{2t\sqrt{1+t^2}-t^2}\right)\log\left(3-\frac{2t}{\sqrt{1+t^2}}\right)}$$

and equality holds for $t=0$.

Proof. We show that there exists $x, y \in G$ such that

$$q(x,y) = c(t) j_G(x,y),$$

where

$$c(t) = \frac{1+t^2-t\sqrt{1+t^2}}{\sqrt{1+t^2}\left(1+t^2-t\sqrt{2t\sqrt{1+t^2}-t^2}\right)\log\left(3-\frac{2t}{\sqrt{1+t^2}}\right)}.$$

By the symmetry of G we may assume $z = te_1$. We choose $x, y \in G$ such that $|x-z| = |y-z|$ and $(x+y)/2$, 0 and z are on the same line. If $x_1, y_1 \geq 0$, then $q(x,y) = q(x-2x_1e_1, y-2y_1e_1)$ and $j_G(x,y) \geq j_G(x-2x_1e_1, y-2y_1e_1)$. If $x_1 \geq 0$ and $y_1 \leq 0$, then by rotation about the origin we can find $x', y' \in G$ such that $|x-y| = |x'-y'|$, $|x| = |x'|$, $|y| = |y'|$ and $x'_1, y'_1 \leq 0$. Now $q(x,y) = q(x',y')$ and $j_G(x,y) \geq j_G(x',y')$. Therefore we may assume that $x_1, y_1 \leq 0$.

For any $x \in G$, we choose $y = 2x_1e_1 - x$ and we show that $q(x,y)/k_G(x,y) \leq c(t)$. We denote $h = |x - x_1e_1|$ and $d = |x_1|$ implying $|x| = \sqrt{h^2+d^2} = |y|$, $|x-y| = 2h$ and $|x-z| = \sqrt{h^2+(t+d)^2} = |y-z|$. We are interested in the function

$$f(d,h) = \frac{q(x,y)}{j_G(x,y)} = \frac{2h}{(1+h^2+d^2)\log(1+u)}$$

for $h>0$, $d>0$ and $u = 2h/(\sqrt{h^2+(t+d)^2})$. We define $f(t,h) = \lim_{d\to t} f(d,h)$ and $f(d,0) = \lim_{h\to 0} f(d,h)$.

By a straightforward computation we obtain

$$\frac{\partial f(d,h)}{\partial d} = \frac{4h\left(v - d\log(1+u)\right)}{(1+d^2+h^2)^2\log^2(1+u)},$$

where

$$v = \frac{h(1+d^2+h^2)(d+t)}{(h^2+(d+t)^2)(2h+\sqrt{h^2+(d+t)^2})},$$

and

$$\frac{\partial f(d,h)}{\partial h} = \frac{2\left((1+d^2-h^2)\log(1+u) - 2v(d+t)\right)}{(1+d^2+h^2)^2\log^2(1+u)}$$

and $\partial f(d,h)/\partial d = 0 = \partial f(d,h)/\partial d$ is equivalent to $d = \sqrt{t^2+1-h^2} - t$.

Now we would like to find the maximum value of the function

$$g(h) = f(\sqrt{t^2+1-h^2}-t, h) = \frac{h}{\log\left(1+\frac{2h}{\sqrt{1+t^2}}\right)(t^2+1-t\sqrt{t^2+1-h^2})}$$

for $h \in (0, \sqrt{t^2+1})$. Instead of finding the maximum of the function $g(h)$ we settle for the fact that maximum of the function $g(h)$ on $(0, \sqrt{t^2+1})$ is greater than or equal to $g(1/(1+t)^2)$, because $(1+t^2-t\sqrt{1+t^2})/\sqrt{1+t^2} \le \sqrt{t^2+1}$. Since

$$h\left(\frac{1+t^2-t\sqrt{1+t^2}}{\sqrt{1+t^2}}\right) = \frac{\left(1+t^2-t\sqrt{1+t^2}\right)\left[\log\left(3-\frac{2t}{\sqrt{1+t^2}}\right)\right]^{-1}}{\sqrt{1+t^2}\left(1+t^2-t\sqrt{2t\sqrt{1+t^2}-t^2}\right)},$$

we can choose $c(t) = h((1+t^2-t\sqrt{1+t^2})/\sqrt{1+t^2})$.

Equality in the claim for $t = 0$ follows from Theorem 3.9. □

3.14. **General domain.** We can use Lemmas 3.10 and 3.12 to estimate the ratio of the metrics in a general domain.

3.15. **Corollary.** *Let $G \subset \mathbb{R}^n$ be a domain and $z \in \partial G$ such that $t = |z|$. Then*

$$c_{G,k} \le u(t),$$

where

$$u(t) = \frac{1}{2} + \frac{1}{4}t(t+\sqrt{4+t^2}).$$

Proof. Since $G \subset \mathbb{R}^n \setminus \{z\}$, we have $k_{\mathbb{R}^n\setminus\{z\}}(x,y) \le k_G(x,y)$. Therefore by Lemma 3.10

$$q(x,y) \le u(t)k_{\mathbb{R}^n\setminus\{z\}}(x,y) \le u(t)k_G(x,y)$$

and the assertion follows. □

3.16. **Corollary.** *Let $G \subset \mathbb{R}^n$ be a domain and $z \in \partial G$ such that $t = |z|$. Then*

$$c_{G,j} \le v(t),$$

where

$$v(t) = \frac{1}{\log 3} + \frac{1}{2\log 3}t(t+\sqrt{4+t^2}).$$

Proof. Since $G \subset \mathbb{R}^n \setminus \{z\}$, we have $j_{\mathbb{R}^n\setminus\{z\}}(x,y) \le j_G(x,y)$. Therefore by Lemma 3.12

$$q(x,y) \le v(t)j_{\mathbb{R}^n\setminus\{z\}}(x,y) \le v(t)j_G(x,y)$$

and the assertion follows. □

3.17. **Open problems.** Theorems 3.8 and 3.9 show that the upper bounds in Lemmas 3.10 and 3.12 are sharp for $z = 0$. This raises two natural questions for the general punctured space.

3.18. **Open problem.** Let $G = \mathbb{R}^n \setminus \{z\}$ be a domain and $z \in \mathbb{R}^n \setminus \{0\}$. Is it true

$$\frac{q(x,y)}{k_G(x,y)} \le \frac{|z|+\sqrt{1+|z|^2}}{2}$$

for all $x, y \in G$?

3.19. **Open problem.** What is the exact value of $c_{G,j}$ for the domain $G = \mathbb{R}^n \setminus \{z\}$, $z \in \mathbb{R}^n \setminus \{0\}$?

4. Quasihyperbolic trigonometry

There are not many domains where an explicit formula for the quasihyperbolic distance is known. The simplest such a domain is the complement of the origin, which we shall study in this section for the case $n = 2$. It turns out that numerous classical results for the plane geometry hold with very minor modifications in this case, too. This raises the general question whether and to what extent the results of this section have counterparts for a general plane domain. This topic is beyond the scope of this present investigation. The proofs are presented keeping possible generalizations in mind. For most of the results a shorter proof would follow from the results of G.J. Martin and B.G. Osgood [20].

We denote the n-dimensional Lebesgue measure by m and the $(n-1)$-dimensional surface measure of S^{n-1} by ω_{n-1}. The *quasihyperbolic volume* of a Lebesgue measurable set $A \subset G$ is defined by

$$m_k(A) = \int_A \frac{dm(z)}{d(z)^n}. \tag{4.1}$$

In the case $n = 2$ we call $m_k(A)$ the *quasihyperbolic area*.

Since (G, k), $G = \mathbb{R}^2 \setminus \{0\}$, is a geodesic metric space, like the usual hyperbolic space, it is possible to consider basic geometry. In item 4.6 we consider basic trigonometric identities of geodesic trigons. In items 4.28, 4.47 and 4.54 we consider the quasihyperbolic area of quasihyperbolic triangles, quadrilaterals and disks.

The main results of this section are the Euclidean model of the quasihyperbolic geometry in (G, k_G) for $G = \mathbb{R}^2 \setminus \{0\}$ introduced in item 4.59 and the following theorem. For the definition of a quasihyperbolic triangle and trigon, we refer to Definition 4.8.

4.2. **Theorem** (Law of Cosines). *Let $x, y, z \in \mathbb{R}^2 \setminus \{0\}$.*
(i) For the quasihyperbolic triangle $\triangle_k(x, y, z)$

$$k(x, y)^2 = k(x, z)^2 + k(y, z)^2 - 2k(x, z)k(y, z)\cos \measuredangle_k(y, z, x).$$

(ii) For the quasihyperbolic trigon $\triangle_k^*(x, y, z)$

$$k(x, y)^2 = k(x, z)^2 + k(y, z)^2 - 2k(y, z)k(z, x)\cos \measuredangle_k(y, z, x) - 4\pi(\pi - \alpha),$$

where $\alpha = \measuredangle(x, 0, y)$.

The Law of Cosines is a fundamental tool in the Euclidean geometry (see [5, Chapter 10]). We show that Theorem 4.2 also has a similar role in the quasihyperbolic trigonometry by proving several corollaries in item 4.28. The Euclidean model was implicit already in the paper of G.J. Martin and B.G. Osgood [20] but as far as we know Theorem 4.2 as well as the consequences have never been published in this form. Very recently J. Väisälä has studied the quasihyperbolic geometry in planar domains [27].

The domains $\mathbb{B}^n$ and $\mathbb{R}^n \setminus \{0\}$ are the extremal domains where the quasihyperbolic metric is defined. The results of the quasihyperbolic metric in these two domains can be used to estimate k_D for other domains $D \subset \mathbb{R}^n$, because by rescaling and translating we may assume either $D \subset \mathbb{R}^n \setminus \{0\}$ or $\mathbb{B}^n \subset D$. Roughly speaking one could expect that the results for $k_{\mathbb{R}^2\setminus\{0\}}$ may have generalizations to other cases, too. For instance, is there a counterpart of the Law of Cosines in Theorem 4.2 for a general domain in the form of an inequality? In this direction we have proved that Theorem 4.2 *(i)* holds as an inequality for the half-plane $\mathbb{H}^2 = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$.

4.3. **Logarithmic spirals.** Next we consider some properties of logarithmic spirals. By (2.3) the geodesic $J_k[x,y]$ is a subset of a logarithmic spiral whose polar equation is $r = ae^{b\theta}$ for $a = |x| \exp(-b \arg x)$ and $b = (1/\phi) \log(|y|/|x|)$, where ϕ is the signed angle between x and y at 0.

For $s \in (0,1)$ and $x, y \in \mathbb{R}^2 \setminus \{0\}$, $|x| \le |y|$, there exists $z \in J_k[x,y]$ such that $k(x,y) = k(x,z) + k(z,y)$ and $k(x,z) = sk(x,y)$. By (2.3) we can choose z to be a point such that $|z| = |x|^{1-s}|y|^s$, $\measuredangle(x,0,z) = s\measuredangle(x,0,y)$ and $\measuredangle(z,0,y) = (1-s)\measuredangle(x,0,y)$.

We consider now some basic properties of the logarithmic spiral. Let us define a *ray* by $R\,(x) = \{z \in \mathbb{R}^2 \setminus \{0\} \colon z = tx,\ t \in (0,\infty)\}$ for any $x \in \mathbb{R}^2 \setminus \{0\}$. The angle between $R\,(z)$ and the tangent of the logarithmic spiral at an intersection point is given by [2, p. 189-190]

$$\arctan \frac{1}{b}. \tag{4.4}$$

Note that the angle $\arctan(1/b)$ does not depend on z and therefore the angle between the ray $R\,(z)$ and the logarithmic spiral is always a constant. In the case $b = 0$ the logarithmic spiral is a ray and in the limiting case $b = \infty$ the logarithmic spiral is a circle.

By (4.4) the logarithmic spirals $r_1 = a_1 e^{b_1\theta}$ and $r_2 = a_2 e^{b_2\theta}$ are orthogonal if $b_1 = -1/b_2$. There are infinitely many logarithmic spirals containing two fixed points. Polar equation of the logarithmic spiral, which contains two distinct points $x = (r_1, 0), y = (r_2, \phi_2) \in \mathbb{R}^2 \setminus \{0\}$, $\phi_2 \in (-\pi, \pi]$, and the quasihyperbolic geodesic $J_k[x,y]$ is

$$r(\phi) = r_1 \exp\left(\frac{\phi}{\phi_2} \log \frac{r_2}{r_1}\right).$$

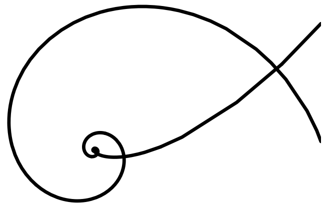

FIGURE 4.5. An example of two orthogonal logarithmic spirals.

For a given logarithmic spiral $S = \{(r, \phi) \colon r = ae^{b\phi}\}$ and a point $p = (r_1, \phi_1) \in S$ we can define a logarithmic spiral S', which is orthogonal to S and contains p, by

$$S' = \{(r, \phi) \in \mathbb{R}^2 \setminus \{0\} \colon r = r_1 e^{(\phi_1 - \phi)/b}\}.$$

An example of two orthogonal logarithmic spirals is represented in Figure 4.5. Note that the family of logarithmic spirals is invariant under stretching, inversions and rotations about the origin.

4.6. **Geodesic trigons.** We define quasihyperbolic triangles and trigons and find basic trigonometric properties of these geodesic trigons.

For $x, y \in \mathbb{R}^2 \setminus \{0\}$ a geodesic $J_k[x, y]$ is unique by (2.3) if $\measuredangle(x, 0, y) < \pi$. If $\measuredangle(x, 0, y) = \pi$, then there are exactly two geodesics joining x and y and they are symmetric about the line that contains x and y.

Let us fix $x, y \in \mathbb{R}^2 \setminus \{0\}$ such that $\measuredangle(x, 0, y) = \alpha_{xy} \in (0, \pi]$. Then $J_k[x, y]$ is a logarithmic spiral, *i.e.* for any $z \in J_k[x, y]$ we have

$$|z| = r(\alpha_{xz}) = |x| \exp\left(\frac{\alpha_{xz}}{\alpha_{xy}} \log\frac{|y|}{|x|}\right),$$

where $\alpha_{xz} = \measuredangle(x, 0, z) \in [0, \alpha_{xy}]$. If $\alpha_{xy} = 0$ then $J_k[x, y]$ is the line segment $[x, y]$.

4.7. **Definition.** For distinct $x, y, z \in \mathbb{R}^2 \setminus \{0\}$ we define the *quasihyperbolic angle* $\measuredangle_k(x, y, z)$ at y to be the Euclidean angle between the geodesics $J_k[x, y]$ and $J_k[y, z]$.

Note that by definition the quasihyperbolic angle is determined by the geodesics and not by the points. By (4.4) we can find an expression for the angle between the geodesic $J_k[x, y]$ and the ray $R(x)$. Namely, the function

$$\alpha(x, y) = \begin{cases} \pi/2, & \text{if } |x| = |y|, \\ \arctan\left(\dfrac{\alpha_{xy}}{|\log(|y|/|x|)|}\right), & \text{if } |x| \neq |y|, \end{cases}$$

describes the angle between the ray $R(x)$ and $J_k[x, y]$. The function α can be used to calculate quasihyperbolic angles between two intersecting geodesics.

4.8. **Definition.** For distinct $x, y, z \in \mathbb{R}^2 \setminus \{0\}$ we define a *geodesic trigon* T to be $J_k[x, y] \cup J_k[y, z] \cup J_k[z, x]$ for fixed *sides* $J_k[x, y]$, $J_k[y, z]$ and $J_k[z, x]$. The *interior* of a geodesic trigon is the set of points in $\mathbb{R}^2 \setminus \{0\}$ that is enclosed by the geodesic trigon. The points x, y and z are called the *vertices* of the geodesic trigon.

If the interior of the geodesic trigon is simply connected we call T *quasihyperbolic triangle* and use notation $\triangle_k(x, y, z)$. Otherwise T is called *quasihyperbolic trigon* and denoted by $\triangle_k^*(x, y, z)$.

Note that $\partial\triangle_k^*(x, y, z)$ is the boundary of a domain $D \subset \mathbb{R}^2$, which contains the origin. Clearly the quasihyperbolic triangle is always unique and it is contained in the closure of a half-plane H with $0 \in \partial H$. Therefore arbitrary points $x, y, z \in \mathbb{R}^2 \setminus \{0\}$ need not form a quasihyperbolic triangle. An example of a quasihyperbolic triangle and a quasihyperbolic trigon is represented in Figure 4.9.

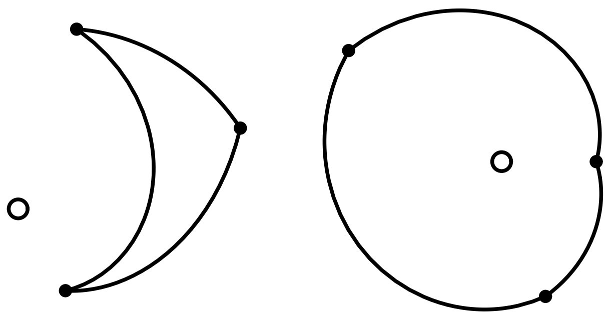

FIGURE 4.9. An example of a quasihyperbolic triangle (left) and a quasihyperbolic trigon (right).

For distinct $x, y, z \in \mathbb{R}^2 \setminus \{0\}$ we fix $J_k[x, y]$, $J_k[y, z]$ and $J_k[z, x]$ and denote by $\alpha_x = \measuredangle_k(z, x, y)$, $\alpha_y = \measuredangle_k(x, y, z)$ and $\alpha_z = \measuredangle_k(y, z, x)$, where each angle α_i is the angle measured from the set that is enclosed by $J_k[x, y] \cup J_k[y, z] \cup J_k[z, x]$. Especially α_x, α_y and α_z are the angles of a quasihyperbolic triangle $\triangle_k(x, y, z)$.

One of the basic facts about hyperbolic geometry is that the sum of the angles of a triangle is less than π [3, 14]. For the quasihyperbolic geometry of $\mathbb{R}^n \setminus \{0\}$ the sum of angles of a quasihyperbolic triangle is π, as in the Euclidean geometry, and the sum of angles of a quasihyperbolic trigon is equal to 3π.

4.10. **Theorem.** *Let $x, y, z \in \mathbb{R}^2 \setminus \{0\}$ be distinct points. If x, y and z form a quasihyperbolic triangle, then $\alpha_x + \alpha_y + \alpha_z = \pi$. Otherwise $\alpha_x + \alpha_y + \alpha_z = 3\pi$.*

Proof. We know that each geodesic in $\mathbb{R}^2 \setminus \{0\}$ is a subset of a logarithmic spiral. We use the fact that the angle between a logarithmic spiral and the ray $R(x)$ is always a constant for any $x \in \mathbb{R}^2 \setminus \{0\}$.

We consider first the case where x, y and z form a quasihyperbolic triangle, *i.e.* $x, y, z \in \overline{H}$ for a half-plane H such that $0 \in \partial H$. We may assume $H = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ and $\arg x \le \arg y \le \arg z$.

If $y \in J_k[x, z]$, then $\alpha_x = \alpha_z = 0$ and $\alpha_y = \pi$ and the claim is clear.

Let us assume $y \notin J_k[x, z]$. Now $\alpha_x = \alpha(x, y) - \alpha(x, z)$, $\alpha_x = \alpha(x, z) - \alpha(x, y)$ or $\alpha_x = \pi - \alpha(x, y) - \alpha(x, z)$ (see Figure 4.11).

If $\alpha_x = \alpha(x, y) - \alpha(x, z)$, then $\alpha_z = \alpha(z, x) - \alpha(z, y)$ and $\alpha_y = \alpha(y, z) + \pi - \alpha(y, x)$. Therefore $\alpha_x + \alpha_y + \alpha_z = \pi$.

If $\alpha_x = \alpha(x, z) - \alpha(x, y)$, then $\alpha_z = \pi - \alpha(z, x) - \alpha(z, y)$ and $\alpha_y = \alpha(y, x) + \alpha(y, z)$ and $\alpha_x + \alpha_y + \alpha_z = \pi$.

Finally, if $\alpha_x = \pi - \alpha(x, y) - \alpha(x, z)$, then $\alpha_z = \alpha(z, x) - \alpha(z, y)$ and $\alpha_y = \alpha(y, x) + \alpha(y, z)$. Therefore $\alpha_x + \alpha_y + \alpha_z = \pi$.

Let us then assume that x, y and z form a quasihyperbolic trigon. Now $\alpha_x = \alpha(x, y) + \alpha(x, z)$, $\alpha_x = \alpha(x, y) + \pi - \alpha(x, z)$, $\alpha_x = \alpha(x, z) + \pi - \alpha(x, y)$ or $\alpha_x = 2\pi - \alpha(x, y) - \alpha(x, z)$ (see Figure 4.12).

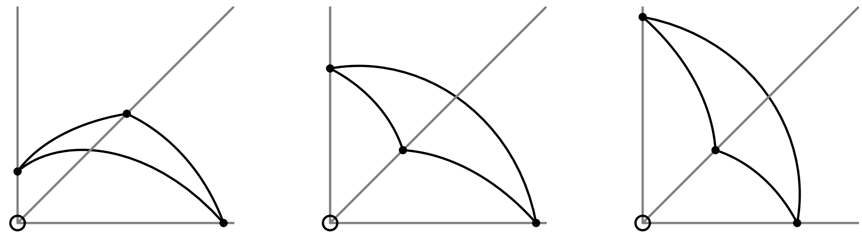

FIGURE 4.11. The three different cases of α_x in the case that x, y and z form a quasihyperbolic triangle. In each case $\arg x = \pi/2$.

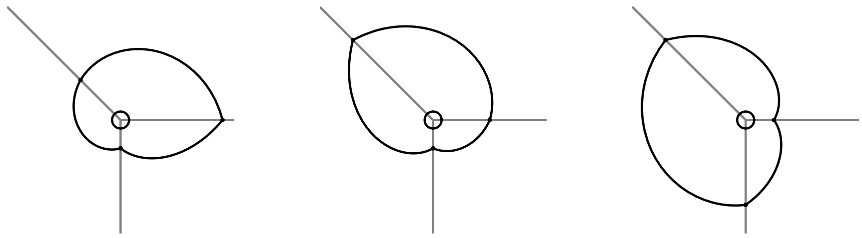

FIGURE 4.12. The three essentially different cases in the case that x, y and z form a quasihyperbolic trigon. In each case $\arg x = 0$.

If $\alpha_x = \alpha(x,y) + \alpha(x,z)$, then

$$(\alpha_y, \alpha_z) = \big(\pi - \alpha(y,x) + \alpha(y,z), 2\pi - \alpha(z,x) - \alpha(z,y)\big)$$

or

$$(\alpha_y, \alpha_z) = \big(2\pi - \alpha(y,x) - \alpha(y,z), \pi - \alpha(z,x) + \alpha(z,y)\big).$$

Either way, we have $\alpha_x + \alpha_y + \alpha_z = 3\pi$.

If $\alpha_x = \alpha(x,y) + \pi - \alpha(x,z)$, then

$$(\alpha_y, \alpha_z) = \big(2\pi - \alpha(y,x) - \alpha(y,z), \alpha(z,x) + \alpha(z,y)\big)$$

or

$$(\alpha_y, \alpha_z) = \big(\pi - \alpha(y,x) + \alpha(y,z), \alpha(z,x) + \pi - \alpha(z,y)\big).$$

In both cases, we have $\alpha_x + \alpha_y + \alpha_z = 3\pi$.

The case $\alpha_x = \alpha(x,y) + \pi - \alpha(x,z)$ can be obtained from the case $\alpha_x = \alpha(x,y) + \pi - \alpha(x,z)$ by changing y and z.

Finally, if $\alpha_x = 2\pi - \alpha(x,y) - \alpha(x,z)$, then

$$(\alpha_y, \alpha_z) = \big(\alpha(y,x) + \pi - \alpha(y,z), \alpha(z,x) + \alpha(z,y)\big)$$

or

$$(\alpha_y, \alpha_z) = \big(\alpha(y,x) + \alpha(y,z), \alpha(z,x) + \pi - \alpha(z,y)\big).$$

In both cases, we have $\alpha_x + \alpha_y + \alpha_z = 3\pi$. $\square$

We introduce then the Pythagorean Theorem for the quasihyperbolic triangles in a special case.

4.13. **Lemma.** *Let* $\triangle_k(x,y,z)$ *be a quasihyperbolic triangle,* $\measuredangle(x,0,y)=0$ *and* $|z|=|x|$. *Then*

$$k(y,z)^2 = k(x,y)^2 + k(x,z)^2.$$

Proof. By (2.3) $k(x,y)=\big|\log|x|-\log|y|\big|$, $k(x,z)=\measuredangle(x,0,z)$ and

$$k(y,z)=\sqrt{\alpha_{yz}^2+(\log|y|-\log|z|)^2},$$

where $\alpha_{yz}=\measuredangle(y,0,z)$. Since $\measuredangle(y,0,z)=\measuredangle(x,0,z)$ and $|x|=|z|$, the assertion follows. □

Before generalizing Lemma 4.13 we introduce a convenient notation for a *sector of an annulus*

$$S_{xy}=\{z\in\mathbb{R}^2\setminus\{0\}\colon |x|<|z|<|y|,\ \measuredangle(z,0,x)+\measuredangle(z,0,y)=\measuredangle(x,0,y)\},$$

where $x,y\in\mathbb{R}^2\setminus\{0\}$, $|x|<|y|$ and $\measuredangle(x,0,y)\in(0,\pi)$.

4.14. **Theorem.** *For any quasihyperbolic triangle the Pythagorean Theorem, the Law of Sines and the Law of Cosines are true. In particular, for a quasihyperbolic triangle* $\triangle_k(x,y,z)$ *with* $\measuredangle_k(y,x,z)=\pi/2$ *we have*

$$k(y,z)^2 = k(x,y)^2 + k(x,z)^2,$$

for a quasihyperbolic triangle $\triangle_k(x,y,z)$

$$\frac{k(x,y)}{\sin\alpha_z}=\frac{k(y,z)}{\sin\alpha_x}=\frac{k(z,x)}{\sin\alpha_y}$$

and

$$k(x,y)^2 = k(x,z)^2 + k(y,z)^2 - 2k(x,z)k(y,z)\cos\alpha_z. \tag{4.15}$$

Proof. Let us consider the smallest possible set S_{ab} that contains the interior of $\triangle_k(x,y,z)$ with $|a|<|b|$. By Theorem 4.10 S_{ab} has a corresponding rectangle S in the metric space $(\mathbb{R}^2,|\cdot|)$ with height $k(a,a|b|/|a|)$ and width $k(a,b|a|/|b|))$ and similarly $\triangle_k(x,y,z)$ has a corresponding triangle $\triangle(x',y',z')$ in $(\mathbb{R}^2,|\cdot|)$ with the same angles. By the definition of S_{ab} we have $x,y,z\in\partial S_{ab}$ and $x',y',z'\in\partial S$ (see Figure 4.16).

By Lemma 4.13 $|x'-y'|=k(x,y)$, $|y'-z'|=k(y,z)$ and $|z'-x'|=k(z,x)$. Now the assertion follows from the Euclidean Pythagorean Theorem, the Euclidean Law of Sines and the Euclidean Law of Cosines. □

4.17. **Remark.** Let $T(x,y,z)$ be a geodesic trigon in a metric space (X,d) and let $C(a,b,c)$ be a *comparison triangle* in $\mathbb{R}^2$ such that $d(x,y)=|a-b|$, $d(y,z)=|b-c|$ and $d(z,x)=|c-a|$. If for all geodesic trigons T and $u,v\in T$

$$d(u,v)\le|\overline{u}-\overline{v}|, \tag{4.18}$$

where $\overline{u}$ and $\overline{v}$ are comparison points of u and v respectively, then (X,d) is called CAT(0) *space* [6]. The inequality (4.18) is called the CAT(0) *condition.*

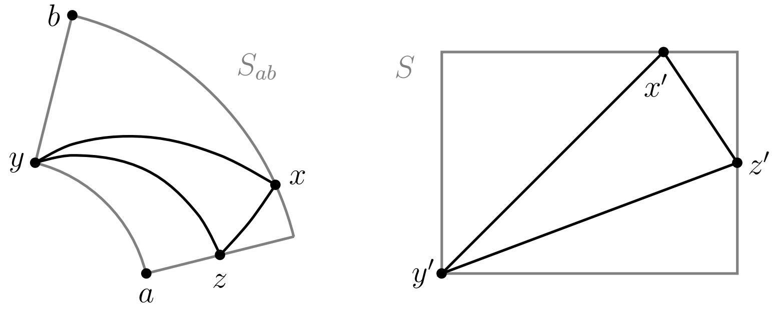

FIGURE 4.16. The points a, b and c in the proof Theorem 4.14.

By Theorem 4.14 the quasihyperbolic triangles in the metric space $(\mathbb{R}^2 \setminus \{0\}, k)$ satisfy the CAT(0) condition. However, the quasihyperbolic trigons need not satisfy the CAT(0) condition. Let us consider the case $x = 1$, $y = e^{i\,2\pi/3}$ and $y = e^{-i\,2\pi/3}$. Now

$$k(x,y) = k(y,z) = k(z,x) = \frac{2\pi}{3}$$

and

$$\measuredangle_k(x,y,z) = \measuredangle_k(y,z,x) = \measuredangle_k(z,x,y) = \pi$$

and the comparison triangle is an equilateral triangle with side length $2\pi/3$.

Let us denote $u = -1$ and $\overline{u} \in \mathbb{R}^2$ be the comparison point of u. Now

$$k(x,u) = \pi > \frac{\pi}{\sqrt{3}} = |\overline{x} - \overline{u}|$$

and therefore $(\mathbb{R}^2 \setminus \{0\}, k)$ is not a CAT(0) space.

Let us now consider trigonometry of the quasihyperbolic trigons.

4.19. **Theorem.** *If $\triangle_k^*(x,y,z)$ is a quasihyperbolic trigon and $\alpha = \measuredangle(x,0,y)$, then*

$$k(x,y)^2 = k(x,z)^2 + k(y,z)^2 - 2k(y,z)k(z,x)\cos\alpha_z - 4\pi(\pi - \alpha).$$

Proof. Let us denote by γ_1 the geodesic from x to y and by $\gamma_2 \neq \gamma_1$ the subarc of a logarithmic spiral such that $a, b \in \gamma$ implies $\measuredangle(a,0,b) \leq 2\pi - \alpha$. Let $u \in \gamma_2$ such a point that $\measuredangle(x,0,u) = \measuredangle(u,0,y)$. Now by integrating the element of length along the curve we obtain

$$\int_{\gamma_2} \frac{|dv|}{d(v)} = \sqrt{(2\pi - \alpha)^2 + \log^2 \frac{|x|}{|y|}}. \tag{4.20}$$

On the other hand, as in the proof of Theorem 4.14 we can show that

$$\left(\int_{\gamma_2} \frac{|dv|}{d(v)} \right)^2 = k(x,z)^2 + k(y,z)^2 - 2k(x,z)k(y,z)\cos\alpha_z. \tag{4.21}$$

Now by (4.20) and (2.3)

$$k(x,y)^2 = \left(\int_{\gamma_2} \frac{|dv|}{d(v)}\right)^2 - 4\pi(\pi-\alpha)$$

and the assertion follows by (4.21). □

4.22. **Corollary.** *Let $\triangle_k^*(x,y,z)$ be a quasihyperbolic trigon, $\alpha_z = \pi/2$ and $\alpha = \measuredangle(x,0,y)$. Then*

$$k(x,y)^2 = k(x,z)^2 + k(y,z)^2 - 4\pi(\pi-\alpha).$$

Proof of Theorem 4.2. The assertion follows from Theorems 4.14 and 4.19. □

The Law of Sines for the Euclidean and the quasihyperbolic triangles is based on the fact that a triangle can be circumscribed by a circle by locating the circumcenter as the point of intersection of the perpendicular bisectors. However, this is not true for the quasihyperbolic trigons in general.

Similarly, other results from the Euclidean trigonometry are true in the quasihyperbolic trigonometry, if the geometric objects involved are contained in a half-plane H with $0 \in \partial H$.

4.23. **Inequality of cosines.** By Theorem 4.19 we see that the Law of Cosines (4.15) is not true for the quasihyperbolic trigons in $\mathbb{R}^2 \setminus \{0\}$. Instead of the Law of Cosines we could consider the following inequality of cosines

$$k(x,y)^2 \geq k(x,z)^2 + k(y,z)^2 - 2k(y,z)k(z,x)\cos\measuredangle_k(y,z,x). \tag{4.24}$$

By Theorem 4.2 inequality (4.24) is true for the quasihyperbolic metric in $\mathbb{R}^2 \setminus \{0\}$. The following lemma shows that it is also true for the quasihyperbolic metric in $\mathbb{H}^2$, because then the quasihyperbolic metric coincides with the hyperbolic metric.

4.25. **Lemma.** *Let $x, y, z \in \mathbb{H}^2$ be distinct points. Then*

$$k_{\mathbb{H}^2}(x,y)^2 \geq k_{\mathbb{H}^2}(x,z)^2 + k_{\mathbb{H}^2}(y,z)^2 - 2k_{\mathbb{H}^2}(y,z)k_{\mathbb{H}^2}(x,z)\cos\gamma,$$

where γ is the Euclidean angle between geodesics $J_k[z,x]$ and $J_k[z,y]$.

Proof. Let us denote $a = k_{\mathbb{H}^2}(x,z)$, $b = k_{\mathbb{H}^2}(y,z)$ and $c = k_{\mathbb{H}^2}(x,y)$. By [3, 7.12]

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma$$

and since the function cosh is increasing on $[0,\infty)$, we need to show that

$$\cosh a \cosh b - \sinh a \sinh b \cos\gamma \geq \cosh\sqrt{a^2+b^2-2ab\cos\gamma}.$$

Let us therefore show that the function

$$f(x) = \cosh a \cosh b - x \sinh a \sinh b - \cosh\sqrt{a^2+b^2-2abx}$$

is non-negative on $[-1,1]$. Clearly

$$\begin{aligned} f(-1) &= \cosh a \cosh b + \sinh a \sinh b - \cosh\sqrt{a^2+b^2+2ab} \\ &= \cosh(a+b) - \cosh|a+b| = 0 \end{aligned}$$

and

$$\begin{aligned} f(1) &= \cosh a \cosh b - \sinh a \sinh b - \cosh \sqrt{a^2+b^2-2ab} \\ &= \cosh(a-b) - \cosh|a-b| = 0. \end{aligned}$$

By a straightforward computation we obtain

$$f''(x) = \frac{a^2b^2\left(\sinh\sqrt{a^2+b^2-2abx} - \sqrt{a^2+b^2-2abx}\cosh\sqrt{a^2+b^2-2abx}\right)}{(a^2+b^2-2abx)^{3/2}}.$$

Because $a,b>0$, we have $a^2+b^2-2abx = (a-b)^2+2ab(1-x) > 0$ for $x \in (-1,1)$ and for $x = \pm 1$ in the case of $a \neq b$. Since $\sinh t < t\cosh t$ for all $t > 0$, we have $f''(x) < 0$ for $x \in [-1,1]$ and therefore f is concave on $[-1,1]$. Since $f(-1) = 0 = f(1)$, the function f is non-negative on $[-1,1]$ and the assertion follows. □

Note that equality in Lemma 4.25 holds true if and only if x, y and z are on the same line that is orthogonal to the boundary of the domain $\mathbb{H}^2$.

4.26. **Remark.** The Law of Cosines for the quasihyperbolic metric is true neither in $G_1 = \mathbb{R}^3 \setminus \{0\}$ nor in $G_2 = \mathbb{R}^2 \setminus \{0,1\}$. This can be seen by observing that the Pythagorean Theorem is not true.

Let us first consider G_1. Choose $x = e_1$, $y = e_2$ and $z = e_3$. Now $k_{G_1}(x,y) = k_{G_1}(y,z) = k_{G_1}(z,x) = \pi/2$ and $\measuredangle_k(x,y,z) = \measuredangle_k(y,z,x) = \measuredangle_k(z,x,y) = \pi/2$. Therefore

$$k_{G_1}(x,y)^2 + k_{G_1}(y,z)^2 = \pi^2/2 > \pi^2/4 = k_{G_1}(z,x)^2$$

and the Pythagorean Theorem is not true.

Let us then consider G_2. We find first a formula for $k_{G_2}(1/2, 1/2+ci)$ when $c > 0$. Since in G_2, $J_k[1/2, 1/2+ci] = [1/2, 1/2+ci]$, we have

$$k_{G_2}(1/2, 1/2+ci) = \int_{1/2}^{1/2+c} \frac{dx}{\sqrt{1/4+x^2}} = \log(2c+\sqrt{4c^2+1}). \tag{4.27}$$

Choose $a = 1/2$, $b = 1/4$ and $c = 1/2 + i/2$. Now by (2.3), $k_{G_2}(a,b) = \log 2$, $k_{G_2}(a,c) = \sqrt{\pi^2+4\log 2}/4$ and by (4.27), $k_{G_2}(b,c) = \log(1+\sqrt{2})$. Because $\measuredangle_k(a,b,c) = \pi/2$ and

$$k_{G_2}(a,b)^2 + k_{G_2}(b,c)^2 = (\log 2)^2(\log(1+\sqrt{2}))^2 > 1 > \frac{\pi^2+4\log 2}{16} = k_{G_2}(b,c)^2,$$

the Pythagorean Theorem is not true.

It is not known whether the Pythagorean inequality $k(x,y)^2 + k(y,z)^2 \le k(x,z)^2$ holds in the domain G_i, $i \in \{1,2\}$, for points $x,y,z \in G_i$ such that $\measuredangle_k(x,y,z) = \pi/2$.

4.28. **Quasihyperbolic area of quasihyperbolic triangle.** Next we consider the quasihyperbolic area defined by (4.1). Since the quasihyperbolic area of quasihyperbolic trigons is always infinity, we concentrate on the quasihyperbolic area of a quasihyperbolic triangle $\triangle_k(x,y,z)$. We may assume that $\alpha_x \ge \max\{\alpha_y, \alpha_z\}$. We consider the side $J_k[y,z]$ as the *base side* and define the *height* of $\triangle_k(x,y,z)$ as

$k_{\mathbb{R}^2\setminus\{0\}}(x,u)$ for $u \in J_k[y,z]$ such that $\measuredangle_k(x,u,y) = \pi/2$. Note that u exists by Theorem 4.14 and it is uniquely defined.

4.29. **Lemma.** *Let $\triangle_k(x,y,z)$ be a quasihyperbolic triangle in $\mathbb{R}^2 \setminus \{0\}$ and $\alpha_x \geq \max\{\alpha_y, \alpha_z\}$. Then the height of $\triangle_k(x,y,z)$ is*

$$\sqrt{k(x,y)^2 - \frac{\big(k(y,z)^2 + k(x,y)^2 - k(x,z)^2\big)^2}{4k(y,z)^2}}.$$

Proof. Since $\measuredangle_k(x,u,y) = \pi/2$ and $\measuredangle_k(x,u,z) = \pi/2$, we have by Theorem 4.14

$$k(x,u)^2 = k(x,y)^2 - k(y,u)^2 \tag{4.30}$$

and

$$k(x,u)^2 = k(x,z)^2 - k(z,u)^2. \tag{4.31}$$

Since $u \in J_k[y,z]$, we have $k(y,z) = k(y,u) + k(u,z)$ and therefore by (4.30) and (4.31)

$$k(x,y)^2 - k(y,u)^2 = k(x,z)^2 - \big(k(y,z) - k(y,u)\big)^2,$$

which is equivalent to

$$k(y,u) = \frac{k(x,y)^2 + k(y,z)^2 - k(x,z)^2}{2k(y,z)}. \tag{4.32}$$

By (4.30) and (4.32) the assertion follows. □

The formula for the height of $\triangle_k(x,y,z)$ given in Lemma 4.29 is equivalent to

$$\sqrt{k(x,z)^2 - \frac{\big(k(y,z)^2 + k(x,z)^2 - k(x,y)^2\big)^2}{4k(y,z)^2}}.$$

We find now a formula for the quasihyperbolic area of two special domains and then extend the result to an arbitrary quasihyperbolic triangle.

4.33. **Lemma.** *The quasihyperbolic area of S_{xy} is*

$$\measuredangle(x,0,y)\log\frac{|y|}{|x|}.$$

Proof. By a simple computation

$$\begin{aligned} m_k(S_{xy}) &= \int_0^{\measuredangle(x,0,y)} \left(\int_{|x|}^{|y|} \frac{dx}{x}\right) d\phi \\ &= \int_0^{\measuredangle(x,0,y)} \log\frac{|y|}{|x|} d\phi \\ &= \measuredangle(x,0,y)\log\frac{|y|}{|x|}. \end{aligned}$$

□

For a fixed S_{xy} let us denote $z = y|x|/|y|$. Now $k(x,z) = \measuredangle(x,0,y)$ and $k(z,y) = \log(|y|/|x|)$. Therefore the quasihyperbolic area of S_{xy} is equal to $k(x,z)k(z,y) = \measuredangle(x,0,y)\log(|y|/|x|)$.

Each S_{xy} is divided into two right quasihyperbolic triangles by the geodesic $J_k[x,y]$, see Figure 4.34. The next result gives the quasihyperbolic area of these quasihyperbolic triangles.

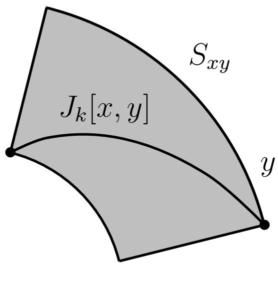

FIGURE 4.34. The set S_{xy} and the geodesic $J_k[x,y]$.

4.35. **Lemma.** *The geodesic $J_k[x,y]$ divides S_{xy} into two quasihyperbolic triangles which both have the quasihyperbolic area*

$$\frac{\measuredangle(x,0,y)}{2}\log\frac{|y|}{|x|}.$$

Proof. We denote $\alpha = \measuredangle(x,0,y)$ and the two quasihyperbolic triangles by T_1 and T_2. We assume that T_2 is closer to the origin. By a simple computation

$$\begin{aligned} m_k(T_1) &= \int_0^\alpha \left(\int_{|x|\exp((\phi/\alpha)\log(|y|/|x|))}^{|y|} \frac{dx}{x} \right) d\phi \\ &= \int_0^\alpha \log\left(\left(\frac{|y|}{|x|}\right)^{1-\phi/\alpha} \right) d\phi \\ &= \frac{\alpha}{2}\log\frac{|y|}{|x|}. \end{aligned} \tag{4.36}$$

The quasihyperbolic area of T_2 is obtained by Lemma 4.33 and (4.36). □

4.37. **Corollary.** *Let $\triangle_k(x,y,z)$ be a quasihyperbolic triangle in $\mathbb{R}^2\setminus\{0\}$ such that $\measuredangle_k(y,x,z) = \pi/2$. Then the quasihyperbolic area of $\triangle_k(x,y,z)$ is*

$$\frac{k(x,y)k(x,z)}{2}.$$

Proof. If either $J_k[x,y]$ or $J_k[x,z]$ is subset of the ray $R(x)$, then the assertion follows by Lemma 4.35. Otherwise we may assume $x,y,z\in\mathbb{H}^2 = \{u\in\mathbb{C}\colon \operatorname{Im} u > 0\}$ and $|y|\le|z|$.

Let us assume $|x| \geq |y|$ and denote $a = z|y|/|z|$, $b = z|x|/|z|$ and $c = y|x|/|y|$ (see Figure 4.38).

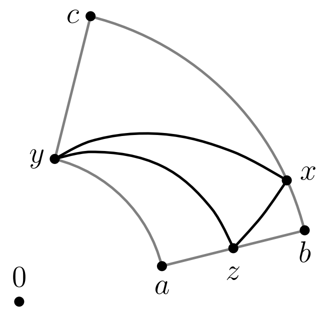

FIGURE 4.38. The points a, b and c in the proof of Corollary 4.37.

By Lemmas 4.33 and 4.35 the quasihyperbolic area of $\triangle_k(x,y,z)$ is

$$\begin{aligned} & k(a,y)k(c,y) - \frac{k(a,y)k(a,z)+k(b,x)k(b,z)+k(c,x)k(c,y)}{2} \\ = & \frac{1}{2}\big(k(a,y)k(c,y)+k(a,y)k(b,z)-k(b,x)k(b,z)-k(c,x)k(c,y)\big) \\ = & \frac{1}{2}\Big(\big(k(b,x)+k(c,x)\big)\big(k(c,y)+k(b,z)\big)-k(b,x)k(b,z)-k(c,x)k(c,y)\Big) \\ = & \frac{1}{2}\big(k(b,x)k(c,y)+k(c,x)k(b,z)\big), \end{aligned} \tag{4.39}$$

because $k(c,y)-k(a,z) = k(b,z)$ and $k(a,y) = k(b,x)+k(c,x)$. By Theorem 4.14 we have

$$\frac{k(x,y)}{k(c,x)} = \frac{k(x,z)}{k(b,z)}, \tag{4.40}$$

$$\frac{k(x,y)}{k(c,y)} = \frac{k(x,z)}{k(b,x)} \tag{4.41}$$

and

$$k(b,x)^2 + k(b,z)^2 = k(x,z)^2. \tag{4.42}$$

By (4.40) and (4.41) the formula (4.39) is equivalent to

$$\frac{1}{2}\left(k(b,x)^2\frac{k(x,y)}{k(x,z)} + k(b,z)^2\frac{k(x,y)}{k(x,z)}\right).$$

By (4.42) the formula (4.39) is equivalent to $k(x,y)k(x,z)/2$ and the assertion follows.

The case $|x| < |z|$ can be considered similarly. □

The next theorem gives Heron's formula for the quasihyperbolic triangles.

4.43. **Theorem.** *Let $\triangle_k(x,y,z)$ be a quasihyperbolic triangle. Then the quasihyperbolic area of $\triangle_k(x,y,z)$ is*

$$\sqrt{s\big(s-k(x,y)\big)\big(s-k(y,z)\big)\big(s-k(z,x)\big)},$$

where $s = \big(k(x,y)+k(y,z)+k(z,x)\big)/2$.

Proof. By Theorem 4.14 the quasihyperbolic triangle $\triangle_k(x,y,z)$ can be separated into two right quasihyperbolic triangles by the height geodesic $J_k[x,u]$, where $u \in J_k[y,z]$ and $\measuredangle_k(x,u,y) = \pi/2$. By the proof of Lemma 4.29 we know that

$$k(y,u) = \frac{k(x,y)^2 + k(y,z)^2 - k(x,z)^2}{2k(y,z)}$$

and

$$k(z,u) = k(y,z) - k(y,u) = \frac{k(x,z)^2 + k(y,z)^2 - k(x,y)^2}{2k(y,z)}.$$

By Corollary 4.37 the quasihyperbolic area of $\triangle_k(x,y,z)$ is

$$\frac{k(x,u)k(y,u)}{2} + \frac{k(x,u)k(z,u)}{2} = \frac{k(x,u)k(y,z)}{2}. \tag{4.44}$$

By Lemma 4.29 and (4.44) the quasihyperbolic area of $\triangle_k(x,y,z)$ is

$$\begin{aligned} &\frac{1}{2}k(y,z)\sqrt{k(x,y)^2 - \frac{\big(k(y,z)^2 + k(x,y)^2 - k(x,z)^2\big)^2}{4k(y,z)^2}} \\ =\ &\frac{1}{4}\sqrt{4k(y,z)^2k(x,y)^2 - \big(k(y,z)^2 + k(x,y)^2 - k(x,z)^2\big)^2} \\ =\ &\frac{1}{4}\sqrt{t\big(t-2k(x,y)\big)\big(t-2k(y,z)\big)\big(t-2k(z,x)\big)}, \end{aligned}$$

where $t = k(x,y)+k(y,z)+k(z,x)$. Since $s = t/2$ the assertion follows. □

Also by the Euclidean Law of Sines the quasihyperbolic area of a quasihyperbolic triangle $\triangle_k(x,y,z)$ is equal to

$$\frac{k(x,y)k(x,z)\sin\alpha_x}{2} = \frac{k(y,x)k(y,z)\sin\alpha_y}{2} = \frac{k(z,x)k(z,y)\sin\alpha_z}{2}. \tag{4.45}$$

4.46. **Remark.** The quasihyperbolic area of a quasihyperbolic triangle defined by vertices is not monotone with respect to the domain. Let us consider the quasihyperbolic triangle with the vertices $x = 0$, $y = -ti$ and $z = -y$ for $t > 0$ in domains $G_1 = \mathbb{R}^2 \setminus \{-1\}$ and $G_2 = \mathbb{R}^2 \setminus \{-1,1\}$. We denote the quasihyperbolic triangle in G_1 by T_1 and in G_2 by T_2. Now T_1 contains an open set whereas T_2 is degenerate, a subset of the line $\{w \in \mathbb{R}^2 \colon \operatorname{Re} w = 0\}$. Therefore, $m_{k_{G_2}}(T_2) = 0$ while $m_{k_{G_1}}(T_1) > 0$. Moreover, $m_{k_{G_1}}(T_1) \to \infty$ as $t \to \infty$.

On the other hand, let us consider the quasihyperbolic triangle with the vertices $x = 0$, $y = -i$, $z = t$ for $t \in (1/2,1)$ and denote the quasihyperbolic triangle in G_1 by T_3 and in G_2 by T_4. Now $m_{k_{G_1}}(T_3) < m_{k_{G_1}}(T_5) < \infty$, where T_5 is the quasihyperbolic triangle with vertices x, y and 1 in G_1. However, $m_{k_{G_2}}(T_4) \to \infty$ as $t \to 1$.

4.47. **Quasihyperbolic area of quasihyperbolic quadrilateral.** We consider next convex quasihyperbolic quadrilaterals.

4.48. **Definition.** A *quasihyperbolic quadrilateral* $xyzu$ is $J_k[x,y] \cup J_k[y,z] \cup J_k[z,u] \cup J_k[u,x]$ for $x,y,z,u \in \mathbb{R}^2 \setminus \{0\}$ and fixed $J_k[x,y]$, $J_k[y,z]$, $J_k[z,u]$ and $J_k[u,x]$, where $\triangle_k(x,y,u)$ and $\triangle_k(y,z,u)$ are quasihyperbolic triangles. If the domain enclosed by the quasihyperbolic quadrilateral $xyzu$ does not contain origin and is convex with respect to the quasihyperbolic metric, then it is a *convex quasihyperbolic quadrilateral.*

The following theorem is Bretschneider's formula for the quasihyperbolic quadrilaterals.

4.49. **Theorem.** *The quasihyperbolic area of a convex quasihyperbolic quadrilateral $xyzu$ is*

$$\sqrt{(p-k_{xy})(p-k_{yz})(p-k_{zu})(p-k_{ux}) - k_{xy}k_{yz}k_{zu}k_{ux}\cos^2\frac{\alpha_x+\alpha_z}{2}},$$

where $k_{ab} = k(a,b)$, $p = (k_{xy}+k_{yz}+k_{zu}+k_{ux})/2$, $\alpha_x = \measuredangle_k(u,x,y)$ and $\alpha_z = \measuredangle_k(y,z,u)$.

Proof. Let us denote the quasihyperbolic area of $xyzu$ by A. Clearly A is equal to the sum of the quasihyperbolic areas of $\triangle_k(x,y,z)$ and $\triangle_k(x,y,u)$. By (4.45)

$$A = \frac{k_{xy}k_{xu}\sin\alpha_x}{2} + \frac{k_{zy}k_{zu}\sin\alpha_z}{2}$$

which is equivalent to

$$4A^2 = k_{xy}^2k_{xu}^2\sin^2\alpha_x + k_{zy}^2k_{zu}^2\sin^2\alpha_z + 2k_{xy}k_{yz}k_{zu}k_{ux}\sin\alpha_x\sin\alpha_z. \tag{4.50}$$

By the Law of Cosines

$$k_{xy}^2 + k_{xu}^2 - 2k_{xy}k_{xu}\cos\alpha_x = k_{zy}^2 + k_{zu}^2 - 2k_{zy}k_{zu}\cos\alpha_z$$

which is equivalent to

$$\frac{(k_{xy}^2+k_{xu}^2-k_{zx}^2-k_{zu}^2)^2}{4} = (k_{xy}k_{xu}\cos\alpha_x - k_{zy}k_{zu}\cos\alpha_z)^2. \tag{4.51}$$

Now (4.50) and (4.51) together give us

$$4A^2 + \frac{(k_{xy}^2+k_{xu}^2-k_{zx}^2-k_{zu}^2)^2}{4} = k_{xy}^2k_{xu}^2 + k_{zy}^2k_{zu}^2 + 2k_{xy}k_{yz}k_{zu}k_{ux}\cos(\alpha_x+\alpha_z)$$

which is equivalent to

$$4A^2 = 4(p-k_{xy})(p-k_{yz})(p-k_{zu})(p-k_{ux}) - 4k_{xy}k_{yz}k_{zu}k_{ux}\cos^2\frac{\alpha_x+\alpha_z}{2}$$

and the assertion follows. □

We also introduce another formula for the quasihyperbolic area of quasihyperbolic quadrilaterals.

4.52. **Theorem.** *The quasihyperbolic area of a convex quasihyperbolic quadrilateral* $xyzu$ *is*

$$\frac{k(x,z)k(y,u)\sin\theta}{2},$$

where θ *is the angle between the geodesics* $J_k[x,z]$ *and* $J_k[y,u]$.

Proof. Let us denote the quasihyperbolic area of $xyzu$ by A, $k_{ab} = k(a,b)$ and the intersection point of the geodesics $J_k[x,z]$ and $J_k[y,u]$ by v. Since $\sin\alpha = \sin(\pi-\alpha)$, we have by (4.45)

$$\begin{aligned} A &= \frac{k_{xv}k_{yv}\sin\theta}{2} + \frac{k_{yv}k_{zv}\sin\theta}{2} + \frac{k_{zv}k_{uv}\sin\theta}{2} + \frac{k_{uv}k_{xv}\sin\theta}{2} \\ &= \sin\theta\frac{\big(k(x,v)+k(z,v)\big)\big(k(y,v)+k(u,v)\big)}{2} \end{aligned}$$

and the assertion follows. □

4.53. **Remark.** We could also consider quasihyperbolic polygons in $\mathbb{R}^2 \setminus \{0\}$ by defining a quasihyperbolic polygon to be the union of given geodesics $J_0, \ldots, J_m$ such that $\cup_{p=0}^m J_p$ is a simple and closed curve in a slit plane and for $p, r \in \{0, \ldots m\}$, $p \neq r$, $J_p \cap J_r$ is a singleton set if $p = r \pm 1 (\mathrm{mod}\ m+1)$ and $\emptyset$ otherwise. The quasihyperbolic area of the quasihyperbolic polygons can be obtained as the Euclidean area of the Euclidean polygons.

4.54. **Quasihyperbolic perimeter and quasihyperbolic area of quasihyperbolic disk.** We determine the quasihyperbolic perimeter and the quasihyperbolic area of quasihyperbolic disk $B_k(x,M)$. The next lemma shows that the quasihyperbolic perimeter of the quasihyperbolic disk is equal to the Euclidean perimeter of the Euclidean disk for small radii.

4.55. **Lemma.** *For* $r \in (0,\pi]$ *the quasihyperbolic perimeter of* $B_k(x,r)$ *is equal to* $2\pi r$.

Proof. We may assume $x = 1$. Let us first consider regular quasihyperbolic n-gon P_n. By Theorem 4.14 the perimeter of P_n is equal to

$$2rn\sin\frac{\pi}{n}$$

and by letting n approach to infinity the assertion follows. □

We consider next the quasihyperbolic area of the quasihyperbolic disk for small radii.

4.56. **Lemma.** *For* $r \in (0,\pi]$ *we have* $m_k\left(B_k(x,r)\right) = \pi r^2$.

Proof. We may assume $x = 1$. Let us denote $\phi = \measuredangle(1, 0, z)$ for $z = (s, \phi) \in \mathbb{R}^2 \setminus \{0\}$. By (2.3)

$$\begin{aligned} m_k\left(B_k(1,r)\right) &= \int_{-r}^{r}\int_{e^{-\sqrt{r^2-\phi^2}}}^{e^{\sqrt{r^2-\phi^2}}} \frac{1}{s} ds d\phi \\ &= 2\int_{-r}^{r}\sqrt{r^2-\phi^2}d\phi \\ &= (r\sqrt{r^2-r^2}+r^2\arctan\frac{r}{\sqrt{r^2-r^2}}) \\ &\quad -(-r\sqrt{r^2-r^2}+r^2\arctan\frac{-r}{\sqrt{r^2-r^2}}) \\ &= \pi r^2. \end{aligned}$$

□

The next theorem gives the quasihyperbolic area of the quasihyperbolic disk for large radii.

4.57. **Theorem.** *For $r > \pi$ the quasihyperbolic area of $B_k(x, r)$ is equal to*

$$2\pi\sqrt{r^2-\pi^2}+2\pi^2\arctan\frac{\pi}{\sqrt{r^2-\pi^2}}.$$

Proof. We may assume $x = 1$. Let us denote $\phi = \measuredangle(1, 0, z)$ for $z = (s, \phi) \in \mathbb{R}^2 \setminus \{0\}$. By (2.3)

$$\begin{aligned} m_k\left(B_k(1,r)\right) &= \int_{-\pi}^{\pi}\int_{a}^{b} \frac{1}{s} ds d\phi \\ &= 2\int_{-\pi}^{\pi}\sqrt{r^2-\phi^2}d\phi \\ &= (\pi\sqrt{r^2-\pi^2}+\pi^2\arctan\frac{\pi}{\sqrt{r^2-\pi^2}}) \\ &\quad -(-\pi\sqrt{r^2-\pi^2}+\pi^2\arctan\frac{-\pi}{\sqrt{r^2-\pi^2}}) \\ &= 2\pi\sqrt{r^2-\pi^2}+2\pi^2\arctan\frac{\pi}{\sqrt{r^2-\pi^2}}, \end{aligned}$$

where $a = e^{-\sqrt{r^2-\phi^2}}$ and $b = e^{\sqrt{r^2-\phi^2}}$. □

4.58. **Corollary.** *For $x \in \mathbb{R}^2 \setminus \{0\}$ and $r > 0$*

$$m_k\left(B_k(x,r)\right) < 4\pi r$$

and for $r > \pi$

$$m_k\left(B_k(x,r)\right) > 2\pi\sqrt{r^2-\pi^2}.$$

Proof. The assertion follows from Lemma 4.56 and Theorem 4.57, because for $r > \pi$

$$2\pi\sqrt{r^2-\pi^2} + 2\pi^2 \arctan \frac{\pi}{\sqrt{r^2-\pi^2}} \le 2\pi\sqrt{r^2-\pi^2} + \pi^3 < 4\pi r.$$

□

4.59. **Euclidean model of quasihyperbolic metric.** We show that the metric space $(\mathbb{R}^2 \setminus \{0\}, k)$ is equivalent to the metric space (C_2, m), where $C_2 = \{x \in \mathbb{R}^3 \colon d(x, \mathbb{R}\cdot e_3) = 1\}$ and m is a metric in C_2. We define a mapping $\lambda\colon \mathbb{R}^2 \setminus \{0\} \to C_2$ by

$$\lambda(x) = \big(\sin(\arg x), \cos(\arg x), \log|x|\big) \tag{4.60}$$

and the metric m for $x, y \in C_2$ by

$$m(x,y) = d\big(\lambda(x), \lambda(y)\big),$$

where d is the Euclidean distance on the surface C_2.

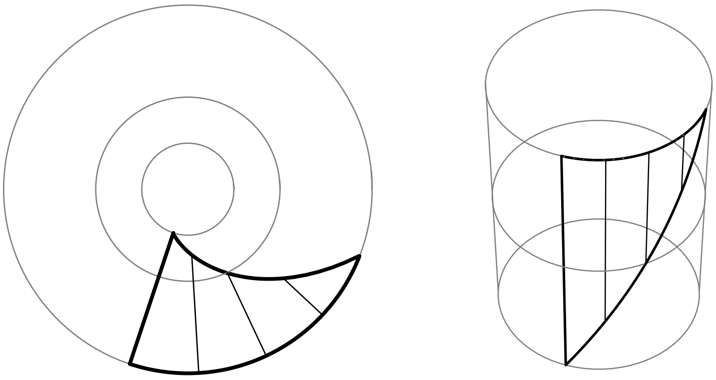

FIGURE 4.61. An example of a geodesic trigon in the metric spaces $(\mathbb{R}^2 \setminus \{0\}, k)$ and (C_2, m).

By a simple computation we obtain

$$m(x,y) = \sqrt{\alpha^2 + \log^2 \frac{|x|}{|y|}},$$

where $\alpha = |\arg x - \arg y| \in [0, \pi)$. By (2.3) and (4.60) a geodesic in the metric $(\mathbb{R}^2 \setminus \{0\}, k)$ is a subset of a helix.

We may consider the surface C_2 as an infinite cylinder and then a geodesic in the metric $(\mathbb{R}^2 \setminus \{0\}, k)$ is a line segment in the infinite cylinder, see Figure 4.61.

4.62. **Open problems.** We pose some open problems concerning the quasihyperbolic trigonometry.

4.63. **Open problem.** Let $G \subset \mathbb{R}^n$ be a domain and $m_G \in \{\rho_G, k_G\}$, where ρ_G is the hyperbolic metric whenever it is defined (see [4]). Does the inequality

$$m_G(x,y)^2 \geq m_G(x,z)^2 + m_G(y,z)^2 - 2m_G(y,z)m_G(z,x)\cos\measuredangle_k(y,z,x)$$

hold for all domains $G \subsetneq \mathbb{R}^n$ and distinct points $x,y,z \in G$?

4.64. **Open problem.** Let $G \subsetneq \mathbb{R}^2 \setminus \{0\}$, $\triangle_1$ be a quasihyperbolic triangle in $\mathbb{R}^2 \setminus \{0\}$ with side lengths a, b and c and $\triangle_2$ be a quasihyperbolic triangle in G with the same side lengths. Can we use Theorem 4.43 to estimate the quasihyperbolic area of $\triangle_2$?

5. Estimations of the quasihyperbolic distance

The explicit formula for the quasihyperbolic distance is known only in a few special domains. In this section we estimate the quasihyperbolic length and distance in some special cases, where the explicit formula is not known.

5.1. **Quasihyperbolic distance in twice punctured plane.** We consider the quasihyperbolic metric k in twice punctured plane $\mathbb{R}^2_{-1,1} = \mathbb{R}^2 \setminus \{-1,1\}$. We find a lower bound for $\ell_k(\gamma)$ in terms of $d(\gamma,\{-1,1\})$, where $\gamma \subset \mathbb{R}^2_{-1,1}$ is a closed rectifiable curve that encloses $\{-1,1\}$.

5.2. **Lemma.** *Let $x \in \mathbb{R}^2_{-1,1}$ with $\operatorname{Re} x < 0$, $F = \{z \in \mathbb{R}^2 \colon \operatorname{Re} z = 0\}$ and $E = \{z \in \mathbb{R}^2 \colon \operatorname{Im} z = 0,\ \operatorname{Re} z < -1\}$. Then*

$$k(E,x) + k(x,F) \geq k(y,F),$$

where $y = -1 - d(x)$.

Proof. By (2.3), $k(E,x) = k(y,x)$ and by the triangle inequality

$$k(y,F) \leq k(y,x) + k(x,F) = k(E,x) + k(x,F)$$

and the assertion follows. □

Let E be as in Lemma 5.2, fix a closed rectifiable curve $\gamma \subset \mathbb{R}^2_{-1,1}$ such that it encloses $\{-1,1\}$ and denote $x = 1 + d(\gamma,\{-1,1\})$. By (2.3) the function

$$\begin{aligned} s(h) &= k(x,hi) + k(hi,E) \\ &= \sqrt{(\pi - \arctan h)^2 + \log^2 \frac{\sqrt{1+h^2}}{d(\gamma,\{-1,1\})}} + \pi - \arctan h, \end{aligned} \tag{5.3}$$

$h \in [0,\infty)$, gives us

$$k(x,E) = \min_{h\geq 0} s(h). \tag{5.4}$$

We use the function $s(h)$ and (5.4) to estimate the quasihyperbolic length of the curve γ. First we obtain lower bounds for $\ell_k(\gamma)$. The lower bound of the following lemma gives a good estimate for $\ell_k(\gamma)$, when $d(\gamma,\{-1,1\})$ is close to 0.

5.5. **Lemma.** *Let $\gamma \subset \mathbb{R}^2_{-1,1}$ be a closed rectifiable curve that encloses $\{-1,1\}$ and denote $d = d(\gamma, \{-1,1\})$. Then*

$$\ell_k(\gamma) \geq \frac{3\pi}{2} + 2\log\frac{\sqrt{2}}{d}.$$

Proof. By (5.4)

$$\ell_k(\gamma) \geq 2k(1+d, E) \geq \min_{h\geq 0} 2g(h)$$

for the function

$$g(h) = \log\frac{\sqrt{1+h^2}}{d} + \pi - \arctan h, \quad h \geq 0.$$

Since $g'(h) = 2(h-1)/(1+h^2) = 0$ is equivalent to $h = 1$, we have

$$g(h) \geq g(1) = \frac{3\pi}{4} + \log\frac{\sqrt{2}}{d}$$

and the assertion follows. □

We define the function $w\colon (0,\infty) \to (0,\infty)$ by

$$w(h) = \sqrt{h^2+1}\exp\left(\frac{\arctan h - \pi}{h}\right). \tag{5.6}$$

Since $w'(h) = w(h)(h+\pi-\arctan h)/h^2 > 0$, the function $w(h)$ is a homeomorphism.

The following two results give a good lower bound for $\ell_k(\gamma)$, when $d(\gamma, \{-1,1\})$ is large.

5.7. **Theorem.** *Let $\gamma \subset \mathbb{R}^2_{-1,1}$ be a closed rectifiable curve enclosing $\{-1,1\}$. Then*

$$\ell_k(\gamma) \geq (\pi - \arctan h)\sqrt{1+\frac{1}{h^2}} + \frac{3\pi}{2},$$

where h is such that

$$d(\gamma, \{-1,1\}) = w(h) \tag{5.8}$$

and $w(h)$ is defined by (5.6).

Proof. Let us denote $F = \{z \in \mathbb{R}^2 \colon \operatorname{Re} z = 0\}$ and $E = \{z \in \mathbb{R}^2 \colon \operatorname{Im} z = 0, \operatorname{Re} z < -1\}$. By the symmetry we may assume that γ is closest to $\{-1,1\}$ in the second quadrant. By trivial estimate we know that in each quadrant the quasihyperbolic length of γ is at least $\pi/2$. Therefore we have

$$\ell_k(\gamma) \geq \inf k(x,y) + 3\pi/2,$$

where $x \in E$, $y \in F$ and $|x+1|, |y+1| \geq d(\gamma, \{-1,1\})$.

By Lemma 5.2 it is sufficient to minimize $k(x,y)$ for $x \in E$, $d(x) = d(\gamma, \{-1,1\})$ and $y = hi$. We denote $f(h) = k(x,y)$ and $d = d(\gamma, \{-1,1\})$. By (2.3)

$$f(h) = \sqrt{(\pi - \arctan h)^2 + \log^2\frac{\sqrt{h^2+1}}{d}}.$$

It is easy to verify that $f'(h) = 0$ is equivalent to (5.8) and therefore $f(h) \geq (\pi - \arctan h)\sqrt{1+h^{-2}}$. $\square$

5.9. **Corollary.** *Define*

$$t_1(d) = (\pi - \arctan(de^{\pi}))\sqrt{1+\frac{e^{-2\pi}}{d^2}}, \quad \textit{for } d > 0,$$

and

$$t_2(d) = \left(\pi - \arctan\frac{\pi}{\log(1/d)}\right)\sqrt{1+\left(\frac{\log(1/d)}{\pi}\right)^2}, \quad \textit{for } d \in (0,1).$$

Moreover, define a decreasing function $t\colon (0,\infty) \to (\pi/2,\infty)$ *by*

$$t(d) = \begin{cases} t_1(d), & d \geq d_0, \\ t_2(d), & d \in (0,d_0), \end{cases}$$

with $d_0 = \sqrt{2}e^{-3\pi/4} \in (0,1)$. *Then for a closed rectifiable curve* $\gamma \subset \mathbb{R}^2_{-1,1}$ *that encloses* $\{-1,1\}$, $d = d(\gamma,\{-1,1\})$, *we have*

$$\ell_k(\gamma) \geq t(d) + \frac{3\pi}{2}.$$

Proof. By Theorem 5.7, $\ell_k(\gamma) \geq g(h) + 3\pi/2$, where $g(h) = (\pi - \arctan h)\sqrt{1+h^{-2}}$, and

$$d = \sqrt{h^2+1}e^{(-\pi+\arctan h)/h}. \tag{5.10}$$

The function $g(h)$ is decreasing on $(0,\infty)$, because

$$g'(h) = -\frac{\pi + h - \arctan h}{h^2\sqrt{1+h^2}} \leq 0.$$

Let us assume $h \geq 1$, which is equivalent to $d \geq d_0$. By (5.10)

$$d \geq he^{(-\pi+\arctan h)/h} \geq he^{-\pi/h} \geq he^{-\pi}$$

and hence $h \leq e^{\pi}d$. Therefore

$$g(h) \geq g(e^{\pi}d) = t_1(d).$$

Let us next assume $h \in (0,1)$, which is equivalent to $d \in (0,d_0)$. Now by (5.10)

$$d = \sqrt{h^2+1}e^{(-\pi+\arctan h)/h} \geq e^{-\pi/h}$$

and hence $h \leq \pi/\log d^{-1}$. Therefore, we have

$$g(h) \geq g(\pi/\log d^{-1}) = t_2(d).$$

By (5.10), $h = 1$ is equivalent to $d = \sqrt{2}e^{-3\pi/4}$ and therefore $\ell_k(\gamma) \geq t(d) + 3\pi/2$. The assertion follows since $t_2(d_0) > 2 > t_1(d_0)$ and $t(d)$ is clearly a decreasing function. $\square$

For $d > 0$ we define

$$p(d) = \inf \ell_k(\gamma),$$

where the infimum is taken over all closed rectifiable curves γ that enclose $\{-1,1\}$ and satisfy $d(\gamma,\{-1,1\}) = d$. Next we find upper bounds for $p(d)$.

Let γ_0 be the closed curve consisting of the left half of $S^1(-1,d)$, line segments $[-1+di, 1+di]$ and $[-1-di, 1-di]$ and right half of $S^1(1,d)$. Clearly $p(d) \le \ell_k(\gamma_0)$, because γ_0 encloses $\{-1,1\}$ and $d(\gamma_0) = d$. By a straightforward computation we obtain $\ell_k(\gamma_0) = 2\pi + 4\log((1+\sqrt{1+d^2})/d)$ and therefore

$$p(d) \le 2\pi + 4\log\frac{1+\sqrt{1+d^2}}{d}. \tag{5.11}$$

We improve the upper bound of (5.11) in the following lemma.

5.12. **Lemma.** *For $d > 0$*

$$p(d) \le 2s(d+1)$$

and

$$p(d) \le \frac{3\pi}{2} + 2\sqrt{\frac{9\pi^2}{16} + \log^2\frac{\sqrt{2}}{d}} = 2s(1),$$

where $s(h)$ is the function defined in (5.3).

Proof. By (5.4)

$$p(d) \le 2\left(\min_{h\ge 0} s(h)\right)$$

for the function $s(h)$. Since $\min_{h\ge 0} s(h) \le s(1)$ and $\min_{h\ge 0} s(h) \le s(1+d)$, the assertion follows. □

Combining the results of Lemma 5.5, Corollary 5.9 and Lemma 5.12 gives us

$$\frac{3\pi}{2} + \max\{l_1(d), l_2(d)\} \le p(d) \le 2\pi + u(d) \tag{5.13}$$

for

$$\begin{aligned} u(d) &= 2s(d+1), \\ l_1(d) &= 2\log\frac{\sqrt{2}}{d}, \\ l_2(d) &= \sqrt{1+\frac{e^{-2\pi}}{d^2}}(\pi - \arctan(de^{\pi})). \end{aligned}$$

Note that $u(d) \to 0$ and $\max\{l_1(d), l_2(d)\} \to \pi/2$ as $d \to \infty$. By Lemma 5.5 and Lemma 5.12 we have

$$\frac{3\pi}{2} + 2\log\frac{\sqrt{2}}{d} \le p(d) \le \frac{3\pi}{2} + 2\left(\log\frac{\sqrt{2}}{d}\right)v(d) \tag{5.14}$$

for

$$v(d) = \sqrt{\frac{9\pi^2}{16(\log(\sqrt{2}/d))^2} + 1}.$$

In particular, $v(d) \geq 1$ and $v(d) \to 1$ as $d \to 0$. The functions of (5.13) and (5.14) are illustrated in Figure 5.15.

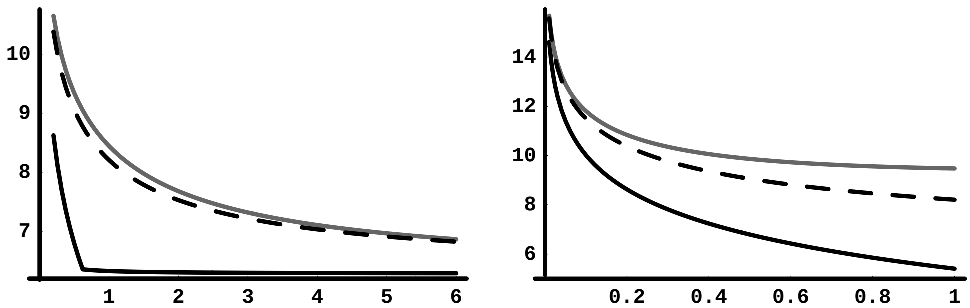

FIGURE 5.15. *Left:* Functions $u(d)$, $p(d)$ (dashed line) and $\max\{l_1(d), l_2(d)\}$ of (5.13). *Right:* Upper and lower bounds of (5.14).

5.16. **Corollary.** *We have* $p(d) - 2\log(1/d) \to 3\pi/2 + \log 2$ *as* $d \to 0$.

Proof. By (5.14)

$$\frac{3\pi}{2} \leq p(d) - u(d) \leq \frac{3\pi}{2} + \sqrt{\frac{9\pi^2}{4} + u(d)^2} - u(d)$$

for the function $u(d) = 2\log(\sqrt{2}/d)$, $d > 0$. Since $u(d) \to \infty$ as $d \to 0$, we have

$$\sqrt{\frac{9\pi^2}{4} + u(d)^2} - u(d) \to 0$$

as $d \to 0$ and the assertion follows. $\square$

5.17. **Quasihyperbolic distance in annulus.** We estimate the quasihyperbolic metric and the distance ratio metric in a Euclidean ball $B^n(r)$, its complement $\mathbb{R}^n \setminus \overline{B^n}(r)$ and in an annulus $B^n(R) \setminus \overline{B^n}(r)$ for $0 < r < R$. Since the quasihyperbolic geodesics in these domains are not known, we estimate the quasihyperbolic distance by using the logarithmic spirals. Before stating our main result we introduce a notation for the Euclidean annulus. For $0 < r < R < \infty$ we denote

$$A(r, R) = \{z \in \mathbb{R}^n \colon r < |z| < R\}.$$

5.18. **Theorem.** *Let* $m \in \{k, j\}$, $R > 1$ *and* $G = A(1/R, R)$. *Then there exists a constant* $c(R) \geq 1$ *such that* $c(R) \to 1$ *as* $R \to \infty$ *and*

$$m_G(x, y) \leq c(R) m_{\mathbb{R}^n \setminus \{0\}}(x, y)$$

for all $x, y \in A(1/\sqrt{R}, \sqrt{R})$.

5.19. **Lemma.** *For* $r>0$*,* $G=\mathbb{R}^n\setminus\overline{B^n}(r)$ *and* $x,y\in G$

$$k_G(x,y)\le k_{\mathbb{R}^n\setminus\{0\}}(x,y)l(x,y),$$

where

$$l(x,y)=\begin{cases}\dfrac{\log(|y|-r)-\log(|x|-r)}{\log|y|-\log|x|}, & |x|\neq|y|,\\[2mm] \dfrac{|x|}{|x|-r}, & |x|=|y|.\end{cases}$$

Proof. Let us first assume $|x|<|y|$. Denote $\alpha=\measuredangle(x,0,y)$, $b=(1/\alpha)\log(|y|/|x|)$ and $a=|x|$. Now $(r(\varphi),\varphi)$ for $r(\varphi)=ae^{b\varphi}$ defines the polar coordinates of the logarithmic spiral from x to y as φ goes from 0 to α. Therefore

$$\begin{aligned} k_G(x,y) &\le \int_0^\alpha \frac{\sqrt{r(\varphi)^2+r'(\varphi)^2}}{r(\varphi)-r}d\varphi\\ &= \sqrt{1+b^2}\int_0^\alpha \frac{r(\varphi)}{r(\varphi)-r}d\varphi\\ &= \frac{\sqrt{1+b^2}}{b}\log\frac{|y|-r}{|x|-r}\\ &= k_{\mathbb{R}^n\setminus\{0\}}(x,y)\frac{\log\frac{|y|-r}{|x|-r}}{\log\frac{|y|}{|x|}}. \end{aligned}$$

If $|x|=|y|$, then we consider the shortest circular arc $\gamma\in S^{n-1}(0,|x|)$ that connects x and y. Now

$$k_G(x,y)\le\int_\gamma\frac{|dz|}{d(z)}=\int_0^\alpha\frac{|x|}{|x|-r}d\varphi=\alpha\frac{|x|}{|x|-r}=k_{\mathbb{R}^n\setminus\{0\}}(x,y)\frac{|x|}{|x|-r},$$

where $\alpha=\measuredangle(x,0,y)$, and the assertion follows. □

5.20. **Theorem.** *For* $r>0$*,* $G=\mathbb{R}^n\setminus\overline{B^n}(r)$ *and* $x,y\in G$

$$k_G(x,y)\le\frac{\min\{|x|,|y|\}}{\min\{|x|,|y|\}-r}k_{\mathbb{R}^n\setminus\{0\}}(x,y).$$

Proof. By Lemma 5.19 we need to show that for all $b\ge a>r$

$$\frac{\log\frac{b-r}{a-r}}{\log(b/a)}\le\frac{a}{a-r},$$

which is equivalent to $g(r)\le g(0)$ for

$$g(r)=(a-r)\log\frac{b-r}{a-r}.$$

By a simple computation we obtain

$$g'(r)=\frac{b-a}{b-r}+\log\frac{b-r}{a-r}\quad\text{and}\quad g''(r)=-\frac{(a-b)^2}{(a-r)(b-r)^2}.$$

Therefore, $g''(r) < 0$ and $g'(r) < g'(0) = 1 - a/b - \log(b/a) \le 0$, where the last inequality holds, because $f(c) = 1 - 1/c - \log c$ is a decreasing function for $c \ge 1$ and $f(0) = 0$. Since $g'(r) \le 0$, the function $g(r)$ is decreasing and $g(r) \le g(0)$. □

Similarly as in the proof of Theorem 5.20 we can obtain that for $R > 0$ and $x, y \in B^n(R)$ with $|x|, |y| \ge R/2$

$$k_{B^n(R)}(x,y) \le \frac{\max\{|x|,|y|\}}{R - \max\{|x|,|y|\}} k_{\mathbb{R}^n \setminus \{0\}}(x,y). \tag{5.21}$$

5.22. **Theorem.** *Let $0 < r < R < \infty$. Then for $x, y \in A(r,R)$*

$$k_{A(r,R)}(x,y) \le \begin{cases} m k_{\mathbb{R}^n\setminus\{0\}}(x,y), & \text{if } |x|,|y| \le (r+R)/2, \\ M k_{\mathbb{R}^n\setminus\{0\}}(x,y), & \text{if } |x|,|y| > (r+R)/2, \\ m k_{\mathbb{R}^n\setminus\{0\}}(x,z) + M k_{\mathbb{R}^n\setminus\{0\}}(z,y), & \text{otherwise,} \end{cases}$$

where $m = \max\{|x|/(|x|-r), |y|/(|y|-r)\}$, $M = \max\{|x|/(R-|x|), |y|/(R-|y|)\}$ and where z is the point of intersection of the logarithmic spiral from x to y and $S^{n-1}((r+R)/2)$.

Proof. The assertion follows from Theorem 5.20, (5.21) and the following fact

$$k_{A(r,R)}(x,y) \le \begin{cases} k_{\mathbb{R}^n\setminus B^n(r)}(x,y), & \text{if } |x|,|y| \le (r+R)/2, \\ k_{B^n(R)}(x,y), & \text{if } |x|,|y| > (r+R)/2, \\ k_{\mathbb{R}^n\setminus B^n(r)}(x,z) + k_{B^n(R)}(z,y), & \text{otherwise.} \end{cases}$$

□

Note that by Theorem 5.22 in an annulus the quasihyperbolic distance between two points, that are far away from the boundary, is roughly at most the quasihyperbolic distance in punctured space. Moreover, for any domain $G \subset \mathbb{R}^n$ with $A(r,R) \subset G$ we have $k_G \le k_{A(r,R)}$ and therefore we can use the upper bound of Theorem 5.22 also for the quasihyperbolic distance in G. On the other hand, $G \subset \mathbb{R}^n \setminus \{0\}$ implies $k_{\mathbb{R}^n\setminus\{0\}} \le k_G$.

5.23. **Corollary.** *Let $R > 1$ and $G = A(1/R, R)$. Then there exists a constant $c(R) \ge 1$ such that $c(R) \to 1$ as $R \to \infty$ and*

$$k_G(x,y) \le c(R) k_{\mathbb{R}^n\setminus\{0\}}(x,y)$$

for all $x, y \in A(1/\sqrt{R}, \sqrt{R})$.

Proof. Let us denote $r = 1/R$, $m = \sqrt{r}/(\sqrt{r} - r)$ and $M = \sqrt{R}/(R - \sqrt{R})$. Now $m \ge \max\{|x|/(|x|-r), |y|/(|y|-r)\}$ and $M \ge \max\{|x|/(R-|x|), |y|/(R-|y|)\}$. If $|x|, |y| \le (r+R)/2$ then by Theorem 5.22

$$k_G(x,y) \le m k_{\mathbb{R}^n\setminus\{0\}}(x,y) \le (m+M) k_{\mathbb{R}^n\setminus\{0\}}(x,y).$$

If $|x|, |y| > (r+R)/2$ then by Theorem 5.22

$$k_G(x,y) \le M k_{\mathbb{R}^n\setminus\{0\}}(x,y) \le (m+M) k_{\mathbb{R}^n\setminus\{0\}}(x,y).$$

If $|x| \le (r+R)/2$ and $|y| > (r+R)/2$ then by Theorem 5.22

$$\begin{aligned} k_G(x,y) &\le m k_{\mathbb{R}^n\setminus\{0\}}(x,z) + M k_{\mathbb{R}^n\setminus\{0\}}(z,y) \\ &\le (m+M)(k_{\mathbb{R}^n\setminus\{0\}}(x,z) + k_{\mathbb{R}^n\setminus\{0\}}(z,y)) \\ &= (m+M)(k_{\mathbb{R}^n\setminus\{0\}}(x,y)). \end{aligned}$$

Because $k_G(x,y) \le (m+M) k_{\mathbb{R}^n\setminus\{0\}}(x,y)$ for all $x,y \in A(1/\sqrt{R},\sqrt{R})$, we may choose

$$c(R) = m + M = 1 + \frac{2}{\sqrt{R}-1}.$$

Now $c(R) \ge 1$, because $R > 1$ and $c(R) \to 1$ as $R \to \infty$. □

The following theorem shows that Corollary 5.23 holds also for the distance ratio metric.

5.24. **Theorem.** *Let $R > 1$ and $G = A(1/R, R)$. Then there exists a constant $c(R) \ge 1$ such that $c(R) \to 1$ as $R \to \infty$ and*

$$j_G(x,y) \le c(R) j_{\mathbb{R}^n\setminus\{0\}}(x,y)$$

for all $x,y \in A(1/\sqrt{R},\sqrt{R})$.

Proof. Since the $x = y$ is clearly true, we assume $x \ne y$. Let us first assume $R \in (1,4]$. Since $|x| \le \sqrt{R}$ and $d_G(x) \ge 1/\sqrt{R} - 1/R$, we have

$$\frac{j_G(x,y)}{j_{\mathbb{R}^n\setminus\{0\}}(x,y)} \le \frac{\log\left(1 + \dfrac{|x-y|}{1/\sqrt{R} - 1/R}\right)}{\log\left(1 + \dfrac{|x-y|}{\sqrt{R}}\right)} = \frac{\log\left(1 + \dfrac{R}{\sqrt{R}-1}|x-y|\right)}{\log\left(1 + \dfrac{1}{\sqrt{R}}|x-y|\right)} \le \frac{R^{3/2}}{\sqrt{R}-1},$$

because $R/(\sqrt{R}-1) > 1/\sqrt{R}$ and $\log(1+az)/\log(1+bz) \in (1, a/b]$ for $a > b > 0$ and $z > 0$.

We may assume $|x| \le |y|$ and denote $r = 1/R$. If $R > 4$ then $R - \sqrt{R} > \sqrt{R}$ and therefore $d(x, \partial G) = |x| - r$ for all $x \in A(\sqrt{r}, \sqrt{R})$. Now

$$\begin{aligned} \frac{j_G(x,y)}{j_{\mathbb{R}^n\setminus\{0\}}(x,y)} &= \frac{\log\left(1 + \dfrac{|x-y|}{|x|-r}\right)}{\log\left(1 + \dfrac{|x-y|}{|x|}\right)} = \frac{\log\left(\dfrac{|x-y| + |x| - r}{|x|-r}\right)}{\log\left(\dfrac{|x-y|+|x|}{|x|}\right)} \\ &\le \frac{|x|}{|x|-r} = 1 + \frac{r}{|x|-r} \\ &\le 1 + \frac{r}{1/\sqrt{r} - r} = 1 + \frac{1}{R^{3/2}-1} \end{aligned}$$

because $\log((z+c)/c)/\log((z+d)/d) \in (1, d/c]$ for $0 < c < d$ and $z > 0$. Now we can choose

$$c(R) = \begin{cases} \dfrac{R^{3/2}}{\sqrt{R}-1}, & R \in (1,4], \\ 1 + \dfrac{1}{R^{3/2}-1}, & R > 4 \end{cases}$$

and since $c(R) \to 1$ as $R \to \infty$, the assertion follows. □

Proof of Theorem 5.18. The assertion follows from Corollary 5.23 and Theorem 5.24. □

5.25. **Remark.** By Corollary 5.23 and Theorem 5.24 one could ask the following question: for $R > 1$ and $G = A(1/R, R)$, does there exists a constant $c(R) \geq 1$ such that $c(R) \to 1$ as $R \to \infty$ and

$$k_G(x,y) \leq c(R) j_{\mathbb{R}^n \setminus \{0\}}(x,y) \tag{5.26}$$

for all $x, y \in A(1/\sqrt{R}, \sqrt{R})$. We show that such $c(R)$ does not exist.

By (3.4) and Theorem 5.24

$$k_G(x,y) \leq \frac{\pi}{\log 3} j_G(x,y) \leq \frac{\pi}{\log 3} c'(R) j_{\mathbb{R}^n \setminus \{0\}}(x,y)$$

where $c'(R) \to 1$ as $R \to \infty$. Now by choosing $c(R) = c'(R)\pi/\log 3$ the condition (5.26) is satisfied but $c(R) \to \pi/\log 3$ as $R \to \infty$. However, by choosing $x \in A(1/\sqrt{R}, \sqrt{R})$ and $y = -x \in A(1/\sqrt{R}, \sqrt{R})$ we have

$$\frac{k_G(x,y)}{j_{\mathbb{R}^n \setminus \{0\}}(x,y)} = \frac{\pi}{\log 3}$$

and this is true for all $R > 1$.

5.27. **Open problems.** In item 5.1 we considered the quasihyperbolic length of a closed curve in punctured plane. We can also consider a similar problem in a finitely punctured plane.

5.28. **Open problem.** Let $z_1, \ldots, z_m \in \mathbb{R}^2$ and γ be a simple and closed curve that encloses the points $z_1, \ldots, z_m$. Find a lower bound for $\ell_{k_G}(\gamma)$, where $G = \mathbb{R}^2 \setminus \{z_1, \ldots, z_m\}$.

6. Properties of metric balls

In this section we consider some properties of metric balls defined by (2.1). The study of convexity of metric balls was motivated by the following open problem posed by M. Vuorinen [31, 8.2] in 2006:

6.1. **Open problem.** Does there exist a constant $r_0 > 0$ such that the metric ball $B_m(x,r)$ is convex in the Euclidean geometry for $m \in \{q,k,j\}$ and for all $r \in (0, r_0)$?

Instead of convexity in 6.1 we are also interested in strict convexity, starlikeness and strict starlikeness of metric balls.

For the spherical metric 6.1 is very simple. It is easy to verify that spherical balls $B_q(x,r)$ are strictly convex (and hence strictly starlike) whenever $r \in (0,(1+|x|^2)^{-1/2})$ or $r > 1$.

6.2. **Convexity properties.** Open problem 6.1 for the distance ratio metric has been solved in [15]. We showed that $B_j(x,r)$ is (strictly) convex whenever $r \in (0,\log 2]$ ($r \in (0,\log 2)$) and strictly starlike with respect to x whenever $r \in (0,\log(1+\sqrt{2})]$. We also showed that $B_j(x,r)$ is always convex in convex domains and always starlike with respect to x in domains that are starlike with respect to x.

Open problem 6.1 for the quasihyperbolic metric was first considered by O. Martio and J. Väisälä [21]. They showed that in convex domains quasihyperbolic balls B_k are always convex. Later the convexity of the quasihyperbolic balls with small radii was considered in punctured space in [16]. We showed that in punctured space quasihyperbolic balls are strictly convex whenever the radius is less than or equal to one. We also showed that the quasihyperbolic balls $B_k(x,r)$ are starlike with respect to x in domains that are starlike with respect to x and strictly starlike with respect to x in the punctured space whenever $r \in (0,\kappa]$, where κ is defined in [16, (4.1)] and has a numerical approximation $\kappa \approx 2.83297$. Recently J. Väisälä solved the convexity problem in planar domains and showed that the quasihyperbolic disk is strictly convex whenever the radius is less than or equal to one [27].

6.3. **Connectivity of metric balls.** If a metric space is geodesic, then all metric balls are connected. For nongeodesic metric spaces the connectivity of metric balls depends on the setting. For example, spherical balls are always connected while j-balls need not be connected [15, Remark 4.9 (2)]. We construct next such a domain that for any $m \in \mathbb{N}$ the j-ball has exactly m components.

Let us first consider the planar case $n = 2$. The generalization to $n > 3$ is straightforward. We denote by m the number of components of the j-ball we want to construct. We assume first $m \geq 9$ and denote the $(m-1)$th roots of unity by $\varepsilon_1,\ldots,\varepsilon_{m-1}$. Let $E_p = \{z \in \mathbb{R}^2 \colon |z| \leq 2, \measuredangle(z,\varepsilon_p,2\varepsilon_p) \leq \pi/(m-1)\}$ for all $p = 1,\ldots,m-1$ and

$$G_m = \mathbb{R}^2 \setminus \bigcup_{p=1}^{m-1} E_p. \tag{6.4}$$

The set G_{12} is illustrated in Figure 6.6.

6.5. **Lemma.** *For $m \geq 9$ and G_m as in (6.4) the j-ball $B_j(0,\log 4)$ has exactly m components.*

Proof. Let us denote $\alpha = \pi/(m-1)$. By geometry $d(E_i,E_{i+1}) = 2\sin(\alpha)$ and ∂E_i consists of a circular arc and two line segments.

Let us first show that $B_j(0,\log 4)$ has a component in $B^2(0,3/2)$. Let $x \in G_m$ with $|x| = 3/2$. Since $2\sin(\alpha) < 1$, we have

$$j(0,x) = \log\left(1+\frac{3}{2d(x)}\right) \geq \log\left(1+\frac{3}{2\sin(\alpha)}\right) > \log 4.$$

We show next that there exists $y \in B_j(0,\log 4)$ such that $|y| > 2$. We choose $y = e^{i\alpha}(\cos(\alpha)+\sqrt{4-\sin^2(\alpha)})$. Now $d(y) = 1 = d(0)$ and $|y| = \cos(\alpha)+\sqrt{4-\sin^2(\alpha)}$. Therefore

$$j(0,y) = \log\left(1+\cos(\alpha)+\sqrt{4-\sin^2(\alpha)}\right) < \log 4,$$

where the last inequality follows from the fact that $\cos(\alpha)+\sqrt{4-\sin^2(\alpha)} < 3$. By similar computation it is easy to verify that $j(0,y_p) < r$ for points $y_p = e^{i\alpha+2\alpha p}(\cos(\alpha)+\sqrt{4-\sin^2(\alpha)})$, $p = 1,\ldots,m-1$.

Let us finally show that each y_p is in a different component of $B_j(0,\log 4)$. By the symmetry of G_m it is sufficient to show that for all $z \in (2,\infty)$ we have $j(0,z) \geq \log 4$. If $|z| \geq 3$, then $j(0,z) = \log(1+|z|) \geq 4$. If $z \in (2,3)$, then $d(z) \in (0,1)$ and

$$j(0,z) = \log\left(1+\frac{|z|}{d(z)}\right) = \log\left(1+\frac{2+d(z)}{d(z)}\right) = \log\frac{2(1+d(z))}{d(z)} > \log 4.$$

Now we have shown that there is one component of $B_{j_{G_m}}(0,\log 4)$ in $B^2(0,3/2)$ and $m-1$ components symmetrically in $\mathbb{R}^2 \setminus B^2(0,3/2)$. By the symmetry of G_m these have to be the only m components of $B_j(0,\log 4)$. $\square$

Let us recall the notation for the Euclidean annulus

$$A(r,R) = \{z \in \mathbb{R}^n \colon r < |z| < R\}$$

for $0 < r < R < \infty$. In the case $m = 2,\ldots,8$ we can choose

$$G_m = \mathbb{R}^2 \setminus \left(\overline{A(1,2)} \setminus \bigcup_{p=0}^{m-2} D_p\right),$$

where $D_p = \{z \in R^2 \colon d(L_t,z) < \sin(\pi/8)\}$ and $L_t = \{u \in \mathbb{R}^2 \colon u = te^{pi\pi/8}, t > 0\}$. By the proof of Lemma 6.5 we can easily see that $B_{j_{G_m}}(0,\log 4)$ has exactly m components. The set G_5 is illustrated in Figure 6.6.

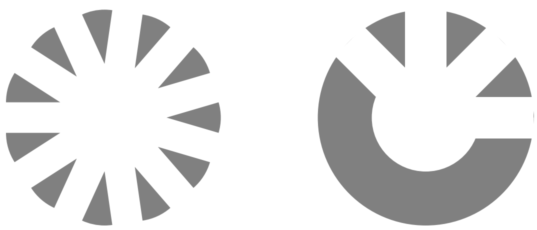

FIGURE 6.6. Examples of the domain G_{12} (left) and G_5. The gray area represents the complement of the domain.

6.7. **Remark.** In Lemma 6.5 we could [15, Remark 4.9] choose any radius $\log(3+\varepsilon)$, $\varepsilon > 0$, instead of $\log 4$. Then we just have to make sure that $d(E_1, E_2)$ is small enough.

We estimate then the size of the j-balls.

6.8. **Lemma.** *Let $G \subset \mathbb{R}^n$ be a domain, $x \in G$, and $r > 0$. Then for each connected component D of $B_j(x,r)$ we have*

$$\operatorname{diam}_k(D) \le c(r,n).$$

Proof. By [23, Theorem 3.8] $B^n(x, d(x)(1-e^{-r})) \subset B_j(x,r) \subset B^n(x, d(x)(e^r-1))$ and therefore

$$\frac{d(B_j(x,r))}{d(B_j(x,r),\partial G)} \le \frac{e^r-1}{1-e^{-r}} = e^r.$$

By [28, proof of Corollary 2.18]

$$\operatorname{diam}_k(D) \le c'(n)\left(2+\frac{d(B_j(x,r))}{d(B_j(x,r),\partial G)}\right)^n \le c'(n)(2+e^r)^n = c(r,n).$$

□

6.9. **Remark.** We consider a lower bound for the constant $c(r,n)$ of Lemma 6.8 in a special case. Let $n = 2$ and define

$$F_0 = \{z \in \mathbb{R}^2 \colon 0 < \operatorname{Re} z < 10,\ \operatorname{Im} z > 0\}$$

and for $m \ge 1$

$$F_m = \{z \in \mathbb{R}^2 \colon \operatorname{Im} z = 2m, \operatorname{Re} z \le 8\},$$
$$F'_m = \{z \in \mathbb{R}^2 \colon \operatorname{Im} z = 2+2m, \operatorname{Re} z \ge 2\}$$

and consider the domain

$$G = F_0 \setminus \left(\bigcup_{m=1}^{\infty} F_m \cup F'_m\right).$$

Let $x = 1+i$, $r = \log(4(1+t))$ and $y_t = 1 + i(3+4t)$ for $t = 1, 2, \ldots$. Now

$$j(x, y_t) = \log(3+4t) < r$$

and therefore $y_t \in B_j(x,r)$. By the selection of r we have $t = e^r/4 - 1$ and therefore

$$\operatorname{diam}_k(B_j(x,r)) \ge 18 + 20t = 5e^r - 2.$$

By Lemma 6.8 $\operatorname{diam}_k(B_j(x,r)) \le c'(2)(1+e^r)^2 = c'(2)(1+2e^r+e^{2r})$.

Let us assume that $G \subset \mathbb{R}^n$ is a domain such that $\mathbb{R}^n \setminus G$ is bounded. We find an estimate for the lower bound of the radius of the quasihyperbolic and distance ratio metric balls that encloses ∂G.

6.10. **Lemma.** *Let $G \subset \mathbb{R}^n$ be a domain such that $\mathbb{R}^n \setminus G \subset B^n(r)$ and fix $x \in \mathbb{R}^n \setminus B^n(r)$. Then $B_k(x,R)$ encloses $\mathbb{R}^n \setminus G$ for all $R > \pi|x|/(|x|-r)$.*

Proof. By Theorem 5.20

$$k_G(x,-x) \le \frac{|x|}{|x|-r} k_{\mathbb{R}^n\setminus\{0\}}(x,-x) = \frac{|x|\pi}{|x|-r}$$

and the assertion follows. □

6.11. **Lemma.** *Let $G \subset \mathbb{R}^n$ be a domain such that $\mathbb{R}^n \setminus G \subset B^n(r)$ and fix $x \in \mathbb{R}^n \setminus B^n(r)$. Then $B_j(x,R)$ encloses $\mathbb{R}^n \setminus G$ for all $R > \log((3|x|-r)/(|x|-r))$.*

Proof. Now

$$j_G(x,-x) \le \log\left(1 + \frac{2|x|}{|x|-r}\right) = \log\left(\frac{3|x|-r}{|x|-r}\right)$$

and the assertion follows. □

6.12. **Open problems.** In view of [27] and [16, p. 197] Open problem 6.1 can be reduced to the following one:

6.13. **Open problem.** Are the quasihyperbolic balls $B_k(x,r)$ convex in the Euclidean geometry in domains that are starlike with respect to x for all $r \in (0,1]$?

We can also ask the uniqueness of geodesics.

6.14. **Open problem.** Are quasihyperbolic geodesics unique in simply connected planar domains?

The convexity problem is linked to uniqueness of the geodesics and prolongation of the geodesics:

6.15. **Open problem.** Does there exist $r_{uni} > 0$ such that for all domains $G \subset \mathbb{R}^n$ each quasihyperbolic geodesic with quasihyperbolic length less than r_{uni} is unique?

6.16. **Open problem.** Does there exist $r_{pro} > 0$ such that for all $x, y \in G$ with $k(x,y) < r_{pro}$ there exist $z \in G$ with $k(x,z) = r_{pro}$ and $J_k[x,y] \subset J_k[x,z]$?

Open problem 6.1 implies 6.15, which implies 6.16, in general and all three open problems are true in the planar case [27]. A domain $G \subset \mathbb{R}^n$ is *close-to-convex* if $\mathbb{R}^n \setminus G$ is a union of half-lines that do not intersect except for the tips. We may modify 6.1 for the close-to-convex domains.

6.17. **Open problem.** Does there exist a constant $r_0 > 0$ such that the metric ball $B_m(x,r)$ is close-to-convex in the Euclidean geometry for $m \in \{k, j\}$ and for all $r \in (0, r_0)$?

6.18. **Remark.** Let us define

$$G = \mathbb{R}^3 \setminus \{z \in \mathbb{R}^3 \colon z = e_3 t, |t| \ge 1\}$$

and consider the metric (G,k). It can be shown that for all $x, y \in \mathbb{R}^2$ we have $J_k[x,y] \subset \mathbb{R}^2$. It was also indicated to the author by O. Martio and J. Väisälä that in this metric the uniqueness constant r_{uni} is less than π, which was a conjecture in [16, p. 201].

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