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PLURIPOTENTIAL THEORY AND CAPACITY INEQUALITIES

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1. Introduction

Pluripotential theory is a nonlinear theory which offers effective methods for the purposes of multidimensional complex analysis. Although its main objects, plurisuperharmonic functions, were introduced in the early 1940's independently by P. Lelong [Lel42a, Lel42b] and K. Oka [Oka42], the main theory is developed during the last quarter of the 20^{th} century. Fundamental works and surveys in pluripotential theory are made by P. Lelong [Lel68], E. Bedford and B. A. Taylor [BT76, BT82], A. Sadullæv [Sad81], U. Cegrell [Ceg88], M. Klimek [Kli91], E. Bedford [Bed93], and C. Kiselman [Kis00].

Traditionally, plurisubharmonic functions rather than plurisuperharmonic ones have been studied in pluripotential theory. This is natural because the generalized complex Monge–Ampère operator acting on a plurisubharmonic function gives a nonnegative value and it is a positive measure without a minus sign attached to the operator. However, we consider plurisuperharmonic functions in this study, and this aspect has also its advantages. These are more or less related to the fact that nonnegative plurisuperharmonic functions can be nonzero and simultaneously have zero boundary values. Of course, this choice is made only for the sake of convenience in this context, and the known results for plurisubharmonic functions are easy to convert to this setting.

The basic concepts and known results in pluripotential theory are introduced in Part 1 of this study. Plurisuperharmonic functions in an open subset of $\mathbb{C}^n$ for $n \geqslant 2$ form a proper subclass of superharmonic functions, while these two classes coincide for $n = 1$. Many of the basic properties are thus carried over directly to the plurisuperharmonic case. However, the main interest in the study of pluripotential theory arises from the difference between these function classes. For example, distribution theory gives a general machinery in classical potential theory that is not available in pluripotential theory.

The idea of the approach in Section 6 is based on a recent article by J. Kinnunen and O. Martio [KM00] where they study the relation between the signed Hardy–Littlewood maximal function and superharmonicity. We define a slightly different maximal function so that the original properties are still valid. However, an iteration of our maximal function produces the least plurisuperharmonic majorant of a function instead of the least superharmonic one.

Capacity is a set function arising in potential theory as the analogue of the physical concept of the electrostatic capacity. Capacity is defined for a condenser, that is, an open set with a relatively compact subset inside. We study some specific properties of the relative Monge–Ampère capacity defined by E. Bedford and B. A. Taylor [BT82] that is a Choquet capacity [Cho54] adapted to pluripotential theory.

Classical potential theory in $\mathbb{R}^m$ and pluripotential theory in $\mathbb{C}^n$ have their own characteristics, but potential theory in the plane is common to the both. A central part of this intersection, capacities in the plane, is studied in Part 2. In fact, these considerations are well-acquainted in classical theory. A fundamental reference for potential theory in the complex plane is the monograph by M. Tsuji [Tsu59]. T. Ransford [Ran95] has recently written a survey of this field. As our contribution to this theory, we show that the Green capacity and the variational 2-capacity coincide with the relative Monge–Ampère capacity in the plane.

Part 3 is devoted to the study of holomorphic mappings and their potential theoretic aspects. It is plain that holomorphic mappings preserve plurisuperharmonic functions but we introduce a push forward function for proper holomorphic mappings. This function preserves plurisuperharmonic functions as well. Moreover, we study integral transformation formulas for holomorphic mappings both with the Lebesgue measure and with the Monge–Ampère operator acting on a plurisuperharmonic function that gives a Radon measure. After these considerations we have enough information to state capacity inequalities for holomorphic mappings. If $f : \Omega \to \Omega'$ is a holomorphic mapping and $E \Subset \Omega$, then we prove under some additional conditions that

$$\begin{aligned} N_{\min}(f,E)\,\mathrm{cap}(f(E),\Omega') &\leqslant M_{\min}(f,E)\,\mathrm{cap}(f(E),\Omega') \\ &\leqslant \mathrm{cap}(E,\Omega) \leqslant N_{\max}(f,\Omega)^n\,\mathrm{cap}(f(E),\Omega'), \end{aligned}$$

where $M_{\min}(f,\cdot)$ denotes the minimal multiplicity and $N_{\min}(f,\cdot)$ and $N_{\max}(f,\cdot)$ denote the crude minimal and maximal multiplicities of f in a subset of Ω.

The theory of quasiregular mappings provides a wide background of this study; both nonconstant quasiregular mappings and proper holomorphic mappings are discrete, open and sense-preserving. For the theory of holomorphic mappings we refer to the books of L. Hörmander [Hör66], Steven G. Krantz [Kra92] and W. Rudin [Rud80]. Proper holomorphic mappings are thoroughly concidered in the articles by R. Remmert and K. Stein [RS60] and by E. Bedford [Bed84]. In the 1960's and early 1970's, quasiregular mappings were intensively studied in Helsinki by the group O. Martio, S. Rickman and J. Väisälä [MRV69, MRV70, MRV71]. In addition to these articles, there are monographs on this theory, for example, by M. Vuorinen [Vuo88], Yu. G. Reshetnyak [Res89], J. Heinonen, T. Kilpeläinen and O. Martio [HKM93] and S. Rickman [Ric93].

2. Notation and terminology

The n-dimensional complex space is denoted by $\mathbb{C}^n$, which is often identified with the $2n$-dimensional Euclidean real space $\mathbb{R}^{2n}$. This identification is made so that the complex coordinates $z_j = x_j + iy_j$ of a point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ give the real coordinates x_j and y_j for $z = (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$. In this context, the general notation $\mathbb{C}^n$ contains also a special case $\mathbb{C}^1 = \mathbb{C}$. The *Euclidean norm* of a point $z \in \mathbb{C}^n$ is given by $|z| = \sqrt{z_1\overline{z_1} + \cdots + z_n\overline{z_n}}$, and the *open ball* with center z and radius $r > 0$ is the set $B(z, r) = \{w \in \mathbb{C}^n : |z - w| < r\}$.

A *domain* in $\mathbb{C}^n$ is defined to be a connected open subset of $\mathbb{C}^n$. If G is a subset of $\mathbb{C}^n$, then a *neighbourhood* of G is a domain in $\mathbb{C}^n$ such that it contains G. The *boundary*, the *interior* and the *closure* of G are denoted by ∂G, $\operatorname{int} G$ and $\overline{G}$, respectively. The notation $G \Subset F$ means that $\overline{G}$ is a compact subset of F. Moreover, the $2n$-dimensional *Lebesgue measure* of G is denoted by $m(G)$ whenever G is Lebesgue measurable.

An open nonempty subset of $\mathbb{C}^n$ is denoted by Ω, in particular Ω may coincide with $\mathbb{C}^n$. The class of Lebesgue p-integrable functions in Ω is denoted by $\mathrm{L}^p(\Omega)$, and the class of locally Lebesgue p-integrable functions in Ω is denoted by $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$. In particular, $\mathrm{L}^\infty(\Omega)$ is the class of bounded functions and $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$ is the class of locally bounded functions in Ω. The *support* of a function $u : \Omega \to \mathbb{R}$, denoted by $\operatorname{spt} u$, is the smallest closed set such that u vanishes outside $\operatorname{spt} u$. The classes of compactly supported continuous and compactly supported infinitely smooth functions in Ω are denoted by $\mathrm{C}_0(\Omega)$ and $\mathrm{C}_0^\infty(\Omega)$, respectively.

The *gradient* of a C^1-function $u : \Omega \to \mathbb{R}$ is the vector

$$\nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_{2n-1} u, \partial_{2n} u) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial y_n}\right).$$

More generally, let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. A function $v = (v_1, \ldots, v_{2n}) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ is the *weak* (or *distributional*) *gradient* of u if

$$\int_\Omega u\, \partial_j \varphi \, dm = -\int_\Omega v_j\, \varphi \, dm$$

for all $\varphi \in \mathrm{C}_0^\infty(\Omega)$ and $j = 1, 2, \ldots, 2n$. Then we denote $v = \nabla u$. If $f : \Omega \to \mathbb{C}^m$ is a C^1-mapping, then we use the standard notation

$$\frac{\partial f_k}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f_k}{\partial x_j} - i\frac{\partial f_k}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial f_k}{\partial \overline{z_j}} = \frac{1}{2}\left(\frac{\partial f_k}{\partial x_j} + i\frac{\partial f_k}{\partial y_j}\right)$$

for every $k = 1, \ldots, m$ and $j = 1, \ldots, n$.

An outer measure μ on $\mathbb{C}^n$ is said to be *regular* if for each set $A \subset \mathbb{C}^n$ there exists a μ-measurable set B such that $A \subset B$ and $\mu(A) = \mu(B)$. A measure μ on $\mathbb{C}^n$ is called *Borel* if every Borel subset of $\mathbb{C}^n$ is μ-measurable, and a Borel regular measure μ on $\mathbb{C}^n$ is called a *Radon measure* if $\mu(K) < \infty$ for each compact subset K of $\mathbb{C}^n$. We say that a sequence of Radon measures μ_k on $\mathbb{C}^n$, $k = 1, 2, \ldots$, is *weak*-convergent* to a Radon measure μ on $\mathbb{C}^n$ if

$$\lim_{k\to\infty} \int_{\mathbb{C}^n} \varphi \, d\mu_k = \int_{\mathbb{C}^n} \varphi \, d\mu$$

for all $\varphi \in \mathrm{C}_0(\mathbb{C}^n)$. Furthermore, the *support* of a measure μ on $\mathbb{C}^n$, denoted by $\operatorname{spt}\mu$, is the set of the points $z \in \mathbb{C}^n$ such that $\mu(U) > 0$ for each open neighbourhood U of z. Then $\operatorname{spt}\mu$ is the smallest closed subset F of $\mathbb{C}^n$ such that $\mu(\mathbb{C}^n \setminus F) = 0$.

Part 1. Plurisuperharmonicity and related topics

In this introductory part we recall the basic concepts and some known results in pluripotential theory. The proofs of the cited results can be found, for example, in the articles by E. Bedford and B. A. Taylor [BT76, BT82] or in the monographs by M. Klimek [Kli91] and by U. Cegrell [Ceg88]. The last section of this part introduces some new considerations related to plurisuperharmonic functions that are based on a recent article by J. Kinnunen and O. Martio [KM00].

3. Plurisuperharmonic functions

Let Ω be an open subset of $\mathbb{C}^n$. Recall that a C^2-function $u : \Omega \to \mathbb{R}$ is said to be *harmonic* in Ω if it satisfies the *homogeneous Laplace equation*

$$\Delta u = \sum_{j=1}^{n} \left(\frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y_j^2} \right) = 0 \quad \text{in } \Omega. \tag{3.1}$$

Note that the Laplace operator can also be given as

$$\Delta u = 4 \sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_j}.$$

Definition. Let $u : \Omega \to (-\infty, \infty]$ be a lower semicontinuous function which is not identically ∞ on any component of Ω. The function u is said to be *superharmonic* in Ω if for every open $G \Subset \Omega$ and every function h which is continuous in $\overline{G}$ and harmonic in G, the following *comparison principle* holds:

$$u|_{\partial G} \geqslant h|_{\partial G} \quad \Longrightarrow \quad u|_G \geqslant h|_G.$$

The family of all superharmonic functions in Ω is denoted by $\mathrm{SH}(\Omega)$. A function v is called *subharmonic* in Ω if $-v$ is superharmonic in Ω. Superharmonicity of a lower semicontinuous function $u : \Omega \to (-\infty, \infty]$ can be defined also by the *mean value principle*:

$$u(z) \geqslant \frac{1}{m(B(z,r))} \int_{B(z,r)} u(w)\, dm(w) \quad \text{whenever } B(z,r) \Subset \Omega.$$

The mean value principle is here an equivalent condition to the comparison principle. Furthermore, a set E is said to be *polar* if there is a neighbourhood U of E and a function $v \in \mathrm{SH}(U)$ such that $E \subset \{z \in U : v(z) = \infty\}$.

Definition. Let $u : \Omega \to (-\infty, \infty]$ be a lower semicontinuous function which is not identically ∞ on any component of Ω. The function u is said to be *plurisuperharmonic* in Ω if for each $z \in \Omega$ and $w \in \mathbb{C}^n$, the function $\lambda \longmapsto u(z + \lambda w)$ is superharmonic or identically ∞ on every component of the set $\{\lambda \in \mathbb{C} : z + \lambda w \in \Omega\}$.

The family of all plurisuperharmonic functions in Ω is denoted by $\mathrm{PSH}(\Omega)$. A function v is called *plurisubharmonic* in Ω if $-v$ is plurisuperharmonic in Ω. It is well-known that plurisuperharmonic functions are locally integrable [Kli91, Corollary 2.9.6]. Further, all plurisuperharmonic functions can be approximated locally from below by infinitely smooth plurisuperharmonic functions:

3.2. Local approximation theorem. [Kli91, Theorem 2.9.2] *Let u be a plurisuperharmonic function in Ω. If $G \Subset \Omega$ is an open set, then there is an increasing sequence of functions $u_j \in C^\infty(G) \cap \mathrm{PSH}(G)$ such that $\lim_{j\to\infty} u_j = u$ in G.*

Even though the definition is not local, plurisuperharmonicity is a local property [Kli91, Theorem 2.9.1]: A function $u \in \mathrm{PSH}(\Omega)$ if and only if each $z \in \Omega$ has a neighbourhood $G \subset \Omega$ such that $u|_G \in \mathrm{PSH}(G)$. In view of this result, Local approximation theorem is a very useful tool with plurisuperharmonic functions.

A set $E \subset \mathbb{C}^n$ is said to be *pluripolar* if there is a neighbourhood U of E and a function $u \in \mathrm{PSH}(U)$ such that $E \subset \{z \in U : u(z) = \infty\}$. It is well-known that pluripolar sets have Lebesgue measure zero [Kli91, Corollary 2.9.10]. Even if the above definition of pluripolarity is local of its nature, *Josefson's theorem* [Jos78] states that every pluripolar set in $\mathbb{C}^n$ is globally pluripolar, i.e., contained in the set $\{u = \infty\}$ for some $u \in \mathrm{PSH}(\mathbb{C}^n)$. Moreover, a property is said to hold *quasieverywhere* in a subset S of $\mathbb{C}^n$ (or for *quasievery* $z \in S$) if it holds everywhere in $S \setminus E$ for some pluripolar set E. It seems that pluripolar sets are the best choice for the small sets in pluripotential theory.

Next we introduce a well-known *sweeping-out process* which is a two step procedure; the result is a plurisuperharmonic function. Suppose that Ω is a domain in $\mathbb{C}^n$. Let $\psi : \Omega \to (-\infty, \infty]$ be a function that is locally bounded below, and let

$$\Phi^{\psi}_{\mathrm{PSH}}(\Omega) = \{u \in \mathrm{PSH}(\Omega) : u \geqslant \psi \text{ in } \Omega\} \tag{3.3}$$

be nonempty. Then the function

$$R^{\psi}_{\mathrm{PSH}} = R^{\psi,\Omega}_{\mathrm{PSH}} = \inf \Phi^{\psi}_{\mathrm{PSH}}(\Omega) \tag{3.4}$$

is called the *reduced function* of ψ in Ω and its lower semicontinuous regularization

$$\hat{R}^{\psi}_{\mathrm{PSH}}(z) = \hat{R}^{\psi,\Omega}_{\mathrm{PSH}}(z) = \liminf_{\substack{w\to z\\ w\in\Omega}} R^{\psi}_{\mathrm{PSH}}(w) \tag{3.5}$$

is called the *regularized reduced function* of ψ in Ω. If $\Phi^{\psi}_{\mathrm{PSH}}(\Omega)$ is empty, then $\hat{R}^{\psi}_{\mathrm{PSH}} \equiv \infty$. It is well-known that the regularized reduced function $\hat{R}^{\psi}_{\mathrm{PSH}}$ is plurisuperharmonic in Ω unless $\hat{R}^{\psi}_{\mathrm{PSH}} \equiv \infty$. In the superharmonic case

$$\Phi^{\psi}_{\mathrm{SH}}(\Omega) = \{u \in \mathrm{SH}(\Omega) : u \geqslant \psi \text{ in } \Omega\}, \tag{3.6}$$

the reduced function R^{ψ}_{SH} is called the *réduite*, and the regularized reduced function $\hat{R}^{\psi}_{\mathrm{SH}}$ is called the *balayage*. In fact, $R^{\psi}_{\mathrm{PSH}} = R^{\psi}_{\mathrm{SH}}$ and $\hat{R}^{\psi}_{\mathrm{PSH}} = \hat{R}^{\psi}_{\mathrm{SH}}$ in the plane, since superharmonic and plurisuperharmonic functions coincide there. Moreover, if $\psi : \Omega \to (-\infty, \infty]$ is locally bounded below, then

$$\hat{R}^{\psi}_{\mathrm{PSH}}(z) \geqslant \psi(z) \text{ for almost every } z \in \Omega. \tag{3.7}$$

Let u be a nonnegative function on a subset E of Ω Then we write

$$\Phi^{u,E}_{\mathrm{PSH}}(\Omega) = \Phi^{\psi}_{\mathrm{PSH}}(\Omega),\ R^{u,E}_{\mathrm{PSH}} = R^{\psi}_{\mathrm{PSH}},\ \text{and } \hat{R}^{u,E}_{\mathrm{PSH}} = \hat{R}^{\psi}_{\mathrm{PSH}},$$

where

$$\psi = \begin{cases} u & \text{on } E, \\ 0 & \text{on } \Omega \setminus E. \end{cases}$$

The function $R^{u,E}_{\mathrm{PSH}}$ is called the *reduced function of u relative to E in* Ω, and the function $\hat{R}^{u,E}_{\mathrm{PSH}}$ is called the *regularized reduced function of u relative to E in* Ω. If $u \equiv c$ is a constant in E, then we write $\hat{R}^{c,E}_{\mathrm{PSH}} = \hat{R}^{u,E}_{\mathrm{PSH}}$, and in the plane again $R^{u,E}_{\mathrm{PSH}} = R^{u,E}_{\mathrm{SH}}$ and $\hat{R}^{u,E}_{\mathrm{PSH}} = \hat{R}^{u,E}_{\mathrm{SH}}$. In addition, $0 \leqslant \hat{R}^{u,E}_{\mathrm{PSH}}(z) \leqslant R^{u,E}_{\mathrm{PSH}}(z)$ for each $z \in \Omega$ and $\hat{R}^{u,E}_{\mathrm{PSH}}$ is plurisuperharmonic in Ω.

Note that the reduced functions $R^{1,E}_{\mathrm{PSH}}$ and $\hat{R}^{1,E}_{\mathrm{PSH}}$ are closely related (but not exactly the same, in general) to the concepts of the *relative extremal function* [Sic81] and the *regularized relative extremal function* or PSH-*measure of E relative to* Ω [Sad81] in pluripotential theory.

4. Complex Monge–Ampère operator

Let Ω be an open set in $\mathbb{C}^n$. The *complex differential form of bidegree* (p, q) or shortly (p, q)*-form* in Ω is a sum

$$\omega = \sum_{I,J} \omega_{IJ}(z)\, dz_I \wedge d\overline{z}_J,$$

where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are increasing p- and q-indices, respectively, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$; here the wedge $\wedge$ denotes the *exterior product*. Moreover, we denote $\Lambda^{p,q}(\Omega) = \{\omega : \omega$ is a (p, q)-form in $\Omega\}$ and $\mathrm{F}(\Omega) \cap \Lambda^{p,q}(\Omega) = \{\omega \in \Lambda^{p,q}(\Omega) :$ the coefficients $\omega_{IJ} \in \mathrm{F}(\Omega)\}$ where $\mathrm{F}(\Omega)$ is some function class in Ω. For more details of the complex differential forms, see [Rud80, Chapter 16] and [Kli91, Section 1.5]

Let $\omega \in \mathrm{C}^1(\Omega) \cap \Lambda^{p,q}(\Omega)$ be given by $\omega = \sum_{I,J} \omega_{IJ}\, dz_I \wedge d\overline{z}_J$. Then the *exterior differential* of ω is given by

$$d\omega = \sum_{I,J} d\omega_{IJ} \wedge dz_I \wedge d\overline{z}_J = \sum_{I,J} (\partial\omega_{IJ} + \overline{\partial}\omega_{IJ}) \wedge dz_I \wedge d\overline{z}_J,$$

and the differential operators $\partial\omega$ and $\overline{\partial}\omega$ are defined as the components of $d\omega$ such that

$$\partial\omega = \sum_{I,J} \partial\omega_{IJ} \wedge dz_I \wedge d\overline{z}_J \quad \text{and} \quad \overline{\partial}\omega = \sum_{I,J} \overline{\partial}\omega_{IJ} \wedge dz_I \wedge d\overline{z}_J.$$

Then $\partial\omega \in \Lambda^{p+1,q}(\Omega)$ and $\overline{\partial}\omega \in \Lambda^{p,q+1}(\Omega)$. In particular, if $f \in \mathrm{C}^1(\Omega)$ is a complex-valued function (this is, a $(0, 0)$-form), then $df = \partial f + \overline{\partial} f$ where

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\, dz_j \quad \text{and} \quad \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}\, d\overline{z}_j.$$

Furthermore, $\partial f \in \Lambda^{1,0}(\Omega)$ and $\overline{\partial} f \in \Lambda^{0,1}(\Omega)$.

We use the standard notation $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$ so that $dd^c = 2i\partial\overline{\partial}$. The *complex Monge–Ampère operator* in $\mathbb{C}^n$ is the nth exterior power of dd^c, that is,

$$(dd^c)^n = \overbrace{dd^c \wedge \cdots \wedge dd^c}^{n \text{ times}}.$$

It is easily seen that if $u \in C^2(\Omega)$, then

$$(dd^c u)^n = 4^n n! \det\left[\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right] dV,$$

where

$$dV(z) = \left(\frac{i}{2}\right)^n dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_n \wedge d\overline{z_n}$$

is the volume form in $\mathbb{C}^n$.

Let $u_1, \ldots, u_k \in C^2(\Omega)$, $1 \leqslant k \leqslant n$. If $\varphi \in C_0^\infty(\Omega) \cap \Lambda^{n-k,n-k}(\Omega)$ is a test form, then

$$\int_\Omega dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \varphi = \int_\Omega u_j \, dd^c u_1 \wedge \cdots \wedge dd^c u_{j-1} \wedge dd^c u_{j+1} \wedge \cdots \wedge dd^c u_k \wedge dd^c \varphi$$

for $j = 1, \ldots, k$ [BT76, Proposition 2.1]. This formula is a kind of integration by parts property for the complex Monge–Ampère operator. Secondly, if $u_1, \ldots, u_k \in L^\infty_{\mathrm{loc}}(\Omega) \cap \mathrm{PSH}(\Omega)$, $1 \leqslant k \leqslant n$, then $-(dd^c u_1 \wedge \cdots \wedge dd^c u_k)$ is a positive (k, k)-current with measure coefficients [BT76, Proposition 2.9]. These two properties have inspired to define the complex Monge–Ampère operator inductively acting on locally bounded plurisuperharmonic functions.

Definition. Suppose that $u_1, \ldots, u_n \in L^\infty_{\mathrm{loc}}(\Omega) \cap \mathrm{PSH}(\Omega)$. If $1 \leqslant k \leqslant n$, then $dd^c u_1 \wedge \cdots \wedge dd^c u_k$ is defined inductively by the formula

$$\int_\Omega dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \varphi = \int_\Omega u_k \, dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c \varphi,$$

where $\varphi \in C_0^\infty(\Omega) \cap \Lambda^{n-k,n-k}(\Omega)$ is any test form. Now $(dd^c)^n$ defined by the previous formula is called the *generalized complex Monge–Ampère operator*.

The definition above defines $-(dd^c)^n$ so that it is a (n, n)-form whose coefficients are Radon measures on Ω, see [BT76]. We usually abbreviate the whole name of this generalized operator as Monge–Ampère operator without the prefix ‘generalized complex’.

Next convergence result says that $-(dd^c)^n$ is continuous under monotone limits. It is known that $-(dd^c)^n$ does not behave well under nonmonotone limits [Ceg83]. For more details of the convergence *in the sense of currents*, this is, in the sense of weak*-convergence of measures, see [Lel68, Kli91, Fed69].

4.1. Weak convergence theorem. [BT82, Theorem 2.1] *Let $u \in L^\infty_{\mathrm{loc}}(\Omega) \cap \mathrm{PSH}(\Omega)$. If $u_j \in L^\infty_{\mathrm{loc}}(\Omega) \cap \mathrm{PSH}(\Omega)$ is a decreasing or increasing sequence of functions such that $\lim_{j\to\infty} u_j = u$ a.e. in Ω (with respect to the Lebesgue measure), then*

$$\lim_{j\to\infty} -(dd^c u_j)^n = -(dd^c u)^n$$

in the sense of currents.

There are even weaker conditions than the previous one to ensure the convergence $-(dd^c u_j)^n \to -(dd^c u)^n$ in the sense of currents, see [Xin96]. Anyway, we need information about the strong convergence of the Monge–Ampère operator. If u is plurisuperharmonic in Ω, then the *nonpolar part* of $-(dd^c u)^n$, denoted by $\mathrm{NP}\,[-(dd^c u)^n]$, is the measure which is zero on $\{u = \infty\}$, and for a Borel set $E \subset \{u < \infty\}$, we have

$$\int_E \mathrm{NP}\,[-(dd^c u)^n] = \lim_{j\to\infty} \int_{E\cap\{u<j\}} -(dd^c \min(u, j))^n.$$

In general, $\mathrm{NP}\,[-(dd^c u)^n]$ is not locally finite, but the following convergence property holds:

4.2. Strong convergence theorem. [BT87, Proposition 4.4] *Let a plurisuperharmonic function u in Ω and a compact subset $K \subset \{u < \infty\}$ be given. If $u_j \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega) \cap \mathrm{PSH}(\Omega)$ is an increasing sequence of functions which converges to u, then*

$$\lim_{j\to\infty} \int_K -(dd^c u_j)^n = \int_K \mathrm{NP}\,[-(dd^c u)^n]\,.$$

Finally, the Monge–Ampère operator satisfies the *superadditivity property* [Kli91, Corollary 3.4.9]: If $u, v \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega) \cap \mathrm{PSH}(\Omega)$, then

$$-\,(dd^c(u+v))^n \geqslant -(dd^c u)^n - (dd^c v)^n. \tag{4.3}$$

5. Relative Monge–Ampère capacity

Definition. [BT82] Let K be a compact set contained in Ω. The *(relative) Monge–Ampère capacity* of K in Ω is defined by the formula

$$\mathrm{cap}(K, \Omega) = \sup_{\substack{u\in\mathrm{PSH}(\Omega)\\ 0\leqslant u\leqslant 1}} \int_K -(dd^c u)^n. \tag{5.1}$$

If E is an arbitrary set in Ω, we put

$$\mathrm{cap}(E, \Omega) = \sup\{\mathrm{cap}(K, \Omega) : K \text{ is a compact subset of } E\}\,. \tag{5.2}$$

Note that the supremum in (5.1) is taken over all plurisuperharmonic functions u with values in the closed interval $0 \leqslant u \leqslant 1$. In the original definition [BT82] the values of plurisuperharmonic functions u are supposed to belong to the open interval $0 < u < 1$, however, these definitions are equivalent, see [Bed93, p. 63].

A pair (E, Ω) is said to be a *condenser* if Ω is an open set and $E \Subset \Omega$ is an arbitrary set. A condenser (E, G) is said to be *in* Ω if $G \subset \Omega$. Moreover, a condenser (E, Ω) is said to be a *Borel condenser* (respectively, a *compact condenser* or an *open condenser*) if E is a Borel set in Ω (respectively, a compact set or an open set in Ω). Of course, both compact and open condensers are Borel condensers.

The Monge–Ampère capacity in Ω is defined for all subsets of Ω; even for the sets $E \subset \Omega$ such that $\mathrm{dist}(E, \partial\Omega) = 0$. However, all condensers (E, Ω) are supposed to satisfy $E \Subset \Omega$ in this study.

We note that the Monge–Ampère capacity is an *inner capacity*. It has the following properties [BT82, Section 3]:

(i) If (E, Ω) is a Borel condenser, then

$$\operatorname{cap}(E,\Omega) = \sup_{\substack{u\in \mathrm{PSH}(\Omega)\\ 0\leqslant u\leqslant 1}} \int_E -(dd^c u)^n.$$

(ii) If $E_1 \subset E_2 \subset \Omega$, then $\operatorname{cap}(E_1,\Omega) \leqslant \operatorname{cap}(E_2,\Omega)$.
(iii) If $E \subset \Omega \subset \Omega'$, then $\operatorname{cap}(E,\Omega) \geqslant \operatorname{cap}(E,\Omega')$.
(iv) If $E_1, E_2, \dots$ are subsets of Ω, then

$$\operatorname{cap}\left(\bigcup_{j=1}^{\infty} E_j, \Omega\right) \leqslant \sum_{j=1}^{\infty} \operatorname{cap}(E_j,\Omega).$$

(v) If $E_1 \subset E_2 \subset \cdots$ are Borel sets in Ω, then

$$\operatorname{cap}\left(\bigcup_{j=1}^{\infty} E_j, \Omega\right) = \lim_{j\to\infty} \operatorname{cap}(E_j,\Omega).$$

The *outer capacity* of a set E in Ω is defined by the formula

$$\operatorname{cap}^*(E,\Omega) = \inf\{\operatorname{cap}(U,\Omega) : U \supset E,\ U \text{ open}\}. \tag{5.3}$$

The set function $E \longmapsto \operatorname{cap}^*(E,\Omega)$, $E \subset \Omega$, is a *Choquet capacity* relative to Ω [BT82, Section 8]. It follows from *Choquet's theorem* [Cho54] that all K-analytic subsets E of Ω (often referred to as *Suslin sets* of Ω) are *capacitable*, that is, $\operatorname{cap}^*(E,\Omega) = \operatorname{cap}(E,\Omega)$. A condenser (E,Ω) is said to be a *capacitable condenser* if E is capacitable. For example, Borel sets are K-analytic, and thus every Borel condenser is a capacitable condenser. The outer capacity is closely related to pluripolar sets: A set $E \subset \Omega \Subset \mathbb{C}^n$ is pluripolar if and only if $\operatorname{cap}^*(E,\Omega) = 0$ [Bed93, (4.3), p. 63]. For the inner capacity this is always true only in one direction: If $E \subset \Omega \Subset \mathbb{C}^n$ is pluripolar, then $\operatorname{cap}(E,\Omega) = 0$.

It is known that plurisuperharmonic functions are *quasicontinuous* with respect to the Monge–Ampère capacity [Bed93, (4.6), p. 64]: If $u \in \mathrm{PSH}(\Omega)$, then for each $\varepsilon > 0$, there is an open subset $U \subset \Omega$ such that $\operatorname{cap}(U,\Omega) < \varepsilon$ and $u|_{\Omega\setminus U}$ is continuous.

6. Maximal operator and plurisuperharmonicity

Let Ω be an open set in $\mathbb{R}^m$. Suppose that $u : \Omega \to (-\infty,\infty]$ is a Lebesgue measurable function which is locally bounded below. The *signed Hardy–Littlewood maximal function* of u is defined as

$$M_\Omega u(x) = \sup \fint_{B(x,r)} u(y)\, dm(y), \tag{6.1}$$

where the supremum is taken over all radii r with $0 < r < \operatorname{dist}(x,\partial\Omega)$. Here

$$\fint_E u\, d\mu = \frac{1}{\mu(E)} \int_E u\, d\mu$$

denotes the mean integral of u over E.

The maximal function $M_\Omega u$ is defined everywhere in Ω with values in $(-\infty, \infty]$ and it is lower semicontinuous. If u is lower semicontinuous, then $u(x) \leqslant M_\Omega u(x)$ for every $x \in \Omega$. If u is not identically ∞ on any component of Ω, then $M_\Omega u(x) = u(x)$ for every $x \in \Omega$ if and only if u is superharmonic. Proofs of these facts can be found in the article by J. Kinnunen and O. Martio [KM00].

Suppose now that Ω is an open set in $\mathbb{C}^n$. Then the definition of the signed Hardy–Littlewood maximal function is valid with the identification $\mathbb{C}^n \approx \mathbb{R}^{2n}$. The next result states the mean value principle for plurisuperharmonic functions in Ω, this is, the super mean value principle for superharmonic functions on all parts of complex lines inside Ω. It follows easily from the definition of plurisuperharmonicity and the super mean value principles for superharmonic functions.

6.2. Plurisuper mean value principle. [Kli91, Theorem 2.9.1] *Let $u : \Omega \to (-\infty, \infty]$ be a lower semicontinuous function which is not identically ∞ on any component of Ω. Then the following conditions are equivalent:*

(i) *u is plurisuperharmonic.*

(ii) *If $z \in \Omega$ and $w \in \mathbb{C}^n$ are such that $0 < |w| < \operatorname{dist}(z, \partial\Omega)$, then*

$$u(z) \geqslant \frac{1}{2\pi} \int_0^{2\pi} u(z + e^{it} w)\, dt = ⨍_{\partial B_2(0,1)} u(z + \lambda w)\, ds(\lambda).$$

(iii) *If $z \in \Omega$ and $w \in \mathbb{C}^n$ are such that $0 < |w| < \operatorname{dist}(z, \partial\Omega)$, then*

$$u(z) \geqslant ⨍_{B_2(0,1)} u(z + \lambda w)\, dm_2(\lambda).$$

The unit disc in $\mathbb{C}$ is denoted here by $B_2(0, 1)$, and s and m_2 denote the length measure and the Lebesgue measure in $\mathbb{C}$, respectively. Note that the previous mean integrals are taken with respect to these measures.

Definition. Suppose that $u : \Omega \to (-\infty, \infty]$ is a Borel function which is locally bounded below. The *signed pluricomplex maximal function* of u is defined as

$$PM_\Omega u(z) = \sup ⨍_{B_2(0,1)} u(z + \lambda w)\, dm_2(\lambda), \tag{6.3}$$

where the supremum is taken over all $w \in \mathbb{C}^n$ such that $0 < |w| < \operatorname{dist}(z, \partial\Omega)$.

Note that every Borel function $u : \Omega \to (-\infty, \infty]$ is measurable on all complex lines in Ω. In addition, u is supposed to be locally bounded below, and hence the maximal function (6.3) is well-defined.

6.4. Remark. If we assume generally that $u : \Omega \to (-\infty, \infty]$ is any Borel function which is locally bounded below, then the signed pluricomplex maximal function $PM_\Omega u$ is not necessarily lower semicontinuous, which is seen by the following example. However, the signed Hardy–Littlewood maximal function $M_\Omega u$ is always lower semicontinuous [KM00, Lemma 2.1 (i)]. This difference between the maximal functions arises from the fact that exceptional values of u in the sets of Lebesgue measure zero have no effect on the values of the signed Hardy–Littlewood maximal function $M_\Omega u$, but they may have effect on the signed pluricomplex maximal function $PM_\Omega u$.

As an example we consider the characteristic function of the closed set $A = \{z \in \mathbb{C}^2 : \frac{1}{2} \leqslant |z_1| \leqslant 1,\ |z_2| = 0\}$. Then u is a Borel function but not lower semicontinuous, and

$$PM_{\mathbb{C}^2}u(0) = \frac{m_2(B_2(0,1)) - m_2(B_2(0,\frac{1}{2}))}{m_2(B_2(0,1))} = \frac{\pi \cdot 1^2 - \pi\left(\frac{1}{2}\right)^2}{\pi \cdot 1^2} = \frac{3}{4}.$$

On the other hand, if (z_j) is a sequence of points in $\mathbb{C}^2$ outside the complex line $\mathbb{C} \times \{0\} \subset \mathbb{C}^2$ converging to the origin, then $PM_{\mathbb{C}^2}u(z_j) = 0$ for every $j = 1, 2, \ldots$, and hence $PM_{\mathbb{C}^2}u$ is not lower semicontinuous at $z = 0$.

However, the signed pluricomplex maximal function of any lower semicontinuous function enjoys the same properties as the signed Hardy–Littlewood maximal function, see [KM00]. These results are presented in the rest of this section.

6.5. Lemma. *Suppose that $u : \Omega \to (-\infty, \infty]$ is a lower semicontinuous function. Then*

(i) *$PM_\Omega u$ is lower semicontinuous, and*
(ii) *$u(z) \leqslant PM_\Omega u(z)$ for every $z \in \Omega$.*

Proof. Since $u : \Omega \to (-\infty, \infty]$ is lower semicontinuous, $PM_\Omega u$ is defined everywhere in Ω. Let $z \in \Omega$ and $w \in \mathbb{C}^n$ be such that $0 < |w| < \operatorname{dist}(z, \partial\Omega)$. Suppose that $(z_j) \subset \Omega$ is a sequence such that $z_j \to z$ as $j \to \infty$. Write $u_0(\lambda) = u(z + \lambda w)$ and $u_j(\lambda) = u(z_j + \lambda w)$ whenever $\lambda \in B_2(0,1)$ and $j > j_0$ for some $j_0 \in \mathbb{N}$. The lower semicontinuity of u implies that for every $\lambda \in B_2(0,1)$

$$u_0(\lambda) = u(z + \lambda w) \leqslant \liminf_{j\to\infty} u(z_j + \lambda w) = \liminf_{j\to\infty} u_j(\lambda),$$

since $z_j + \lambda w \to z + \lambda w$ as $j \to \infty$. Since u is locally bounded below, it follows from Fatou's lemma that

$$\begin{aligned} \fint_{B_2(0,1)} u(z + \lambda w)\, dm_2(\lambda) &= \fint_{B_2(0,1)} u_0(\lambda)\, dm_2(\lambda) \\ &\leqslant \fint_{B_2(0,1)} \liminf_{j\to\infty} u_j(\lambda)\, dm_2(\lambda) \leqslant \liminf_{j\to\infty} \fint_{B_2(0,1)} u_j(\lambda)\, dm_2(\lambda) \\ &= \liminf_{j\to\infty} \fint_{B_2(0,1)} u(z_j + \lambda w)\, dm_2(\lambda) \leqslant \liminf_{j\to\infty} PM_\Omega u(z_j). \end{aligned}$$

This yields $PM_\Omega u(z) \leqslant \liminf_{j\to\infty} PM_\Omega u(z_j)$, and thus $PM_\Omega u$ is lower semicontinuous.

To prove (ii), fix $z \in \Omega$. We can suppose that $u(z) < \infty$. Let $\varepsilon > 0$. Since u is lower semicontinuous, there is $r > 0$ such that $u(z') > u(z) - \varepsilon$ for every $z' \in B(z, r) \subset \Omega$. Hence for every $w \in \mathbb{C}^n$ such that $|w| < r$

$$PM_\Omega u(z) \geqslant \fint_{B_2(0,1)} u(z + \lambda w)\, dm_2(\lambda) > \fint_{B_2(0,1)} (u(z) - \varepsilon)\, dm_2(\lambda) = u(z) - \varepsilon.$$

Because $z \in \Omega$ and $\varepsilon > 0$ were chosen arbitrarily, it follows that $PM_\Omega u(z) \geqslant u(z)$ for every $z \in \Omega$. □

6.6. Remark. If $u : \Omega \to (-\infty, \infty]$ is a lower semicontinuous function, then we know by [KM00, Lemma 2.1 (ii)] and the previous lemma that $u(z) \leqslant M_\Omega u(z)$ and $u(z) \leqslant PM_\Omega u(z)$ for all $z \in \Omega$. However, the signed Hardy–Littlewood maximal function $M_\Omega u$ and the signed pluricomplex maximal function $PM_\Omega u$ are not comparable in the sense that neither $M_\Omega u(z) \leqslant PM_\Omega u(z)$ nor $PM_\Omega u(z) \leqslant M_\Omega u(z)$ holds for all $z \in \Omega$, in general. This is easy to see with the following two examples.

Firstly, suppose that $u : \mathbb{C}^2 \to \{0, 1\}$ is the characteristic function of a domain $\frac{1}{2} < |z| < 1$ in $\mathbb{C}^2$, this is,

$$u(z) = \begin{cases} 1, & \frac{1}{2} < |z| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then u is lower semicontinuos, and

$$M_{\mathbb{C}^2} u(0) = \frac{m(B(0,1)) - m(B(0,\frac{1}{2}))}{m(B(0,1))} = \frac{\frac{1}{2}\pi^2 \cdot 1^4 - \frac{1}{2}\pi^2 \left(\frac{1}{2}\right)^4}{\frac{1}{2}\pi^2 \cdot 1^4} = \frac{15}{16}.$$

Moreover, like the function u in Remark 6.4, our present u satisfies

$$PM_{\mathbb{C}^2} u(0) = \frac{m_2(B_2(0,1)) - m_2(B_2(0,\frac{1}{2}))}{m_2(B_2(0,1))} = \frac{3}{4},$$

and thus $PM_{\mathbb{C}^2} u(0) < M_{\mathbb{C}^2} u(0)$.

On the other hand, we can construct a lower semicontinuous function u in $\mathbb{C}^2$ such that $M_{\mathbb{C}^2} u(z) < PM_{\mathbb{C}^2} u(z)$ for some $z \in \mathbb{C}^2$. Let $\varepsilon > 0$. Define a function $u_\varepsilon : \mathbb{C}^2 \to \{0, 1\}$ as

$$u_\varepsilon(z) = u_\varepsilon(z_1, z_2) = \begin{cases} 1, & \frac{1}{2} < |z| < 1 \text{ and } |z_2| < \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Then u_ε is the characteristic function of an open set which contains the complex interval $\frac{1}{2} < |z_1| < 1$. Obviously, $M_{\mathbb{C}^2} u_\varepsilon(0) \searrow 0$ as $\varepsilon \searrow 0$. However, the signed pluricomplex maximal function again satisfies

$$PM_{\mathbb{C}^2} u_\varepsilon(0) = \frac{m_2(B_2(0,1)) - m_2(B_2(0,\frac{1}{2}))}{m_2(B_2(0,1))} = \frac{3}{4}.$$

Therefore, if $\varepsilon > 0$ is small enough, then $M_{\mathbb{C}^2} u_\varepsilon(0) < PM_{\mathbb{C}^2} u_\varepsilon(0)$.

6.7. Theorem. *Suppose that $u : \Omega \to (-\infty, \infty]$ is a lower semicontinuous function which is not identically ∞ on any component of Ω. Then the following properties hold:*

(i) *u is superharmonic in Ω if and only if $u(z) = M_\Omega u(z)$ for all $z \in \Omega$.*
(ii) *u is plurisuperharmonic in Ω if and only if $u(z) = PM_\Omega u(z)$ for all $z \in \Omega$.*
(iii) *u is superharmonic but not plurisuperharmonic in Ω if and only if $u(z) = M_\Omega u(z)$ for all $z \in \Omega$ and there is at least one point $w \in \Omega$ such that $u(w) < PM_\Omega u(w)$.*

Proof. The property (i) holds by the result [KM00, Lemma 2.2]. The following proof of (ii) follows the idea of the proof of [KM00, Lemma 2.2], but the plurisuper mean value principle 6.2 (iii) is used instead of the usual super mean value principle.

Suppose first that u is plurisuperharmonic. Then by the plurisuper mean value principle 6.2 (iii)

$$u(z) \geqslant \fint_{B_2(0,1)} u(z+\lambda w)\, dm_2(\lambda)$$

for every $z \in \Omega$ and $w \in \mathbb{C}^n$ such that $0 < |w| < \operatorname{dist}(z, \partial\Omega)$. Taking the supremum over all such w we have $u(z) \geqslant PM_\Omega u(z)$ for every $z \in \Omega$. On the other hand, the lower semicontinuity of plurisuperharmonic functions together with Lemma 6.5 (ii) implies that $u(z) \leqslant PM_\Omega u(z)$ for every $z \in \Omega$. Therefore $PM_\Omega u(z) = u(z)$ for every $z \in \Omega$.

Suppose then that $PM_\Omega u(z) = u(z)$ for every $z \in \Omega$. This assumption and the definition of the $PM_\Omega u$ imply that

$$\fint_{B_2(0,1)} u(z+\lambda w)\, dm_2(\lambda) \leqslant PM_\Omega u(z) = u(z)$$

for every $z \in \Omega$ and $w \in \mathbb{C}^n$ such that $0 < |w| < \operatorname{dist}(z, \partial\Omega)$. This is the required plurisuper mean value principle 6.2 (iii) and hence u is plurisuperharmonic.

The property (iii) follows easily from the properties (i) and (ii) together with Lemma 6.5 (ii). □

6.8. Corollary. *Suppose that u and v are plurisuperharmonic functions in Ω. If $u(z) \geqslant v(z)$ for almost every $z \in \Omega$, then $PM_\Omega u(z) \geqslant PM_\Omega v(z)$ for every $z \in \Omega$.*

Proof. It is known [AG01, Corollary 3.2.7] that if u and v are superharmonic (in particular, plurisuperharmonic) and if $u(z) \geqslant v(z)$ for almost every $z \in \Omega$, then $u(z) \geqslant v(z)$ for every $z \in \Omega$. This yields by Theorem 6.7 (ii) that

$$PM_\Omega u(z) = u(z) \geqslant v(z) = PM_\Omega v(z)$$

for every $z \in \Omega$. □

However, Corollary 6.8 is a consequence of a much more general result:

6.9. Lemma. *Suppose that $u : \Omega \to (-\infty, \infty]$ is a Borel function which is locally bounded below and that $v : \Omega \to (-\infty, \infty]$ is a lower semicontinuous function. If $u(z) \geqslant v(z)$ for almost every $z \in \Omega$, then $PM_\Omega u(z) \geqslant PM_\Omega v(z)$ for every $z \in \Omega$.*

Proof. Since v is lower semicontinuous, there is an increasing sequence of continuous functions v_j such that $v_j \to v$ in Ω. Then

$$\lim_{j\to\infty} PM_\Omega v_j(z) = PM_\Omega v(z)$$

for every $z \in \Omega$. For this note that $PM_\Omega v_j(z) \leqslant PM_\Omega v(z)$ for each j, and

$$\lim_{j\to\infty} \fint_{B_2(0,1)} v_j(z+\lambda w)\, dm_2(\lambda) = \fint_{B_2(0,1)} v(z+\lambda w)\, dm_2(\lambda)$$

for each $z \in \Omega$ and $w \in \mathbb{C}^n$ such that $0 < |w| < \operatorname{dist}(z, \partial\Omega)$. Hence for all $z \in \Omega$ the Lebesgue monotone convergence theorem gives the opposite inequality

$$PM_\Omega v(z) \leqslant \lim_{j\to\infty} PM_\Omega v_j(z).$$

To complete the proof, let $z \in \Omega$ and $\varepsilon > 0$. For each $j = 1, 2, \ldots$, pick $w_j \in \mathbb{C}^n$ with $0 < |w_j| < \operatorname{dist}(z, \partial\Omega)$ such that

$$PM_\Omega v_j(z) < \fint_{B_2(0,1)} v_j(z + \lambda w_j)\, dm_2(\lambda) + \varepsilon.$$

Now $u(z) \geqslant v_j(z)$ almost everywhere in Ω, and using the measure on complex lines in $\mathbb{C}^n$ and the properties of the *Haar measure*, see [Fed69], we can choose for each j a complex line $T(w'_j) = \{z + \lambda w'_j : \lambda \in \mathbb{C}\}$ such that

$$m_2(T(w'_j) \cap \{z \in \Omega : u(z) < v_j(z)\}) = 0$$

and that w'_j is arbitrarily close to w_j. Then the continuity of v_j yields

$$\begin{aligned} PM_\Omega u(z) &\geqslant \fint_{B_2(0,1)} u(z + \lambda w'_j)\, dm_2(\lambda) \geqslant \fint_{B_2(0,1)} v_j(z + \lambda w'_j)\, dm_2(\lambda) \\ &\geqslant \fint_{B_2(0,1)} v_j(z + \lambda w_j)\, dm_2(\lambda) - \varepsilon. \end{aligned}$$

Thus

$$PM_\Omega u(z) \geqslant PM_\Omega v_j(z) - 2\varepsilon,$$

and letting $j \to \infty$ and $\varepsilon \to 0$ we obtain the desired result. □

Suppose that a function $u : \Omega \to (-\infty, \infty]$ is locally bounded below. We say that the function $\tilde{u}_{\mathrm{SH}}$ is the *least superharmonic almost everywhere majorant* of u in Ω if

(i) $\tilde{u}_{\mathrm{SH}}$ is superharmonic in Ω,
(ii) $\tilde{u}_{\mathrm{SH}}(z) \geqslant u(z)$ for almost every $z \in \Omega$, and
(iii) if v is another function satisfying (i) and (ii), then $v(z) \geqslant \tilde{u}_{\mathrm{SH}}(z)$ for almost every $z \in \Omega$.

In the same way, the function $\tilde{u}_{\mathrm{PSH}}$ is said to be the *least plurisuperharmonic almost everywhere majorant* of u in Ω if

(i') $\tilde{u}_{\mathrm{PSH}}$ is plurisuperharmonic in Ω,
(ii') $\tilde{u}_{\mathrm{PSH}}(z) \geqslant u(z)$ for almost every $z \in \Omega$, and
(iii') if v is another function satisfying (i') and (ii'), then $v(z) \geqslant \tilde{u}_{\mathrm{PSH}}(z)$ for almost every $z \in \Omega$.

6.10. Remark. The least (pluri)superharmonic almost everywhere majorant doesn't always exist, since a locally bounded below function may have value $+\infty$ in a set of positive Lebesgue measure. On the other hand, we know that every (pluri)superharmonic function can have value $+\infty$ only in a (pluri)polar set. Hence the conditions (ii) and (ii') are not always satisfied with any (pluri)superharmonic function. However, this problem is disregarded here with a convention that we allow (pluri)superharmonic functions be identically $+\infty$ in Ω. To be more precise, if $\tilde{u}_{\mathrm{PSH}}$ (resp. $\tilde{u}_{\mathrm{SH}}$) doesn't exist, then we set $\tilde{u}_{\mathrm{PSH}} \equiv +\infty$ (resp. $\tilde{u}_{\mathrm{SH}} \equiv +\infty$). Of course, this has to be understood in the components of Ω separately. (Pluri)superharmonic functions that are allowed to be identically $+\infty$ are sometimes called *(pluri)hyperharmonic*.

Secondly, the inequality in (iii') actually holds everywhere in Ω, because if v satisfies (i') and (ii'), then the condition $v(z) \geqslant \tilde{u}_{\mathrm{PSH}}(z)$ for almost every $z \in \Omega$ implies that $v(z) \geqslant \tilde{u}_{\mathrm{PSH}}(z)$ for every $z \in \Omega$ as v and $\tilde{u}_{\mathrm{PSH}}$ are plurisuperharmonic.

The same is true in the superharmonic case for the inequality in (iii), this is, $v(z) \geqslant \tilde{u}_{\mathrm{SH}}(z)$ for every $z \in \Omega$ [KM00, Remark 2.4]. From this it follows that $\tilde{u}_{\mathrm{PSH}}(z) \geqslant \tilde{u}_{\mathrm{SH}}(z)$ for all $z \in \Omega$, because the least plurisuperharmonic almost everywhere majorant $\tilde{u}_{\mathrm{PSH}}$ satisfies the conditions (i) and (ii) of the least superharmonic almost everywhere majorant of u.

We show that the least plurisuperharmonic almost everywhere majorant can be constructed by iterating the signed pluricomplex maximal function; the similar construction is known to be possible for the least superharmonic almost everywhere majorant by iterating the signed Hardy–Littlewood maximal function [KM00, Theorem 2.5].

Let $u : \Omega \to (-\infty, \infty]$ be a lower semicontinuous function. We write

$$PM_\Omega^{(k)} u(z) = \underbrace{PM_\Omega \circ PM_\Omega \circ \cdots \circ PM_\Omega}_{k \text{ times}} u(z), \quad k = 1, 2, \ldots .$$

Since the maximal functions $PM_\Omega^{(k)} u$, $k = 1, 2, \ldots$, are lower semicontinuous, we see by using Lemma 6.5 (ii) that

$$PM_\Omega^{(k)} u(z) \leqslant PM_\Omega^{(k+1)} u(z), \quad k = 1, 2, \ldots ,$$

for every $z \in \Omega$. Hence $(PM_\Omega^{(k)} u(z))_k$ is an increasing sequence of functions and it converges for every $z \in \Omega$ (the limit may be ∞). Moreover, we denote

$$PM_\Omega^{(\infty)} u(z) = \lim_{k \to \infty} PM_\Omega^{(k)} u(z)$$

for every $z \in \Omega$.

The following two theorems combine the concepts of the signed pluricomplex maximal function, the least plurisuperharmonic almost everywhere majorant and the regularized reduced function defined in Section 3.

6.11. Theorem. *Suppose that $u : \Omega \to (-\infty, \infty]$ is a lower semicontinuous function which is not identically ∞ on any component of Ω. Then*

$$PM_\Omega^{(\infty)} u(z) = \tilde{u}_{\mathrm{PSH}}(z) \tag{6.12}$$

for every $z \in \Omega$.

Proof. This proof follows the idea of the proof of [KM00, Theorem 2.5]. We show that

$$PM_\Omega^{(\infty)} u(z) = PM_\Omega PM_\Omega^{(\infty)} u(z)$$

for every $z \in \Omega$. Because $\{PM_\Omega^{(k)} u\}_k$ is an increasing sequence of lower semicontinuous functions, the limit is lower semicontinuous. Hence we have by Lemma 6.5 (ii) that

$$PM_\Omega^{(\infty)} u(z) \leqslant PM_\Omega PM_\Omega^{(\infty)} u(z)$$

for every $z \in \Omega$.

Let $z \in \Omega$ be fixed. For all $w \in \mathbb{C}^n$ such that $0 < |w| < \operatorname{dist}(z, \partial\Omega)$, we have by the Lebesgue monotone convergence theorem and the definition of the plurimaximal funcion that

$$\begin{aligned}\fint_{B_2(0,1)} PM_\Omega^{(\infty)} u(z + \lambda w)\, dm_2(\lambda) &= \lim_{k\to\infty} \fint_{B_2(0,1)} PM_\Omega^{(k)} u(z + \lambda w)\, dm_2(\lambda) \\ &\leqslant \lim_{k\to\infty} PM_\Omega^{(k+1)} u(z) = PM_\Omega^{(\infty)} u(z),\end{aligned}$$

because all lower semicontinuous functions are locally bounded below. Taking the supremum over all $w \in \mathbb{C}^n$ such that $0 < |w| < \operatorname{dist}(z, \partial\Omega)$ on the left hand side we obtain

$$PM_\Omega PM_\Omega^{(\infty)} u(z) \leqslant PM_\Omega^{(\infty)} u(z)$$

for every $z \in \Omega$. By Theorem 6.7 (ii), $PM_\Omega^{(\infty)} u$ is plurisuperharmonic in Ω.

Suppose then that v is a plurisuperharmonic function in Ω such that $u(z) \leqslant v(z)$ for almost every $z \in \Omega$. Then by Lemma 6.9 we have $PM_\Omega u(z) \leqslant PM_\Omega v(z)$ for every $z \in \Omega$. By Lemma 6.5 (i) and Theorem 6.7 (ii), $PM_\Omega u$ is lower semicontinuous in Ω and $PM_\Omega v$ is plurisuperharmonic in Ω, and thus by induction we see that $PM_\Omega^{(k)} u(z) \leqslant PM_\Omega^{(k)} v(z)$ for every $k = 1, 2, \ldots$ and $z \in \Omega$. Hence

$$PM_\Omega^{(\infty)} u(z) = \lim_{k\to\infty} PM_\Omega^{(k)} u(z) \leqslant \lim_{k\to\infty} PM_\Omega^{(k)} v(z) = v(z)$$

for every $z \in \Omega$, because by Theorem 6.7 (ii) we see that $PM_\Omega^{(k)} v(z) = v(z)$ for every $k = 1, 2, \ldots$ and every $z \in \Omega$. This completes the proof. □

6.13. Theorem. *Suppose that $u : \Omega \to (-\infty, \infty]$ is a lower semicontinuous function. Then*

$$PM_\Omega^{(\infty)} u(z) = \hat{R}^u_{\mathrm{PSH}}(z) \tag{6.14}$$

for every $z \in \Omega$.

Proof. This proof follows the idea of the proof of [KM00, Proposition 2.7]. The regularized reduced function $\hat{R}^u_{\mathrm{PSH}}$ is plurisuperharmonic in Ω and $\hat{R}^u_{\mathrm{PSH}}(z) \geqslant u(z)$ for almost every $z \in \Omega$ by (3.7). Hence Theorem 6.11 and the definition of the least plurisuperharmonic almost everywhere majorant of u together with Remark 6.10 imply that

$$\hat{R}^u_{\mathrm{PSH}}(z) \geqslant \tilde{u}_{\mathrm{PSH}}(z) = PM_\Omega^{(\infty)} u(z)$$

for every $z \in \Omega$.

Since $u(z) \leqslant PM_\Omega u(z)$ for every $z \in \Omega$, we see by Theorem 6.7 (ii) that

$$u(z) \leqslant PM_\Omega^{(k)} u(z), \quad k = 1, 2, 3, \ldots,$$

for every $z \in \Omega$ and consequently $PM_\Omega^{(\infty)} u(z) \geqslant u(z)$ for every $z \in \Omega$. This implies that $PM_\Omega^{(\infty)} u \in \Phi^u_{\mathrm{PSH}}(\Omega)$ where $\Phi^u_{\mathrm{PSH}}(\Omega)$ is defined by (3.3), and hence

$$\hat{R}^u_{\mathrm{PSH}}(z) \leqslant PM_\Omega^{(\infty)} u(z)$$

for every $z \in \Omega$. We have proved that

$$PM_\Omega^{(\infty)} u(z) = \hat{R}^u_{\mathrm{PSH}}(z)$$

for every $z \in \Omega$. □

The least superharmonic almost everywhere majorant $\tilde{u}_{\mathrm{SH}}$ and the usual superharmonic balayage $\hat{R}^u_{\mathrm{SH}}$ satisfy the corresponding properties

$$M^{(\infty)}_\Omega u(z) = \tilde{u}_{\mathrm{SH}}(z) \quad \text{and} \quad M^{(\infty)}_\Omega u(z) = \hat{R}^u_{\mathrm{SH}}(z)$$

for every $z \in \Omega$, see [KM00, Theorem 2.5 and Proposition 2.7]. The function $M^{(\infty)}_\Omega u$ is defined iteratively like $PM^{(\infty)}_\Omega u$.

Part 2. Capacities in the plane

This part is devoted to the study of capacities in the complex plane, that is, in $\mathbb{C} = \mathbb{R}^2$. This approach based on classical methods is supposed to describe the relationship between the Monge–Ampère capacity and some classical capacities in potential theory. Of course, this relationship has been on the background when the concept of the Monge–Ampère capacity was developed.

7. Potential theoretic preliminaries

We derive first a well-known fact that the complex Monge–Ampère operator is essentially just the Laplace operator. This holds only in the plane, and in the higher dimensional spaces the relation between these operators is much more complicated. If $u \in \mathrm{C}^2(\Omega)$, then a straightforward calculation gives

$$\begin{aligned} dd^c u &= 2i\partial\overline{\partial}u = 2i\,\frac{\partial^2 u}{\partial z \partial \overline{z}}\, dz \wedge d\overline{z} = 2i\,\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)(-2i\,dx \wedge dy) \\ &= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx \wedge dy = \Delta u\, dV. \end{aligned}$$

The *generalized* (or *distributional*) *Laplacian* Δu of an arbitrary superharmonic function u in Ω is defined by the formula

$$\int_\Omega \varphi \Delta u = \int_\Omega u \Delta\varphi\, dm, \tag{7.1}$$

where $\varphi \in \mathrm{C}^\infty_0(\Omega)$ is a nonnegative test function. If u is not smooth enough (not a C^2-function), the right-hand side of (7.1) still makes sense, because all superharmonic functions are locally integrable. If u is a superharmonic function in Ω, then $-\Delta u$ is known to be a Radon measure on Ω. In the plane the family of plurisuperharmonic functions equals to the family of superharmonic functions. Therefore, if K is a compact subset of Ω in the plane, then the Monge–Ampère capacity of the condenser (K, Ω) can be defined as

$$\operatorname{cap}(K, \Omega) = \sup_{\substack{u \in \mathrm{SH}(\Omega) \\ 0 \leqslant u \leqslant 1}} \int_K -\Delta u. \tag{7.2}$$

We set

$$\mathrm{W}(K, \Omega) = \{u \in \mathrm{C}^\infty_0(\Omega) : u \geqslant 1 \text{ on } K\}$$

and

$$W_0(K,\Omega) = \{u \in W_0^{1,p}(\Omega) \cap C(\Omega) : u \geqslant 1 \text{ on } K\},$$

where $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the *Sobolev space* $W^{1,p}(\Omega)$, $1 < p < \infty$. Recall that a function u belongs to $W^{1,p}(\Omega)$ if $u \in L^p(\Omega)$ and the weak first order partial derivatives of u belong also to $L^p(\Omega)$.

Definition. Let K be a compact subset of Ω. The *(variational) p-capacity* of the condenser (K,Ω) is defined by

$$\operatorname{cap}_p(K,\Omega) = \inf_{u\in W(K,\Omega)} \int_\Omega |\nabla u|^p \, dm. \tag{7.3}$$

If $U \subset \Omega$ is open, then

$$\operatorname{cap}_p(U,\Omega) = \sup_{\substack{K\subset U \\ \text{is compact}}} \operatorname{cap}_p(K,\Omega), \tag{7.4}$$

and for an arbitrary set $E \subset \Omega$

$$\operatorname{cap}_p(E,\Omega) = \inf_{\substack{E\subset U\subset\Omega \\ U \text{ is open}}} \operatorname{cap}_p(U,\Omega). \tag{7.5}$$

It is well-known that the set $W(K,\Omega)$ of admissible functions in (7.3) can be replaced by $W_0(K,\Omega)$. For more details of the p-capacity, see [HKM93].

A *potential* is some integral operator acting on a space of signed measures. Such an operator is defined by a *kernel* $K(z,w)$ which is usually dependent only on the distance $|z - w|$. The importance of superharmonic functions is, roughly speaking, that every superharmonic function can be represented as the sum of a potential and of a harmonic function. We shall study Green potentials and Green kernels associated with domains in the plane. Historically, Green potentials were preceded by logarithmic potentials in the plane and Newton potentials in the higher dimensional spaces, that is, in $\mathbb{R}^m$ where $m \geqslant 3$. Recently, M. Carlehed [Car99] has studied pluricomplex potentials in $\mathbb{C}^n$ with general dimensions, not only in the plane.

Recall that if $y \in \mathbb{R}^m$, $m \geqslant 2$, then the function

$$u_y(x) = \begin{cases} -\log|x-y|, & x \neq y,\ m = 2, \\ |x-y|^{2-m}, & x \neq y,\ m \geqslant 3, \\ +\infty, & x = y, \end{cases} \tag{7.6}$$

is harmonic in $\mathbb{R}^m \setminus \{y\}$ and superharmonic in $\mathbb{R}^m$, see [AG01, Corollary 3.2.10]. Let Ω be an open subset of $\mathbb{R}^m$. The *Green function* for Ω with pole at y is a function $g_\Omega : \Omega \times \Omega \to [0,+\infty]$ defined so that $g_\Omega(\cdot, y) = u_y - h_y$ where h_y is the greatest harmonic minorant of u_y on Ω, see [AG01, Section 4.1]. By the extended maximum principle for harmonic functions [AG01, Theorem 1.2.4], each set Ω has at most one Green function with a given pole. If the Green function for an open set Ω exists, then Ω is said to be *Greenian*.

Suppose now that Ω is an open set in the plane $\mathbb{C}$. By the preceding definition, the Green function for Ω with pole at w is the function $g_\Omega(z,w) = u_w(z) + h_w(z)$ where $u_w(z) =$

$-\log|z-w|$ if $z \neq w$, and $u_w(z) = +\infty$ if $z = w$. A domain Ω in the plane is Greenian if and only if $\partial\Omega$ is nonpolar [Ran95, Theorem 4.4.2]. Moreover, it is known that if $\partial\Omega$ is polar, then Ω is necessarily unbounded. This is because the boundary of a bounded open set contains always a continuum that cannot be polar. A non-Greenian open set in the plane is therefore always unbounded. Conversely, every bounded open set in the plane is necessarily Greenian.

Definition. Let Ω be a Greenian set, and let μ be a Radon measure on Ω. Then the *Green potential* of μ in Ω is defined by

$$U^\mu_\Omega(z) = \frac{1}{2\pi}\int_\Omega g_\Omega(z,w)\,d\mu(w) \qquad (z \in \Omega), \tag{7.7}$$

and the *Green energy* of μ in Ω is given by

$$I_\Omega(\mu) = \frac{1}{2\pi}\int_\Omega\int_\Omega g_\Omega(z,w)\,d\mu(w)\,d\mu(z) = \int_\Omega U^\mu_\Omega(z)\,d\mu(z). \tag{7.8}$$

If μ and ν are two Radon measures on Ω, then the *mutual Green energy* of μ and ν in Ω is

$$I_\Omega(\mu,\nu) = \frac{1}{2\pi}\int_\Omega\int_\Omega g_\Omega(z,w)\,d\mu(w)\,d\nu(z) = \int_\Omega U^\mu_\Omega(z)\,d\nu(z). \tag{7.9}$$

It is clear that $I_\Omega(\mu,\mu) = I_\Omega(\mu)$. Note that in the literature, the Green energy of a Radon measure μ on Ω is denoted also by $\|\mu\|^2$. If Ω is a Greenian domain and if μ is a Radon measure in Ω, then U^μ_Ω is nonnegative and either superharmonic or identically $+\infty$ in Ω [Hel69, Lemma 6.1].

7.10. Maria–Frostman domination principle. [Hel69, Theorem 8.43] *Let Ω be a Greenian set, and let μ be a Radon measure on Ω such that U^μ_Ω is finite. If u is a positive superharmonic function on Ω such that $u \geqslant U^\mu_\Omega$ quasieverywhere on* $\operatorname{spt}\mu$*, then $u \geqslant U^\mu_\Omega$ on Ω.*

7.11. Reciprocity theorem. [Hel69, Theorem 6.14] *Let Ω be a Greenian set. If μ and ν are two Radon measures on Ω, then the mutual energy of μ and ν in Ω satisfies*

$$I_\Omega(\mu,\nu) = \int_\Omega U^\mu_\Omega\,d\nu = \int_\Omega U^\nu_\Omega\,d\mu = I_\Omega(\nu,\mu). \tag{7.12}$$

Definition. Let Ω be a Greenian set. If K is a compact subset of Ω, then the unique Radon measure μ_K on Ω for which $\hat{R}^{1,K}_{\mathrm{SH}} = U^{\mu_K}_\Omega$ is called the *capacitary measure* for K in Ω. The *Green capacity* of K relative to Ω is defined by

$$C_\Omega(K) = \mu_K(K). \tag{7.13}$$

The balayage $\hat{R}^{1,K}_{\mathrm{SH}}$ is called the *capacitary function* for K in Ω.

Let u be an arbitrary superharmonic function in an open subset Ω of $\mathbb{C}^n$. Then $-\Delta u$ is a Radon measure on Ω, but it is not necessarily compactly supported in Ω and it can be infinite for some noncompact subset of Ω. But if both Ω and u are bounded, then $-\Delta u$ is a finite Radon measure on Ω. In general, every Radon measure on Ω with compact support is finite in Ω. The converse is not true because a finite Radon measure is not necessarily compactly supported in Ω.

Let Ω be a Greenian set in the plane. If μ is a finite Radon measure on Ω, then

$$-\Delta U_\Omega^\mu = \mu \quad \text{on } \Omega, \tag{7.14}$$

this means that $-\Delta U_\Omega^\mu$ and μ give the same measure on Ω [AG01, Theorem 4.3.8]. Moreover, if μ is a Radon measure on Ω with compact support, then

$$I_\Omega(\mu) = \int_\Omega |\nabla U_\Omega^\mu(z)|^2 \, dm(z) \tag{7.15}$$

provided that $I_\Omega(\mu) < \infty$, see [Lan72].

8. Connections between capacities in the plane

Let E be a subset of Ω. If Ω is Greenian, then we set

$$\begin{aligned}\mathfrak{M}^+(E,\Omega) = \{\mu : \mu &\text{ is a Radon measure on } \Omega \text{ such that } \operatorname{spt}\mu \subset E \text{ and} \\ &\text{that } U_\Omega^\mu \leqslant 1\}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{M}^+(E,\Omega)^* = \{\mu : \mu &\text{ is a Radon measure on } \Omega \text{ such that } \operatorname{spt}\mu \subset E \text{ and} \\ &\text{that } U_\Omega^\mu \geqslant 1 \text{ quasieverywhere on } E\}.\end{aligned}$$

If Ω is not Greenian, then we set $\mathfrak{M}^+(E,\Omega) = \mathfrak{M}^+(E,\Omega)^* = \emptyset$. In one sense, $\mathfrak{M}^+(E,\Omega)^*$ is the complement of $\mathfrak{M}^+(E,\Omega)$ in the space of Radon measures on Ω such that $\operatorname{spt}\mu \subset E$. However, $\mathfrak{M}^+(E,\Omega) \cap \mathfrak{M}^+(E,\Omega)^*$ can be nonempty, and hence it is not exactly correct to say that $\mathfrak{M}^+(E,\Omega)^*$ is the complement of $\mathfrak{M}^+(E,\Omega)$.

8.1. Theorem. *Let Ω be a Greenian set in the plane. If K is a compact set in Ω, then* $\operatorname{cap}(K,\Omega) = C_\Omega(K)$.

Proof. Let K be a compact set in Ω. Denote

$$\tilde{U}_\Omega^\mu(z) = \int_\Omega g(z,w)\, d\mu(w), \qquad z \in \Omega,$$

where μ is any Radon measure on Ω. Then $\hat{R}_{\mathrm{SH}}^{1,K} = \frac{1}{2\pi}\tilde{U}_\Omega^{\mu_K}$, see [Hel69, pp. 137–138]. It holds by [AG01, Theorem 4.3.8] that

$$-\Delta U_\Omega^{\mu_K} = \frac{1}{2\pi}\left(-\Delta \tilde{U}_\Omega^{\mu_K}\right) = \mu_K, \tag{8.2}$$

Moreover, by [Hel69, Lemma 7.19]

$$C_\Omega(K) = \sup\{\mu(K) : U_\Omega^\mu \leqslant 1,\ \mu \text{ is a Radon measure on } \Omega \text{ with } \operatorname{spt}\mu \subset K\}.$$

Let $\mu \in \mathfrak{M}^+(K,\Omega)$ be arbitrary. Since U_Ω^μ is superharmonic in Ω and $0 \leqslant U_\Omega^\mu \leqslant 1$, it follows from the definition of the Monge–Ampére capacity that

$$\left(-\Delta U_\Omega^\mu\right)(K) \leqslant \operatorname{cap}(K,\Omega).$$

This is equivalent to

$$\frac{1}{2\pi}\left(-\Delta\tilde{U}^{\mu}_{\Omega}\right)(K) \leqslant \operatorname{cap}(K,\Omega),$$

and (8.2) gives

$$\mu(K) \leqslant \operatorname{cap}(K,\Omega).$$

Hence

$$C_{\Omega}(K) \leqslant \operatorname{cap}(K,\Omega).$$

It remains to show that $\operatorname{cap}(K,\Omega) \leqslant C_{\Omega}(K)$. Suppose that u is superharmonic in Ω with $0 \leqslant u \leqslant 1$. Riesz decomposition theorem, see [AG01, Theorem 4.4.1], yields

$$u(z) = \tilde{U}^{\mu}_{\Omega}(z) + h(z), \quad z \in \Omega,$$

where $\mu = -\frac{1}{2\pi}\Delta u$ and h is harmonic in Ω (more precisely, h is the greatest harmonic minorant of u in Ω). It follows that

$$-\Delta u = -\Delta\tilde{U}^{\mu}_{\Omega} = 2\pi\mu,$$

and thus

$$(-\Delta u)(K) = 2\pi\mu(K).$$

Suppose now that $\mu_1 = 2\pi\mu$. Then

$$\tilde{U}^{\mu_1}_{\Omega}(z) = \int_{\Omega} g_{\Omega}(z,w)\,d\mu_1(w) = 2\pi\int_{\Omega} g_{\Omega}(z,w)\,d\mu(w) = 2\pi\tilde{U}^{\mu}_{\Omega}(z).$$

Since $0 \leqslant \tilde{U}^{\mu}_{\Omega} \leqslant 1$, we have $0 \leqslant \tilde{U}^{\mu_1}_{\Omega} \leqslant 2\pi$, and hence $0 \leqslant \frac{1}{2\pi}\tilde{U}^{\mu_1}_{\Omega} \leqslant 1$ which implies $0 \leqslant U^{\mu_1}_{\Omega} \leqslant 1$. Define now $\mu_2 = \mu_1|_K$. Then $\mu_2(K) = \mu_1(K)$, $\operatorname{spt}\mu_2 \subset K$ and $0 \leqslant U^{\mu_2}_{\Omega} \leqslant 1$, because

$$0 \leqslant U^{\mu_2}_{\Omega} \leqslant U^{\mu_1}_{\Omega} \leqslant 1.$$

We have obtained

$$(-\Delta u)(K) = \int_K (-\Delta u) = 2\pi\mu(K) = \mu_1(K) = \mu_2(K) \leqslant C_{\Omega}(K),$$

and hence

$$\operatorname{cap}(K,\Omega) \leqslant C_{\Omega}(K).$$

□

8.3. Lemma. *Let Ω be an open set in the plane. If K is a compact subset of Ω, then*

$$\operatorname{cap}(K,\Omega) = \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} \mu(K) = \inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} \mu(K).$$

Proof. We can suppose that Ω is connected. If Ω is not Greenian, then $\mathfrak{M}^+(K,\Omega) = \mathfrak{M}^+(K,\Omega)^* = \emptyset$ (by the definition) and

$$\sup_{\mu\in\mathfrak{M}^+(K,\Omega)} \mu(K) = \inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} \mu(K) = 0$$

trivially. On the other hand, all bounded superharmonic functions in Ω are now constant (Extended maximum principle for superharmonic functions [Ran95, Theorem 3.6.9]), which implies again that

$$\operatorname{cap}(K,\Omega) = \sup_{\substack{u\in \mathrm{SH}(\Omega)\\ 0\leqslant u\leqslant 1}} \int_K -\Delta u = 0.$$

Hence we can suppose that Ω is Greenian. Then the previous Theorem 8.1 together with the details in its proof yields

$$\operatorname{cap}(K,\Omega) = \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} \mu(K).$$

Let $\mu_1 \in \mathfrak{M}^+(K,\Omega)$ and $\mu_2 \in \mathfrak{M}^+(K,\Omega)^*$. Suppose that $U \subset \Omega$ is an open set such that $K \subset U \subset \overline{U} \subset \Omega$. Then $\hat{R}^{1,\overline{U}}_{\mathrm{SH}}$ is the Green potential of a Radon measure ν_0 on Ω such that $\operatorname{spt}\nu_0 \subset \overline{U}$. Note that $U^{\nu_0}_\Omega = \hat{R}^{1,\overline{U}}_{\mathrm{SH}} \equiv 1$ on K and that both μ_1 and μ_2 are supported on K. Maria–Frostman domination principle 7.10 implies that $U^{\mu_2}_\Omega \geqslant U^{\mu_1}_\Omega \geqslant 0$ on the whole Ω. This yields by Reciprocity theorem 7.11 that

$$\begin{aligned}\mu_1(K) &= \int_K d\mu_1 = \int_\Omega U^{\nu_0}_\Omega\, d\mu_1 = \int_\Omega U^{\mu_1}_\Omega\, d\nu_0\\ &\leqslant \int_\Omega U^{\mu_2}_\Omega\, d\nu_0 = \int_\Omega U^{\nu_0}_\Omega\, d\mu_2 = \int_K d\mu_2 = \mu_2(K).\end{aligned}$$

Therefore

$$\sup_{\mu\in\mathfrak{M}^+(K,\Omega)} \mu(K) \leqslant \inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} \mu(K).$$

On the other hand, we have seen that there is a Radon measure ν on Ω such that $U^\nu_\Omega = \hat{R}^{1,K}_{\mathrm{SH}}$. Now $\nu \in \mathfrak{M}^+(K,\Omega)\cap\mathfrak{M}^+(K,\Omega)^*$, see [Hel69, Theorem 7.39], and this implies that

$$\inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} \mu(K) \leqslant \nu(K) \leqslant \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} \mu(K).$$

We have obtained that

$$\inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} \mu(K) = \nu(K) = \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} \mu(K),$$

and the theorem is proved. □

8.4. Lemma. *Let Ω be an open set in the plane. If K is a compact subset of Ω, then*

$$\operatorname{cap}(K,\Omega) = \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} I_\Omega(\mu) = \inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} I_\Omega(\mu).$$

Proof. We can suppose that Ω is Greenian. Suppose that $U \Subset \Omega$ is an open set such that $K \subset U$. Let Radon measures ν and ν_0 be given so that $U^\nu_\Omega = \hat{R}^{1,K}_{\mathrm{SH}}$, $\operatorname{spt}\nu \subset K$, $U^{\nu_0}_\Omega = \hat{R}^{1,\overline{U}}_{\mathrm{SH}}$ and $\operatorname{spt}\nu_0 \subset \overline{U}$. In addition, let $\mu_1 \in \mathfrak{M}^+(K,\Omega)$ and $\mu_2 \in \mathfrak{M}^+(K,\Omega)^*$ be given. Then

$$I_\Omega(\mu_1) = \int_\Omega U^{\mu_1}_\Omega\, d\mu_1 \leqslant \int_K d\mu_1 = \mu_1(K).$$

Write

$$F = \{z \in \operatorname{spt}\mu_2 : U_\Omega^{\mu_2}(z) < 1\},$$

then F is polar. Now

$$\begin{aligned} I_\Omega(\mu_2) &= \int_\Omega U_\Omega^{\mu_2}\, d\mu_2 = \int_K U_\Omega^{\mu_2}\, d\mu_2 = \int_{K\setminus F} U_\Omega^{\mu_2}\, d\mu_2 + \int_{K\cap F} U_\Omega^{\mu_2}\, d\mu_2 \\ &\geqslant \int_{K\setminus F} d\mu_2 + 0 = \mu_2(K \setminus F) = \mu_2(K), \end{aligned}$$

since $\mu_2(F) = 0$ by [Hel69, Theorem 11.2] and

$$\int_\Omega U_\Omega^{\mu_2}\, d\mu_2 = \int_K U_\Omega^{\mu_2}\, d\mu_2 < \infty,$$

because $U_\Omega^{\mu_2} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Thus

$$\begin{aligned} \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} I_\Omega(\mu) &\leqslant \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} \mu(K) = \operatorname{cap}(K,\Omega) \\ &= \inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} \mu(K) \leqslant \inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} I_\Omega(\mu), \end{aligned}$$

where the equalities follow from the previous Lemma 8.3. Furthermore, $\nu \in \mathfrak{M}^+(K,\Omega) \cap \mathfrak{M}^+(K,\Omega)^*$, and this implies that

$$\inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} I_\Omega(\mu) \leqslant I_\Omega(\nu) \leqslant \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} I_\Omega(\mu). \tag{8.5}$$

Hence

$$\inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} I_\Omega(\mu) = \operatorname{cap}(K,\Omega) = \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} I_\Omega(\mu). \tag{8.6}$$

□

8.7. Remark. The formulas (8.5) and (8.6) imply that the Radon measure ν is the *Green equilibrium measure* for K in Ω, that is,

$$I_\Omega(\nu) = \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} I_\Omega(\mu). \tag{8.8}$$

8.9. Lemma. *Let Ω be an open set in the plane. If K is a compact subset of Ω, then*

$$\operatorname{cap}(K,\Omega) = \sup_{\mu\in\mathfrak{M}^+(K,\Omega)} \int_\Omega |\nabla U_\Omega^\mu(z)|^2\, dm(z) = \inf_{\mu\in\mathfrak{M}^+(K,\Omega)^*} \int_\Omega |\nabla U_\Omega^\mu(z)|^2\, dm(z).$$

Proof. Follows from Lemma 8.4 and equation (7.15). □

What is then the connection between the Monge–Ampère capacity $\operatorname{cap}(K,\Omega)$ and the 2-capacity $\operatorname{cap}_2(K,\Omega)$ in the plane? One important observation here is that the Green potentials of Radon measures on Ω with finite energy belong to $\mathrm{W}^{1,2}_0(\Omega)$, see [Tre75, Proposition 30.8]. Our answer to the question stated above is as follows:

8.10. Theorem. *Let Ω be a bounded open set in the plane. If K is a compact subset of Ω, then*

$$\operatorname{cap}(K,\Omega) = \operatorname{cap}_2(K,\Omega). \tag{8.11}$$

Proof. Note that Ω is Greenian since Ω is supposed to be bounded. Suppose that ν is the Green equilibrium measure for K in Ω, that is, a Radon measure on Ω such that $\hat{R}^{1,K}_{\mathrm{SH}} = U^{\nu}_{\Omega}$. By [HKM93, Theorem 8.6], we have

$$\operatorname{cap}_2(K,\Omega) = \int_{\Omega} |\nabla \hat{R}^{1,K}_{\mathrm{SH}}|^2 \, dm,$$

and hence Lemma 8.4 and the equations (7.15) and (8.8) yield

$$\operatorname{cap}(K,\Omega) = I_{\Omega}(\nu) = \int_{\Omega} |\nabla U^{\nu}_{\Omega}(z)|^2 \, dm(z) = \int_{\Omega} |\nabla \hat{R}^{1,K}_{\mathrm{SH}}(z)|^2 \, dm(z) = \operatorname{cap}_2(K,\Omega).$$

□

8.12. Remark. We have shown that

$$\operatorname{cap}(K,\Omega) = C_{\Omega}(K) = \operatorname{cap}_2(K,\Omega)$$

whenever K is a compact subset of an open set Ω in the plane. Strictly speaking, $C_{\Omega}(K)$ is defined only for the Greenian sets Ω. However, we could set

$$C_{\Omega}(K) = \sup_{\mu \in \mathfrak{M}^+(K,\Omega)} \mu(K)$$

as a definition of the Green capacity, and then the previous chain of equations holds in the plane without any restrictions.

Secondly, the classical approach in this section holds also in the higher dimensional spaces $\mathbb{R}^m$, $m \geqslant 3$. This is because we didn't use any plane analytical methods here. Therefore, the given arguments show that the Green capacity $C_{\Omega}(K)$ is equal to the 2-capacity $\operatorname{cap}_2(K,\Omega)$ whenever Ω is an open set in $\mathbb{R}^m$ and K is a compact subset of Ω. However, the same is not true for the Monge–Ampère capacity $\operatorname{cap}(K,\Omega)$ whenever Ω is a subset of $\mathbb{C}^n$, $n \geqslant 2$, because the distributional considerations applied here are no longer valid with plurisuperharmonic functions.

Part 3. Holomorphic mappings

The first section of this part concerns discrete and open mappings, the second section arbitrary holomorphic mappings and the third section concentrates on proper holomorphic mappings. Integral transformation formulas for holomorphic mappings are considered in the section before the last one. These formulas serve the main goal of this work, capacity inequalities for holomorphic mappings which are presented in the last section.

Holomorphic mappings are the main objects in the study of several complex variables. On the other hand, proper ones are quite general holomorphic mappings, but they have some important advantages in relation to potential theory. We frequently use the well-known fact that proper holomorphic mappings are discrete and open.

9. Discrete and open mappings

For the known results of discrete and open mappings we refer to the articles by O. Martio, S. Rickman and J. Väisälä [MRV69, MRV70, MRV71]. Here we only recall the basic terminology connected to these mappings.

Definition. Let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be

(a) *discrete* if $f^{-1}(y)$ consists of isolated points in X for every $y \in Y$,
(b) *open* if $f(A)$ is open for every open $A \subset X$, and
(c) *closed* if $f(A)$ is closed for every closed $A \subset X$.

In this section the term 'f is open' includes the assumption that f is also continuous.

Let Ω be an open subset of $\mathbb{R}^m$, $m \geqslant 2$, and let $f : \Omega \to \mathbb{R}^m$ be a continuous mapping. A point $x \in \Omega$ is called a *branch point* of f if f is not a homeomorphism in any neighbourhood of x; the *branch set*, denoted by B_f, is the collection of all branch points of f. The *real Jacobian* of a C^1-mapping $f : \Omega \to \mathbb{R}^m$ is defined by

$$\mathrm{J}_{\mathbb{R}} f = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix}.$$

The *topological degree* $\mu(y, f, D)$ *of* f *at* y is defined whenever $D \Subset \Omega$ is a domain and $y \in \mathbb{R}^m \setminus f(\partial D)$. The degree $\mu(y, f, D)$ is integer valued and it has the following properties:

(i) $\mu(y, f, D)$ is constant in each component of $\mathbb{R}^m \setminus f(\partial D)$.
(ii) If $y \in f(D)$ and the restriction of f to $\overline{D}$ is one-to-one, then $|\mu(y, f, D)| = 1$.
(iii) If $y \in D$ and id is the identity mapping, then $\mu(y, \mathrm{id}, D) = 1$.
(iv) If $\mu(y, f, D_i)$ is defined for all $i = 1, \dots, k$ and if $D_1, \dots, D_k$ are mutually disjoint domains such that $f^{-1}(y) \cap D \subset \bigcup_{i=1}^{k} D_i \subset D$, then

$$\mu(y, f, D) = \sum_{i=1}^{k} \mu(y, f, D_i).$$

(v) If f ja g are connected with a homotopy h_t, $0 \leqslant t \leqslant 1$, such that $\mu(y, h_t, D)$ is defined for $0 \leqslant t \leqslant 1$, then $\mu(y, f, D) = \mu(y, g, D)$.

The standard reference for the topological degree is the monograph by T. Radó and P. V. Reichelderfer [RR55]. If Ω is a domain in $\mathbb{R}^m$ and if for all domains $D \Subset \Omega$ and $y \in f(D) \setminus f(\partial D)$ we have $\mu(y, f, D) > 0$, then f is called *sense-preserving*. If $\mu(y, f, D) < 0$ for all such y and D, then f is called *sense-reversing*. This characterization of the orientation is an extension of the more familiar case when f is C^1-differentiable. Then the orientation is usually defined by means of the sign of the real Jacobian $\mathrm{J}_{\mathbb{R}} f$.

A domain $D \Subset \Omega$ is called a *normal domain* of f if $f(\partial D) = \partial f(D)$. A normal domain U is a *normal neighbourhood* of $x \in U$ if $\{x\} = U \cap f^{-1}(f(x))$. If U is a normal neighbourhood of x, then the topological degree $\mu(f(x), f, U)$ is defined and independent of U, and we write $i(x, f) = \mu(f(x), f, U)$. We call this number the *local topological index* of f at x. Moreover, $U(x, f, r)$ denotes the x-component of $f^{-1}(B(f(x), r))$.

Let $f : \Omega \to \mathbb{R}^m$ be a mapping, and let $\#A$ denote the number of points in a set $A \subset \mathbb{R}^m$. If $y \in \mathbb{R}^m$ and $E \subset \Omega$, then

(a) $N(y, f, E) = \#\{f^{-1}(y) \cap E\}$ is called the *crude multiplicity* of y in E,
(b) $N_{\max}(f, E) = \sup_{y\in\mathbb{R}^m} N(y, f, E)$ is called the *crude maximal multiplicity* of f in E,
(c) $N_{\min}(f, E) = \inf_{y\in f(E)} N(y, f, E)$ is called the *crude minimal multiplicity* of f in E,
(d) $M(y, f, E) = \sum_{x\in f^{-1}(y)\cap E} i(x, f)$ is called the *multiplicity* of y in E, and
(e) $M_{\min}(f, E) = \inf_{y\in f(E)} M(y, f, E)$ is called the *minimal multiplicity* of f in E.

Each value of the crude multiplicity functions $N(y, f, E)$, $N_{\max}(f, E)$ and $N_{\min}(f, E)$ is a nonnegative integer or $+\infty$. The same is true for the multiplicity functions $M(y, f, E)$ and $M_{\min}(f, E)$ if the mapping f is open, discrete and sense-preserving. Note also that the value of both crude multiplicity functions $N_{\min}(f, E)$ and $N_{\max}(f, E)$ may be $+\infty$. This occurs, for example, if f is a constant mapping with value $c \in \mathbb{R}^m$ and E is a domain in Ω; then $N(c, f, E) = +\infty$ and $f(E) = c$.

If a continuous mapping $f : \Omega \to \mathbb{R}^m$ satisfy the *(Lusin's) condition* (N), this is, the image of every set Lebesgue measure zero is a set of Lebesgue measure zero, then the crude multiplicity function $y \mapsto N(y, f, E)$ is Lebesgue measurable in $\mathbb{R}^m$ for every Lebesgue measurable set $E \subset \Omega$, see [RR55, IV.1.4. Corollary 1] and [FG95, Theorem 5.5]. If $E \subset \Omega$ is a Borel set, then $y \mapsto N(y, f, E)$ is Lebesgue measurable in $\mathbb{R}^n$ for every continuous mapping $f : \Omega \to \mathbb{R}^m$ [RR55, IV.1.2. Theorem], but it is not necessarily Borel measurable due to this result. However, we need a multiplicity function which behaves well under all holomorphic mappings with Borel measures and Borel sets. Moreover, it is required that the modified multiplicity function has values sufficiently close (equal almost everywhere) to the crude multiplicity function. A multiplicity function fulfilling these conditions is introduced as follows.

Let $f : \Omega \to \mathbb{R}^m$ be a continuous mapping, and let $E \subset \Omega$ be a Borel set. Since $y \mapsto N(y, f, E)$ is Lebesgue measurable in $\mathbb{R}^m$ and every Lebesgue measurable set A is contained in a Borel set of equal measure, see [Zie89, Remark 1.2.3], there is an integer-valued function $y \mapsto N^*(y, f, E)$ with the following properties:

(i) $y \mapsto N^*(y, f, E)$ is Borel measurable in $\mathbb{R}^m$,
(ii) $N^*(y, f, E) \geqslant N(y, f, E)$ for every $y \in \mathbb{R}^m$,
(iii) $N^*(y, f, E) = N(y, f, E)$ for almost every $y \in \mathbb{R}^m$, and
(iv) $\sup_{y\in\mathbb{R}^m} N^*(y, f, E) = \sup_{y\in\mathbb{R}^m} N(y, f, E)$.

If f is a (local) homeomorphism, then it is obvious that $N(y, f, E)$ is Borel measurable in $\mathbb{R}^m$, and hence $N^*(y, f, E) = N(y, f, E)$ for every $y \in \mathbb{R}^m$.

All holomorphic mappings satisfy the condition (N) as they are infinitely smooth, see Section 10 and [Res89, Chapter II, Corollary of Theorem 2.2]. If a continuous mapping f satisfies condition (N), then the image of a Lebesgue measurable set is always Lebesgue measurable, see [Res89, Chapter II, Theorem 2.1]. Therefore, the image of a Borel set under any holomorphic mapping is always Lebesgue measurable. However, it is sometimes essential to know that the image is also Borel; this is not true for all continuous mappings, in general. Next topological theorem ensures that all discrete, open and sense-preserving mappings $f : \Omega \to \mathbb{R}^n$ satisfy this property.

9.1. Theorem. *Let $f : \Omega \to \mathbb{R}^m$ be discrete, open and sense-preserving. If E is a Borel set in Ω, then $f(E)$ is a Borel set in $\mathbb{R}^m$.*

Proof. The proof is based on the fact that Ω can be decomposed into sets on which f is injective and the image of every Borel set is known to be Borel. We can write $E = (E \cap (\Omega \setminus B_f)) \cup (E \cap B_f)$, and it is thus enough to show that $f(E \cap (\Omega \setminus B_f))$ and $f(E \cap B_f)$ are Borel sets.

Since B_f is closed in Ω, $\Omega \setminus B_f$ is open. Hence we can express $\Omega \setminus B_f$ as a countable union of closed balls $\overline{B}(x_j, r_j)$, $j = 1, 2, \ldots$,

$$\Omega \setminus B_f = \bigcup_{j=1}^{\infty} \overline{B}(x_j, r_j)$$

such that $f|_{\overline{B}(x_j,r_j)}$ is a homeomorphism. Consequently, f maps Borel sets of $\Omega \setminus B_f$ into Borel sets, and hence $f(E \cap (\Omega \setminus B_f))$ is Borel because $E \cap (\Omega \setminus B_f)$ is included in $\Omega \setminus B_f$.

It remains to show that $f(E \cap B_f)$ is Borel. Denote for each $k = 1, 2, \ldots$ the set

$$A_k = \{x \in \Omega : i(x, f) \leqslant k\}.$$

We show that A_k is open for all k. If $k = 1$, then $A_k = \Omega \setminus B_f$, which is clearly open. Suppose then that $k \geqslant 2$. Fix an arbitrary $x_0 \in A_k$. Then $i(x_0, f) \leqslant k$. By [MRV69, Lemma 2.9 and Lemma 2.12] there is a normal neighbourhood U of x_0 such that

$$\sum_{x \in f^{-1}(y)} i(x, f) = N_{\max}(f, U) = i(x_0, f)$$

for all $y \in f(U)$. This implies that $i(x, f) \leqslant i(x_0, f) \leqslant k$ for all $x \in U$, and thus $U \subset A_k$. Therefore A_k is open for each k.

Suppose that $x \in B_f$ and that $i(x, f) = k$. Then there is an open set $U \subset \Omega$ such that $x \in U$ and $U \subset A_k$, since $x \in A_k$ and A_k is open. Pick a normal neighbourhood U_x of x such that its closure satisfies $\overline{U}_x \subset V_x \subset U$ where V_x is another normal neighbourhood of x. Note that $i(y, f) \leqslant k$ for every $y \in \overline{U}_x$. Denote

$$B_k(x) = \{y \in \overline{U}_x : i(y, f) = k\}.$$

Since $B_k(x) = \overline{U}_x \setminus A_{k-1}$, $B_k(x)$ is compact in $\overline{U}_x$. Moreover, $f|_{B_k(x)} : B_k(x) \to f(B_k(x))$ is a homeomorphism. To see that $f|_{B_k(x)}$ is one-to-one, let $y_1 \in B_k(x)$. Suppose that $y_2 \in B_k(x)$ is another preimage of $f(y_1)$. Pick now normal neighbourhoods V_{y_1} and V_{y_2} of y_1 and y_2, respectively, such that $V_{y_i} \subset V_x$, $i = 1, 2$, and $V_{y_1} \cap V_{y_2} = \emptyset$. Now $f(y_1) \in f(V_x)$, and by [MRV69, Lemma 2.12] and the property (iv) of the topological degree

$$\begin{aligned} k = i(x, f) = N_{\max}(f, V_x) &= \mu(f(y_1), f, V_x) \geqslant \mu(f(y_1), f, V_{y_1}) + \mu(f(y_1), f, V_{y_2}) \\ &= N_{\max}(f, V_{y_1}) + N_{\max}(f, V_{y_2}) = i(y_1, f) + i(y_2, f) = 2k, \end{aligned}$$

which is impossible because $k \geqslant 2$. Hence the restriction $f|_{B_k(x)}$ is one-to-one, and $f|_{B_k(x)}$ is a homeomorphism.

Finally, for every $k = 2, 3, \ldots$, choose a countable set $\{x_j : j = 1, 2, \ldots\}$ of points in B_f such that $i(x_j, f) = k$ and that

$$\{y \in B_f : i(y, f) = k\} = \bigcup_{j=1}^{\infty} B_k(x_j).$$

Then

$$B_f = \bigcup_{k=2}^{\infty} \{y \in B_f : i(y, f) = k\} = \bigcup_{k=2}^{\infty} \bigcup_{j=1}^{\infty} B_k(x_j).$$

This means that B_f can be covered with countably many compact sets $B_k(x_j)$ on which the restriction of f is a homeomorphism. Hence $f(E \cap B_f)$ is Borel, and the claim follows. □

Prof. J. Väisälä informed me about a result in his notes from the early 1960's: If X and Y are Hausdorff spaces and if X is locally compact with a countably base, then every open and discrete mapping preserves Borel sets. He has proved the result without any degree theory.

We constructed the multiplicity function $y \mapsto N^*(y, f, E)$ so that it is Borel measurable for every continuous mapping $f : \Omega \to \mathbb{R}^m$ whenever $E \subset \Omega$ is a Borel set. The following result guarantees that the crude multiplicity function $y \mapsto N(y, f, E)$ behaves well under discrete, open and sense-preserving mappings.

9.2. Lemma. *Let $f : \Omega \to \mathbb{R}^m$ be discrete, open and sense-preserving. If $E \subset \Omega$ is a Borel set, then $y \mapsto N(y, f, E)$ is Borel measurable in $\mathbb{R}^m$.*

Proof. This result can be proved following the proof of [FG95, Theorem 5.5], but a slight modification is required. First of all, we can suppose that E is bounded (just like in the original proof). The main difference is that we have to choose a pairwise nonintersecting partition of E such that the partition sets E_i are Borel sets (Lebesgue measurable in the original proof). It is required that $\operatorname{diam}(E_i) \leqslant \frac{1}{k}$ for every $i = 1, \ldots, n(k)$ where k is a positive constant. This kind of partition mentioned above is always possible to find. For example, we construct first a net of closed and open nonoverlapping m-cubes C_i with $\operatorname{diam}(C_i) = \frac{1}{k}$ side by side covering the whole set E. Then we set $E_i = C_i \cap E$, and all of the sets E_i are now Borel. Moreover, $E = \cup_i E_i$ and $E_i \cap E_j = \emptyset$, $i \neq j$, as required. The images $f(E_i)$ are now Borel sets by Theorem 9.1. We use this fact instead of [FG95, Lemma 5.4] in our modified proof. □

10. Holomorphic functions and mappings

Definition. Let Ω be an open subset of $\mathbb{C}^n$.

(i) A function $f : \Omega \to \mathbb{C}$ is said to be *holomorphic* in Ω if $f \in C^1(\Omega)$ and f satisfies the *Cauchy–Riemann equations*

$$\bar{\partial}_j f = \frac{\partial f}{\partial \bar{z}_j} = 0 \text{ for each } j = 1, \ldots, n.$$

(ii) A mapping $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ is *holomorphic* in Ω if f_k is a holomorphic function for all $k = 1, \ldots, m$.

(iii) A mapping $f : \Omega \to \mathbb{C}^n$ is said to be *biholomorphic* if it is a holomorphic homeomorphism with holomorphic inverse $f^{-1} : f(\Omega) \to \Omega$.

It is remarkable that holomorphic mappings are possible to define as $\mathbb{C}$-*differentiable* mappings without any smoothness assumptions; f is separately $\mathbb{C}$-differentiable is equivalent to f is separately holomorphic, thus by *Hartogs theorem* Cauchy–Riemann equations imply the condition $f \in C^1(\Omega)$ and even more: $f \in C^\infty(\Omega)$. Our definition is therefore as general as the definition by means of $\mathbb{C}$-differentiability.

Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping. The *complex Jacobian* of f is defined by

$$J_{\mathbb{C}} f = \det \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{bmatrix}.$$

Easy matrice calculations show that $J_{\mathbb{R}} f(z) = |J_{\mathbb{C}} f(z)|^2$ for all $z \in \Omega$. Thus the real Jacobian of a holomorphic mapping is always nonnegative.

10.1. Inverse mapping theorem. *Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping, and suppose that $z \in \Omega$. Then f is biholomorphic from a neighbourhood of z onto a neighbourhood of $f(z)$ if and only if $J_{\mathbb{C}} f(z) \neq 0$.*

The following lemma presents three useful properties of arbitrary holomorphic mappings. Recall that a pluripolar set $E \subset \Omega$ is called *complete* in Ω if there is $u \in PSH(\Omega)$ such that $E = \{z \in \Omega : u(z) = \infty\}$.

10.2. Lemma. *Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping. Then the following properties hold:*

(i) *If $z \in \Omega$ is such that $J_{\mathbb{C}} f(z) \neq 0$, then there exists a neighbourhood U of z such that $\mu(w, f, U) = 1$ for all $w \in f(U)$.*
(ii) *$i(z, f) = 1$ for every $z \in \Omega \setminus B_f$.*
(iii) *If $J_{\mathbb{C}} f \not\equiv 0$ in each component of Ω, then B_f is completely pluripolar in Ω.*

Proof. Inverse mapping theorem implies the property (i), which yields the property (ii). Note that $i(z, f)$ is defined for every $z \in \Omega \setminus B_f$ since f is a local homeomorphism in $\Omega \setminus B_f$.

Property (iii): Since f is holomorphic in Ω, the complex Jacobian $J_{\mathbb{C}} f$ is a holomorhic function from Ω to $\mathbb{C}$. Thus $-\log |J_{\mathbb{C}} f| \in PSH(\Omega)$ as $J_{\mathbb{C}} f \not\equiv 0$, and from Inverse mapping theorem 10.1 it follows that

$$B_f = \{z \in \Omega : J_{\mathbb{C}} f(z) = 0\} = \{z \in \Omega : -\log |J_{\mathbb{C}} f(z)| = \infty\}$$

is pluripolar in Ω. □

By the first item of the next lemma, nontrivial holomorphic substitutions preserve plurisuperharmonicity. In other words, plurisuperharmonic functions can be *pulled back* with holomorphic mappings. The second item states that the complex Monge–Ampère operator $(dd^c)^n$ acting on C^2-functions has the *invariance property* under holomorphic mappings. Both of these results are well-known, see [Kli91].

10.3. Lemma. *Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping. If G is a neighbourhood of $f(\Omega)$, then the following properties hold:*

(i) *If u is plurisuperharmonic in G, then $u \circ f$ is plurisuperharmonic or identically $+\infty$ in every component of Ω.*

(ii) *If $u \in C^2(G)$, then*

$$(dd^c(u \circ f))^n(z) = |J_{\mathbb{C}} f(z)|^2 (dd^c u)^n(f(z))$$

for every $z \in \Omega$.

11. Proper holomorphic mappings

Definition. Let X and Y be topological spaces. A continuous mapping $f : X \to Y$ is said to be *proper* if $f^{-1}(K)$ is compact in X for every compact set K in Y.

A homeomorphism f from X onto Y is always proper. Thus all biholomorphic mappings $f : \Omega \to f(\Omega)$ are proper. However, there are proper holomorphic mappings between domains Ω and Ω' in $\mathbb{C}^n$ which are not biholomorphic. A mapping $f : \Omega \to \Omega'$ is proper if and only if f maps $\partial\Omega$ to $\partial\Omega'$ in the following sense: If $(z_j) \subset \Omega$ is a sequence with $\lim_{j\to\infty} \operatorname{dist}(z_j, \partial\Omega) = 0$, then $\lim_{j\to\infty} \operatorname{dist}(f(z_j), \partial\Omega') = 0$. From this it follows that if f extends continuously to the boundary $\partial\Omega$, then $f(\partial\Omega) \subset \partial\Omega'$.

We recall some properties of proper holomorphic mappings, see the articles by R. Remmert and K. Stein [RS60] and E. Bedford [Bed84], and a book by W. Rudin [Rud80, Chapter 15].

11.1. Lemma. *Let $f : \Omega \to \Omega'$ be a proper holomorphic mapping. Then the following properties hold:*

(i) *f is discrete, open and closed.*

(ii) *f is sense-preserving.*

(iii) *$f^{-1}(w)$ is finite for all $w \in \Omega'$.*

(iv) *$f(\Omega) = \Omega'$.*

(v) *$N(w, f, \Omega) = N_{\max}(f, \Omega)$ for every $w \in \Omega' \setminus f(B_f)$.*

(vi) *$N(w, f, \Omega) < N_{\max}(f, \Omega)$ for every $w \in f(B_f)$.*

(vii) *$f(B_f)$ is completely pluripolar in Ω'.*

(viii) *$f^{-1}(f(B_f))$ is completely pluripolar in Ω.*

Proof. Properties (i), (iii), (iv), (v) and (vi), see [Rud80, Chapter 15]. Property (ii) follows from Lemma 10.2 (i) and the fact that discrete and open mappings are either sense-preserving or sense-reversing [Che64, Che65, Väi66].

Properties (vii) and (viii): By [Rud80, Theorem 15.1.9], $f(B_f)$ is a *zero-variety* in Ω', that is,

$$f(B_f) = \{z \in \Omega' : h(z) = 0\}$$

for some holomorphic function h in Ω' such that $h \not\equiv 0$. It follows that $-\log|h| \in \mathrm{PSH}(\Omega')$, and

$$f(B_f) = \{z \in \Omega' : h(z) = 0\} = \{z \in \Omega' : -\log|h(z)| = \infty\}$$

is pluripolar in Ω'. Further, $h \circ f$ is holomorphic and nonzero in Ω because $f(\Omega) = \Omega'$. Hence $-\log|h \circ f| \in \mathrm{PSH}(\Omega)$, and thus

$$f^{-1}(f(B_f)) = \{z \in \Omega : (h \circ f)(z) = 0\} = \{z \in \Omega : -\log|(h \circ f)(z)| = \infty\}$$

is pluripolar in Ω. □

11.2. Remark. Suppose that $f : \Omega \to \mathbb{C}^n$ is a nonconstant holomorphic mapping. Let $D \Subset \Omega$ be a normal domain of f. Then the restriction $f|_D : D \to f(D)$ is clearly holomorphic. Since D is a normal domain of f, we have $f(\partial D) = \partial f(D)$. Thus $f^{-1}(K)$ is compact in D for every compact $K \subset f(D)$. Hence $f|_D : D \to f(D)$ is proper and holomorphic.

11.3. Lemma. *Let $f : \Omega \to \Omega'$ be a proper holomorphic mapping. Suppose that $w \in \Omega'$ and that $\varepsilon > 0$. Then we can find $r > 0$ and mutually disjoint normal neighbourhoods $U_1, \ldots, U_k$ of the points $z_1, \ldots, z_k$ in the set $\{z \in \Omega : f(z) = w\}$ such that*

$$f^{-1}(B(w, r)) = U_1 \cup \cdots \cup U_k$$

and $\operatorname{diam} U_i < \varepsilon$ *for every $i = 1, \ldots, k$. In particular, if $w \in \Omega' \setminus f(B_f)$, then there is a neighbourhood W_0 of w such that for every neighbourhood W of w such that $W \subset W_0$, the following conditions are satisfied:*

(i) $W \cap f(B_f) = \emptyset$.
(ii) *The components of $f^{-1}(W)$ form a collection $U_1, \ldots, U_k$ where each U_i is a neighbourhood of $z_i \in \{z \in \Omega : f(z) = w\}$, $i = 1, \ldots, k = N_{\max}(f, \Omega)$.*
(iii) *f defines biholomorphic mappings $f_i = f|_{U_i} : U_i \to W$.*

Proof. The first part of the theorem follows immediately from [MRV69, Lemma 2.5 and Lemma 2.9]. If $w \in \Omega' \setminus f(B_f)$, then by Inverse mapping theorem, the restriction

$$f|_{f^{-1}(\Omega' \setminus f(B_f))} : f^{-1}(\Omega' \setminus f(B_f)) \longrightarrow \Omega' \setminus f(B_f)$$

is locally biholomorphic, and the first part of the theorem yields the result. □

Suppose that $f : \Omega \to \Omega'$ is a proper holomorphic mapping. M. Klimek [Kli82, Lemma 4.1] has proved that plurisubharmonic functions u in Ω can be pushed forward to Ω' with a function

$$v(w) = \max_{z \in f^{-1}(w)} u(z).$$

In case of plurisuperharmonic functions u in Ω, this function is modified to the form

$$v(w) = \min_{z \in f^{-1}(w)} u(z).$$

Next theorem presents a new way how to push forward plurisuperharmonic functions under proper holomorphic mappings. The following *push forward function* (11.5) is by O. Martio [Mar70, Lemma 5.4], originally used with nonconstant quasiregular mappings $f : \Omega \to \mathbb{R}^m$ and $u \in C_0^\infty(\Omega)$; then it is known that $v \in C_0(f(\Omega))$ and $\operatorname{spt} v \subset f(\operatorname{spt} u)$.

11.4. Theorem. *Let $f : \Omega \to \Omega'$ be a proper holomorphic mapping. If u is a plurisuperharmonic function in Ω, then the formula*

$$v(w) = \frac{1}{N_{\max}(f,\Omega)} \sum_{z\in f^{-1}(w)} i(z,f)\,u(z) \qquad (w \in \Omega') \tag{11.5}$$

defines a plurisuperharmonic function in Ω'. Moreover, if $0 \leqslant u \leqslant 1$ in Ω, then $0 \leqslant v \leqslant 1$ in Ω'.

Recall that a set $E \subset \Omega$ is *removable* with respect to a property in $\Omega \setminus E$ if the property can be removed to hold in the whole of Ω. To prove the previous theorem, we need the following lemma which is a removability result for plurisuperharmonic functions. In fact, all relatively closed pluripolar sets are removable (see, e.g., [Lel68]), but the next lemma gives also an explicit function for this purpose.

11.6. Lemma. [Kli91, Theorem 2.9.22] *Let a relatively closed subset E of Ω be $\mathbb{R}^{2n}$-polar. If a function $u \in \mathrm{PSH}(\Omega \setminus E)$ satisfies $\liminf_{w\to z} u(w) > -\infty$ for all $z \in E$, then*

$$\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in \Omega \setminus E, \\ \liminf_{\substack{w\to z \\ w\notin E}} u(w) & \text{if } z \in E \end{cases}$$

defines a plurisuperharmonic function in Ω.

Proof of Theorem 11.4. Since $\frac{1}{N_{\max}(f,\Omega)}$ is a positive constant, it is enough to show that the function $\sum i(z,f)\,u(z)$ is plurisuperharmonic in Ω'. We will denote also this modified function by v. Suppose that $G \Subset \Omega'$ is open. Then $f^{-1}(G) \Subset \Omega$ and it is open because f is proper and continuous. By Theorem 3.2, there is an increasing sequence of functions $u_j \in C^\infty(f^{-1}(G)) \cap \mathrm{PSH}\left(f^{-1}(G)\right)$ such that $\lim_{j\to\infty} u_j = u$ pointwise in $f^{-1}(G)$. In particular, every u_j is continuous, and that is a sufficient assumption for the functions u_j. Write

$$v_j(w) = \sum_{z\in f^{-1}(w)} i(z,f)\,u_j(z)$$

for $w \in G$. Then $v_j \nearrow v$ on G since $i(z,f) \geqslant 1$. Therefore, if every $v_j \in \mathrm{PSH}(G)$, then $v \in \mathrm{PSH}(G)$ or $v \equiv \infty$ on G. As $u \not\equiv \infty$ on every component of $f^{-1}(G)$, also $v \not\equiv \infty$ on every component of G. Thus $v \in \mathrm{PSH}(G)$ and further $v \in \mathrm{PSH}(\Omega')$. As a consequence, it is enough to show that the theorem is true for continuous plurisuperharmonic functions.

So let $u \in C(\Omega) \cap \mathrm{PSH}(\Omega)$. We show that v is continuous in Ω'. Let $\varepsilon > 0$ and $w_0 \in \Omega'$. We can write $f^{-1}(w_0) = \{z_1, \ldots, z_k\}$ and we know that $k \leqslant N_{\max}(f,\Omega)$. By Lemma 11.3, we can choose a small enough $r > 0$ such that

$$f^{-1}\left(B(w_0,r)\right) = U_1 \cup \cdots \cup U_k$$

and

$$|u(z_i) - u(z)| < \frac{\varepsilon}{N_{\max}(f,\Omega)k}$$

for all $z \in U_i$, where $U_i \cap U_j = \emptyset,\ i \neq j$, and U_i is a normal neighbourhood of z_i for every $i = 1, \dots, k$. Suppose that $w \in B(w_0, r)$. Then by [MRV69, Lemma 2.12],

$$\sum_{z \in f^{-1}(w) \cap U_i} i(z, f) = \mu(w, f, U_i) = N_{\max}(f, U_i) = i(z_i, f).$$

Hence

$$\begin{aligned} v(w) &= \sum_{z \in f^{-1}(w)} i(z, f)\, u(z) = \sum_{i=1}^{k} \sum_{z \in f^{-1}(w) \cap U_i} i(z, f)\, u(z) \\ &< \sum_{i=1}^{k} \left(u(z_i) + \frac{\varepsilon}{N_{\max}(f, \Omega) k} \right) i(z_i, f) \leqslant v(w_0) + \varepsilon, \end{aligned}$$

since the additivity property holds for the topological degree. Similarly, we obtain

$$v(w) > v(w_0) - \varepsilon,$$

and hence $v \in \mathrm{C}(\Omega')$.

Let now $w_0 \in \Omega' \setminus f(B_f)$ and write $f^{-1}(w_0) = \{z_1, \dots, z_k\}$ as before. Now $k = N_{\max}(f, \Omega)$. By Lemma 11.3, there exists a neighbourhood $W \subset \Omega' \setminus f(B_f)$ of w_0, and mutually disjoint neighbourhoods $U_1, \dots, U_k$ of $z_1, \dots, z_k$ such that

$$f_i = f|_{U_i} : U_i \longrightarrow W$$

is biholomorphic for each $i = 1, \dots, k$ and

$$f^{-1}(W) = U_1 \cup \cdots \cup U_k.$$

Now

$$v_i(w) = \sum_{z \in f^{-1}(w) \cap U_i} i(z, f)\, u(z) = i(f_i^{-1}(w), f)\, u(f_i^{-1}(w)) = \left(u \circ f_i^{-1} \right)(w)$$

is plurisuperharmonic in W, since plurisuperharmonicity is preserved under biholomorphic substitutions and $i(z, f) = 1$ in $\Omega \setminus B_f$. Furthermore,

$$v(w) = \sum_{i=1}^{k} v_i(w) = \sum_{i=1}^{k} \left(u \circ f_i^{-1} \right)(w) \tag{11.7}$$

is plurisuperharmonic in W, and thus $v \in \mathrm{PSH}\left(\Omega' \setminus f(B_f) \right)$.

Finally, let $V \Subset \Omega'$ be open. Since $f(B_f)$ is completely pluripolar in Ω', its subset $E = V \cap f(B_f)$ is completely pluripolar in V. In addition, $v \in \mathrm{PSH}(V \setminus E)$, v is bounded in V and

$$\liminf_{\substack{\xi \to w \\ \xi \notin f(B_f)}} v(\xi) = v(w)$$

for all $w \in f(B_f)$, because v is continuous. Thus $v \in \mathrm{PSH}(V)$ by Lemma 11.6, and from this it follows that $v \in \mathrm{PSH}(\Omega')$ because plurisuperharmonicity is a local property.

To prove the last part of the theorem, suppose that $0 \leqslant u \leqslant 1$ in Ω. Then it is clear that $v \geqslant 0$ in Ω since $\frac{1}{N_{\max}(f,\Omega)} > 0$ and $i(z, f) \geqslant 1$ for all $z \in \Omega$. Let $w \in \Omega'$. Then by [MS75, Lemma 3.3],

$$v(w) = \frac{1}{N_{\max}(f,\Omega)} \sum_{z\in f^{-1}(w)} i(z, f)\, u(z) \leqslant \frac{1}{N_{\max}(f,\Omega)} \sum_{z\in f^{-1}(w)} i(z, f) = \frac{N_{\max}(f,\Omega)}{N_{\max}(f,\Omega)} = 1,$$

as $u(z) \leqslant 1$ for all $z \in \Omega$. □

12. Integral transformation formulas

An integral transformation formula

$$\int_E (u \circ f)(z)\, \mathrm{J}_{\mathbb{R}} f(z)\, dm(z) = \int_{\mathbb{R}^m} N(w, f, E)\, u(w)\, dm(w) \tag{12.1}$$

is known to be valid for Lebesgue measurable sets E in Ω, bounded and measurable functions u in $\mathbb{R}^m$ and Sobolev mappings $f \in \mathrm{W}^{1,m}(\Omega)$ such that $\mathrm{J}_{\mathbb{R}} f > 0$ a.e. in Ω (with respect to the Lebesgue measure) where Ω is supposed to be an open and bounded set in $\mathbb{R}^m$, see [FG95, Theorem 5.34]. If u is supposed to be either nonnegative or nonpositive, then the formula (12.1) holds without any boundedness assumptions. From (12.1) we obtain the following result.

12.2. Theorem. *Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping. If E is a Lebesgue measurable subset of Ω, then*

$$\int_E (u \circ f)(z) |\mathrm{J}_{\mathbb{C}} f(z)|^2\, dm(z) = \int_{f(E)} N(w, f, E)\, u(w)\, dm(w) \tag{12.3}$$

holds for every Lebesgue measurable function u in $f(E)$ with nonnegative or nonpositive values.

Proof. We may assume that Ω is a domain. Note that $|\mathrm{J}_{\mathbb{C}} f|^2 = \mathrm{J}_{\mathbb{R}} f$ and that $f(E)$ is Lebesgue measurable since f satisfies the condition (N). If $\mathrm{J}_{\mathbb{C}} f \equiv 0$, then both sides of (12.3) are equal to zero because f maps E to a set of measure zero [FG95, Lemma 1.4 (Sard's Lemma)]. If $\mathrm{J}_{\mathbb{C}} f \not\equiv 0$, then by Lemma 10.2 (iii), $B_f = \{z \in \Omega : \mathrm{J}_{\mathbb{R}} f(z) = 0\}$ is pluripolar in Ω and hence of Lebesgue measure zero. Thus $\mathrm{J}_{\mathbb{R}} f(z) > 0$ a.e. in Ω as $\mathrm{J}_{\mathbb{R}} f$ is known to be nonnegative for holomorphic mappings, and the result follows from the formula (12.1). □

For proper holomorphic mappings $f : \Omega \to \Omega'$ we have $N(w, f, \Omega) = N_{\max}(f, \Omega)$ for almost every $w \in \Omega'$. Hence Theorem 12.2 gives the following result.

12.4. Corollary. *Let $f : \Omega \to \Omega'$ be a proper holomorphic mapping. Then*

$$\int_\Omega (u \circ f)(z)\, |\mathrm{J}_{\mathbb{C}} f(z)|^2\, dm(z) = N_{\max}(f,\Omega) \int_{\Omega'} u(w)\, dm(w) \tag{12.5}$$

for each nonnegative or nonpositive Lebesgue measurable function u in Ω'.

We are interested in integral transformation formulas for the generalized complex Monge–Ampère operator acting on locally bounded plurisuperharmonic functions. The following results are stated so that a minus sign is attached to the operator; then $-(dd^c)^n$ gives a (positive) measure and the known results for Radon measures as well as the applications in the rest of this study are relevant without any changes. The multiplicity function $y \mapsto N^*(y, f, E)$ defined in Section 9 is used in our first integral transformation formula. However, this formula presents only an inequality for integral tranformations. Fortunately the equation is needed only for biholomorphic mappings when we later prove capacity inequalities for holomorphic mappings.

12.6. Theorem. *Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping. Suppose that K is a compact subset of Ω. If G is a neighbourhood of $f(K)$, then*

$$\int_K -(dd^c(u \circ f))^n \geqslant \int_{f(K)} -N^*(w, f, E)\,(dd^c u)^n \tag{12.7}$$

for each $u \in \mathrm{L}^\infty_{\mathrm{loc}}(G) \cap \mathrm{PSH}(G)$.

Proof. Let $u \in \mathrm{L}^\infty_{\mathrm{loc}}(G) \cap \mathrm{PSH}(G)$ and let G be a neighbourhood of $f(K)$. Note that since K is compact and f is continuous, $f(K)$ is compact and hence $f(K) \Subset G$ because G is open. Pick a neighbourhood U of $f(K)$ such that $f(K) \subset U \Subset G$, then $K \subset f^{-1}(U)$. By Local approximation theorem 3.2, it is possible to choose an increasing sequence of functions $u_j \in \mathrm{C}^\infty(U) \cap \mathrm{PSH}(U)$ such that $\lim_{j\to\infty} u_j = u$ pointwise in U. Now by Lemma 10.3 (i), $u \circ f \in \mathrm{L}^\infty_{\mathrm{loc}}(f^{-1}(U)) \cap \mathrm{PSH}(f^{-1}(U))$ as well as $u_j \circ f \in \mathrm{C}^\infty(f^{-1}(U)) \cap \mathrm{PSH}(f^{-1}(U))$ for each j because $f \in \mathrm{C}^\infty(\Omega)$. Moreover, it is clear that $\lim_{j\to\infty}(u_j \circ f) = u \circ f$ pointwise in $f^{-1}(U)$.

Since $-(dd^c u_j)^n$ is now nonnegative and continuous (thus Lebesgue measurable) in U and $w \mapsto N(w, f, A)$ is Lebesgue measurable and $w \mapsto N^*(w, f, A)$ is Borel measurable whenever A is Borel, the invariance property (Lemma 10.3 (ii)) and Theorem 12.2 yield that for all Borel sets $A \subset f^{-1}(U)$ such that $f(A)$ is a Borel set, we have

$$\begin{aligned}
\int_A -(dd^c(u_j \circ f))^n &= \int_A -|\mathrm{J}_{\mathbb{C}} f(z)|^2 (dd^c u_j)^n (f(z)) \\
&= \int_A -|\mathrm{J}_{\mathbb{C}} f(z)|^2\, 4^n n!\, \det\left[\frac{\partial^2 u_j}{\partial z_i \partial \overline{z}_k}(f(z))\right] dV(z) \\
&= \int_{f(A)} -N(w, f, A)\, 4^n n!\, \det\left[\frac{\partial^2 u_j}{\partial z_i \partial \overline{z}_k}(w)\right] dV(w) \\
&= \int_{f(A)} -N^*(w, f, A)\, 4^n n!\, \det\left[\frac{\partial^2 u_j}{\partial z_i \partial \overline{z}_k}(w)\right] dV(w) \\
&= \int_{f(A)} -N^*(w, f, A)\, (dd^c u_j)^n
\end{aligned}$$

for each j, since $N^*(w, f, A) = N(w, f, A)$ for almost every $w \in f(A)$.

Make a partition of $f(K)$ such that

$$f(K)^*_k = \{w \in f(K) : N^*(w, f, K) = k\}, \quad 1 \leqslant k \leqslant \infty.$$

Then

$$f(K)=\bigcup_{k=1}^{\infty} f(K)^*_k \cup f(K)^*_\infty$$

and the sets $f(K)^*_k$ are Borel because the multiplicity function $w \mapsto N^*(w,f,K)$ is Borel measurable.

Suppose first that

$$\int_{f(K)^*_\infty} -(dd^c u)^n = 0. \tag{12.8}$$

Let $\varepsilon>0$ and let $l\in\mathbb{N}$. There are compact sets $Q_k \subset f(K)^*_k$ such that

$$\int_{Q_k} -(dd^c u)^n > \int_{f(K)^*_k} -(dd^c u)^n - \frac{2\varepsilon}{l(l+1)}$$

for every $k=1,2,\ldots,l$. Strong convergence theorem 4.2 yields now

$$\begin{aligned}\int_K -(dd^c(u\circ f))^n &= \lim_{j\to\infty}\int_K -(dd^c(u_j\circ f))^n = \lim_{j\to\infty}\int_{f(K)} -N^*(w,f,K)\,(dd^c u_j)^n \\ &\geqslant \lim_{j\to\infty}\sum_{k=1}^{l}\int_{f(K)^*_k} -N^*(w,f,K)\,(dd^c u_j)^n \geqslant \lim_{j\to\infty}\sum_{k=1}^{l}\int_{Q_k} -N^*(w,f,K)\,(dd^c u_j)^n \\ &= \sum_{k=1}^{l} k\lim_{j\to\infty}\int_{Q_k} -(dd^c u_j)^n \geqslant \sum_{k=1}^{l} k\int_{Q_k} -(dd^c u)^n \\ &> \sum_{k=1}^{l} k\left(\int_{f(K)^*_k} -(dd^c u)^n - \frac{2\varepsilon}{l(l+1)}\right) = \sum_{k=1}^{l}\int_{f(K)^*_k} -N^*(w,f,K)\,(dd^c u)^n - \varepsilon.\end{aligned}$$

Since this holds for arbitrarily chosen l and ε which are independent, it follows that

$$\begin{aligned}\int_K -(dd^c(u\circ f))^n &\geqslant \sum_{k=1}^{\infty}\int_{f(K)^*_k} -N^*(w,f,K)\,(dd^c u)^n - \varepsilon \\ &= \int_{f(K)\setminus f(K)^*_\infty} -N^*(w,f,K)\,(dd^c u)^n - \varepsilon \\ &= \int_{f(K)} -N^*(w,f,K)\,(dd^c u)^n - \varepsilon,\end{aligned}$$

because we supposed that (12.8) holds. This yields

$$\int_K -(dd^c(u\circ f))^n \geqslant \int_{f(K)} -N^*(w,f,K)\,(dd^c u)^n.$$

Suppose then that

$$\int_{f(K)^*_\infty} -(dd^c u)^n > 0. \tag{12.9}$$

Since $f(K)^*_\infty$ is a Borel set, we may choose a compact set $Q_\infty \subset f(K)^*_\infty$ such that

$$\int_{Q_\infty} -(dd^c u)^n > 0. \tag{12.10}$$

Since then also $K \cap f^{-1}(Q_\infty)$ is compact, we see as in the previous part of this proof that

$$\int_K -(dd^c(u \circ f))^n \geqslant \int_{K\cap f^{-1}(Q_\infty)} -(dd^c(u \circ f))^n \geqslant \int_{Q_\infty} -N^*(w, f, K)\,(dd^c u)^n = \infty$$

because $N^*(w, f, K) = \infty$ in Q_∞ and (12.10) holds. It is clear now that (12.7) is satisfied also in the case (12.9), and the theorem is proved. □

12.11. Corollary. *Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping. Suppose that K is a compact subset of Ω. If G is a neighbourhood of $f(K)$, then*

$$\int_K -(dd^c(u \circ f))^n \geqslant N_{\min}(f, K) \int_{f(K)} -(dd^c u)^n \tag{12.12}$$

for each $u \in \mathrm{L}^\infty_{\mathrm{loc}}(G) \cap \mathrm{PSH}(G)$.

Proof. This result follows from the previous theorem, since $N_{\min}(f, K) \leqslant N(w, f, K) \leqslant N^*(w, f, K)$ for all $w \in f(K)$, and the operator $-(dd^c)^n$ acting on locally bounded plurisuperharmonic functions is nonnegative. □

12.13. Corollary. *Let $f : \Omega \to \mathbb{C}^n$ be a biholomorphic mapping, and let $E \Subset \Omega$ be a Borel set. If G is a neighbourhood of $f(E)$ such that $f(E) \Subset G$, then*

$$\int_E -(dd^c(u \circ f))^n = \int_{f(E)} -(dd^c u)^n \tag{12.14}$$

for each $u \in \mathrm{L}^\infty_{\mathrm{loc}}(G) \cap \mathrm{PSH}(G)$.

Proof. Note that $f(E)$ is a Borel set and $N(w, f, E) = 1$ for each $w \in f(E)$, $u \circ f \in \mathrm{L}^\infty_{\mathrm{loc}}(f^{-1}(G)) \cap \mathrm{PSH}(f^{-1}(G))$ and $E \Subset f^{-1}(G)$.

Suppose first that E is compact. It follows from Corollary 12.11 that

$$\begin{aligned}\int_E -(dd^c(u \circ f))^n &\geqslant \int_{f(E)} -(dd^c u)^n = \int_{f(E)} -(dd^c(u \circ (f \circ f^{-1})))^n \\ &= \int_{f(E)} -(dd^c((u \circ f) \circ f^{-1})))^n \geqslant \int_{f^{-1}(f(E))} -(dd^c(u \circ f))^n \\ &= \int_E -(dd^c(u \circ f))^n.\end{aligned}$$

Therefore the equation (12.14) is satisfied for compact sets.

Suppose then that E is Borel. Let $\varepsilon > 0$. There is a compact set $K \subset E$ such that

$$\int_{E\setminus K} -(dd^c(u \circ f))^n < \varepsilon.$$

Since $f(K)$ is compact and contained in $f(E)$, the previous part of this proof yields

$$\int_{f(E)} -(dd^c u)^n \geqslant \int_{f(K)} -(dd^c u)^n = \int_K -(dd^c(u \circ f))^n > \int_E -(dd^c(u \circ f))^n - \varepsilon.$$

As ε was chosen arbitrarily, we have

$$\int_{f(E)} -(dd^c u)^n \geqslant \int_E -(dd^c(u \circ f))^n. \tag{12.15}$$

Correspondingly, we can choose a compact set $Q \subset f(E)$ such that

$$\int_{f(E)\setminus Q} -(dd^c u)^n < \varepsilon.$$

Since $f^{-1}(Q)$ is compact and contained in E, the first part of this proof gives

$$\begin{aligned}\int_E -(dd^c(u \circ f))^n &\geqslant \int_{f^{-1}(Q)} -(dd^c(u \circ f))^n = \int_{f(f^{-1}(Q))} -(dd^c u)^n \\ &= \int_Q -(dd^c u)^n > \int_{f(E)} -(dd^c u)^n - \varepsilon,\end{aligned}$$

and again

$$\int_E -(dd^c(u \circ f))^n \geqslant \int_{f(E)} -(dd^c u)^n. \tag{12.16}$$

The inequalities (12.15) and (12.16) yield the equation (12.14). □

12.17. Theorem. *Let $f : \Omega \to \Omega'$ be a proper holomorphic mapping. Suppose that $E \Subset \Omega$ is a Borel set. If G is a neighbourhood of $f(E)$ such that $f(E) \Subset G$, then*

$$\int_E -(dd^c(u \circ f))^n = \int_{f(E)} -N(w, f, E)\,(dd^c u)^n \geqslant M_{\min}(f, E) \int_{f(E)} -(dd^c u)^n \tag{12.18}$$

for each $u \in \mathrm{L}^\infty_{\mathrm{loc}}(G) \cap \mathrm{PSH}(G)$.

Proof. Note first that by Theorem 9.1, the image set $f(E)$ is Borel. The multiplicity functions $M(w, f, E)$ and $M_{\min}(f, E)$ are now defined since f is discrete and open. Moreover, we may choose $N^*(w, f, E) = N(w, f, E)$ for every $w \in \mathbb{C}^n$ as the crude multiplicity function $w \mapsto N(w, f, E)$ is Borel measurable in $\mathbb{C}^n$ by Lemma 9.2.

We prove first the equation in (12.18). Make a partition of $f(E)$ like in the proof of Theorem 12.6 such that

$$f(E)_k = \{w \in f(E) : N(w, f, E) = k\}, \quad 1 \leqslant k \leqslant N_{\max}(f, E).$$

Then the sets $f(E)_k$ are disjoint and Borel, and

$$\bigcup_{k=1}^{N_{\max}(f,E)} f(E)_k = f(E).$$

The set $f(E) \setminus f(B_f)$ is Borel since $f(B_f)$ is closed in Ω', and hence the sets $f(E)_k \setminus f(B_f)$ are Borel. Each point $w_k \in f(E)_k \setminus f(B_f)$ has k distinct inverse images in E, $k = 1, \ldots N_{\max}(f, E)$. By Lemma 11.3, we can choose a countable number of points

$w_{k,j} \in f(E)_k \setminus f(B_f)$ and disjoint Borel sets $f(E)_{k,j} \subset f(E)_k \setminus f(B_f)$, $j = 1, 2, \ldots$, such that $w_{k,j} \in f(E)_{k,j}$, each $f(E)_{k,j}$ is contained in a domain $D_{k,j} \subset \Omega' \setminus f(B_f)$,

$$f(E)_k \setminus f(B_f) = \bigcup_{j=1}^{\infty} f(E)_{k,j}$$

and f defines biholomorphic mappings

$$f_{k,j,i} = f|_{U_{k,j,i}} : U_{k,j,i} \longrightarrow D_{k,j},$$

where each $U_{k,j,i} \subset \Omega \setminus B_f$ is a neighbourhood of $z_{k,j,i} = f^{-1}_{k,j,i}(w_{k,j})$ for every $k = 1, \ldots N_{\max}(f, E)$, $j = 1, 2, \ldots$ and $i = 1, \ldots, N_{\max}(f, \Omega)$. Note that the sets $E_{k,j,i} = E \cap f^{-1}_{k,j,i}(f(E)_{k,j})$ are Borel, mutually disjoint and contained in $E \setminus f^{-1}(f(B_f))$. Moreover,

$$E \setminus f^{-1}(f(B_f)) = \bigcup_{k=1}^{N_{\max}(f,E)} \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{N_{\max}(f,\Omega)} E_{k,j,i} = \bigcup_{k=1}^{N_{\max}(f,E)} \bigcup_{j=1}^{\infty} E_{k,j},$$

where we denote

$$E_{k,j} = \bigcup_{i=1}^{N_{\max}(f,\Omega)} E_{k,j,i}.$$

Corollary 12.13 yields now

$$\begin{aligned}\int_{E_{k,j}} -(dd^c(u \circ f))^n &= \sum_{i=1}^{N_{\max}(f,\Omega)} \int_{E_{k,j,i}} -(dd^c(u \circ f))^n = \sum_{i=1}^{N_{\max}(f,\Omega)} \int_{f(E_{k,j,i})} -(dd^c u)^n \\ &= \int_{f(E)_{k,j}} -\left(\sum_{i=1}^{N_{\max}(f,\Omega)} \chi_{f(E_{k,j,i})} \right) (dd^c u)^n = k \int_{f(E)_{k,j}} -(dd^c u)^n\end{aligned}$$

for every $k = 1, \ldots, N_{\max}(f, E)$ and $j = 1, 2, \ldots$, because each point $w \in f(E)_{k,j}$ belongs to k of the sets $f(E_{k,j,i})$. Here χ_A denotes the charasteristic function of a set A. Since $f(B_f)$ and $f^{-1}(f(B_f))$ are pluripolar by Lemma 11.1 (vii,viii), we obtain by [Kli91, Proposition 4.6.4] that

$$\begin{aligned}\int_{f(E)} - N(w, f, E)\,(dd^c u)^n &= \int_{f(E)\setminus f(B_f)} -N(w, f, E)\,(dd^c u)^n \\ &= \sum_{k=1}^{N_{\max}(f,E)} \sum_{j=1}^{\infty} \int_{f(E)_{k,j}} -N(w, f, E)\,(dd^c u)^n = \sum_{k=1}^{N_{\max}(f,E)} \sum_{j=1}^{\infty} k \int_{f(E)_{k,j}} -(dd^c u)^n \\ &= \sum_{k=1}^{N_{\max}(f,E)} \sum_{j=1}^{\infty} \int_{E_{k,j}} -(dd^c(u \circ f))^n = \int_{E\setminus f^{-1}(f(B_f))} -(dd^c(u \circ f))^n \\ &= \int_{E} -(dd^c(u \circ f))^n.\end{aligned}$$

Finally we prove the inequality in (12.18). If $w \in f(E)$ is such that $w \notin f(B_f)$, then $f^{-1}(w)$ does not meet $E \cap B_f$ and, moreover, since $i(z, f) = 1$ for every $z \in E \setminus B_f$ by

Lemma 10.2 (ii), we have

$$N(w, f, E) = N(w, f, E \setminus B_f) = M(w, f, E \setminus B_f) = M(w, f, E) \geqslant M_{\min}(f, E).$$

This yields

$$\begin{aligned}\int_{f(E)} - N(w, f, E)\,(dd^c u)^n &= \int_{f(E)\setminus f(B_f)} -N(w, f, E)\,(dd^c u)^n \\ &\geqslant M_{\min}(f, E) \int_{f(E)\setminus f(B_f)} -(dd^c u)^n = M_{\min}(f, E) \int_{f(E)} -(dd^c u)^n.\end{aligned}$$

□

13. Capacity inequalities

In this section we state capacity inequalities for holomorphic mappings. The first inequality holds for all holomorphic mappings $f : \Omega \to \mathbb{C}^n$ even if the 'correct' class of holomorphic mappings within this context seems to be the proper ones.

13.1. Theorem. *Let $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping. If (K, G) is a compact condenser in Ω, then*

$$\operatorname{cap}(K, G) \geqslant N_{\min}(f, K)\operatorname{cap}(f(K), G') \tag{13.2}$$

for each open set $G' \subset \mathbb{C}^n$ such that $f(G) \subset G'$.

Proof. It is assumed that (K, G) is a compact condenser in Ω, but it may happen that $f(G)$ is not open, and hence $(f(K), f(G))$ is not necessarily a condenser in $\mathbb{C}^n$. However, it is required that $f(G)$ is a subset of an open set G' and hence $(f(K), G')$ is a compact condenser. By Lemma 10.3 (i), $u \circ f \in \mathrm{PSH}(G)$ for all $u \in \mathrm{PSH}(G')$ with values between zero and one, and Corollary 12.11 yields

$$\begin{aligned}\operatorname{cap}(f(K), G') &= \sup_{\substack{u\in \mathrm{PSH}(G')\\ 0\leqslant u\leqslant 1}} \int_{f(K)} -(dd^c u)^n \\ &\leqslant \sup_{\substack{u\in \mathrm{PSH}(G')\\ 0\leqslant u\leqslant 1}} \frac{1}{N_{\min}(f, K)} \int_K -(dd^c(u \circ f))^n \\ &\leqslant \frac{1}{N_{\min}(f, K)} \sup_{\substack{v\in \mathrm{PSH}(G)\\ 0\leqslant v\leqslant 1}} \int_K -(dd^c v)^n \\ &= \frac{1}{N_{\min}(f, K)} \operatorname{cap}(K, G).\end{aligned}$$

□

13.3. Theorem. *Let $f : \Omega \to \Omega'$ be a proper holomorphic mapping. If (E, G) is a Borel condenser in Ω, then*

$$\operatorname{cap}(E, G) \geqslant M_{\min}(f, E)\operatorname{cap}(f(E), f(G)). \tag{13.4}$$

Proof. Note first that $f(E)$ is Borel by Theorem 9.1 and $f(G)$ is open since the mapping f is open by Lemma 11.1 (i). Hence $(f(E), f(G))$ is a Borel condenser in Ω' . The proof of (13.4) goes now along the same lines as the proof of (13.2) but Theorem 12.17 is used instead of Corollary 12.11. □

It is an immediate consequence for the previous result that if G' is an open subset of Ω' such that $f(G) \subset G'$, then

$$\operatorname{cap}(E, G) \geqslant M_{\min}(f, E) \operatorname{cap}(f(E), G'),$$

because by a basic property of the Monge–Ampère capacity (see Section 5)

$$\operatorname{cap}(f(E), f(G)) \geqslant \operatorname{cap}(f(E), G').$$

Note also that the estimate in (13.4) is slightly better than in (13.2). If f is proper holomorphic and thus $M(w, f, E)$ and $M_{\min}(f, E)$ are defined, then $M(w, f, E) \geqslant N(w, f, E) \geqslant 1$ and thus $M_{\min}(f, E) \geqslant N_{\min}(w, f, E)$ for every $w \in f(E)$.

The following corollary presents a sufficient setting to confirm that the Monge–Ampère capacity is decreasing under a proper holomorphic mapping. However, the estimate here is much weaker than the estimate in the previous theorem.

13.5. Corollary. *Let $f : \Omega \to \Omega'$ be a proper holomorphic mapping. If (E, G) is a capacitable condenser in Ω, then*

$$\operatorname{cap}(E, G) \geqslant \operatorname{cap}(f(E), f(G)). \tag{13.6}$$

Proof. Suppose that (E, G) is a capacitable condenser in Ω, i.e., $\operatorname{cap}^*(E, G) = \operatorname{cap}(E, G)$ is satisfied. Let $\varepsilon > 0$. We can choose an open set $U \Subset G$ such that $E \subset U$ and

$$\operatorname{cap}(U, G) - \operatorname{cap}(E, G) < \varepsilon.$$

Now (U, G) is an open condenser in Ω, and 13.3 yields

$$\operatorname{cap}(E, G) > \operatorname{cap}(U, G) - \varepsilon \geqslant \operatorname{cap}(f(U), f(G)) - \varepsilon \geqslant \operatorname{cap}(f(E), f(G)) - \varepsilon,$$

because $f(E) \subset f(U)$. Hence

$$\operatorname{cap}(E, G) \geqslant \operatorname{cap}(f(E), f(G)),$$

as ε was chosen arbitrarily. □

Next corollary states the strongest possible connection between the capacities of a condenser and its image condenser.

13.7. Corollary. *Let $f : \Omega \to \mathbb{C}^n$ be a biholomorphic mapping. If (E, G) is a capacitable condenser in Ω, then*

$$\operatorname{cap}(E, G) = \operatorname{cap}(f(E), f(G)). \tag{13.8}$$

Proof. Since both $G = f^{-1}(f(G))$ and $f(G)$ are open, the previous corollary yields

$$\operatorname{cap}(E, G) \geqslant \operatorname{cap}(f(E), f(G)) \geqslant \operatorname{cap}(f^{-1}(f(E)), f^{-1}(f(G)) = \operatorname{cap}(E, G).$$

□

Our next result is a converse capacity inequality for proper holomorphic mappings.

13.9. Theorem. *Let $f : \Omega \to \Omega'$ be a proper holomorphic mapping. If (E, Ω) is a condenser, then*

$$\operatorname{cap}(E, \Omega) \leqslant N_{\max}(f, \Omega)^n \operatorname{cap}(f(E), \Omega'). \tag{13.10}$$

Proof. The first part of this proof follows the idea of the second part of the proof of Theorem 12.17, but the present proof is organized in a slightly different way. Suppose first that (E, Ω) is a Borel condenser. The set $f(E) \setminus f(B_f)$ is Borel, since $f(E)$ is Borel by Theorem 9.1 and $f(B_f)$ is closed in Ω'. Now each $w \in f(E) \setminus f(B_f)$ has $k = N_{\max}(f, \Omega)$ distinct inverse images in Ω.

By Lemma 11.3, we can choose a countable number of points $w_j \in f(E) \setminus f(B_f)$ and disjoint Borel sets $A_j \subset f(E) \setminus f(B_f)$, $j = 1, 2, \ldots$, such that $w_j \in A_j$, each A_j is contained in a domain $D_j \subset \Omega' \setminus f(B_f)$,

$$f(E) \setminus f(B_f) = \bigcup_{j=1}^{\infty} A_j \tag{13.11}$$

and f defines biholomorphic mappings

$$f_{j,i} = f|_{U_{j,i}} : U_{j,i} \longrightarrow D_j,$$

where each $U_{j,i} \subset \Omega \setminus B_f$ is a neighbourhood of $z_{j,i} = f_{j,i}^{-1}(w_j)$ for every $j = 1, 2, \ldots$ and $i = 1, \ldots, k$. Note that the sets $f_{j,i}^{-1}(A_j)$ are mutually disjoint,

$$\bigcup_{j=1}^{\infty} \bigcup_{i=1}^{k} f_{j,i}^{-1}(A_j) = f^{-1}(f(E)) \setminus f^{-1}(f(B_f)) \supset E \setminus f^{-1}(f(B_f)), \tag{13.12}$$

and $f^{-1}(f(B_f))$ is pluripolar by Lemma 11.1 (viii).

It follows from the formula (11.7) that the push forward function v defined by the formula (11.5) satisfies

$$v(w) = \frac{1}{N_{\max}(f, \Omega)} \sum_{z \in f^{-1}(w)} i(z, f)\, u(z) = \frac{1}{N_{\max}(f, \Omega)} \sum_{i=1}^{k} (u \circ f_{j,i}^{-1})(w)$$

for all $w \in D_j$, $j = 1, 2, \ldots$, and thus

$$dd^c v = \frac{1}{N_{\max}(f, \Omega)} \sum_{i=1}^{k} dd^c (u \circ f_{j,i}^{-1})$$

in every D_j. Corollary 12.13 and the superadditivity property (4.3) of the generalized complex Monge–Ampère operator yield

$$
\begin{aligned}
\int_{A_j} -(dd^c v)^n &= \int_{A_j} -\Big(\frac{1}{N_{\max}(f,\Omega)} \sum_{i=1}^{k} dd^c(u \circ f_{j,i}^{-1})\Big)^n \\
&\geqslant \int_{A_j} \frac{1}{N_{\max}(f,\Omega)^n} \sum_{i=1}^{k} -(dd^c(u \circ f_{j,i}^{-1}))^n \\
&= \frac{1}{N_{\max}(f,\Omega)^n} \sum_{i=1}^{k} \int_{f_{j,i}^{-1}(A_j)} -(dd^c u)^n
\end{aligned}
\tag{13.13}
$$

for each A_j and $u \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega) \cap \mathrm{PSH}(\Omega)$.

Theorem 11.4 shows that for each $u \in \mathrm{PSH}(\Omega)$ such that $0 \leqslant u \leqslant 1$, the function v satisfies the properties $v \in \mathrm{PSH}(\Omega')$ and $0 \leqslant v \leqslant 1$. By Lemma 11.1 (vii,viii) and [Kli91, Proposition 4.6.4], we obtain from (13.11), (13.12) and (13.13)

$$
\begin{aligned}
\operatorname{cap}(f(E),\Omega') &= \sup_{\substack{u' \in \mathrm{PSH}(\Omega') \\ 0 \leqslant u' \leqslant 1}} \int_{f(E)} -(dd^c u')^n \geqslant \int_{f(E)} -(dd^c v)^n \\
&= \sum_{j=1}^{\infty} \int_{A_j} -(dd^c v)^n \geqslant \frac{1}{N_{\max}(f,\Omega)^n} \sum_{j=1}^{\infty} \sum_{i=1}^{k} \int_{f_{j,i}^{-1}(A_j)} -(dd^c u)^n \\
&= \frac{1}{N_{\max}(f,\Omega)^n} \int_{f^{-1}(f(E))} -(dd^c u)^n \geqslant \frac{1}{N_{\max}(f,\Omega)^n} \int_{E} -(dd^c u)^n,
\end{aligned}
$$

because the Radon measures $-(dd^c v)^n$ and $-(dd^c u)^n$ are countably additive. Since the previous inequality holds for all $u \in \mathrm{PSH}(\Omega)$ with $0 \leqslant u \leqslant 1$, we have

$$
\operatorname{cap}(f(E),\Omega') \geqslant \frac{1}{N_{\max}(f,\Omega)^n} \sup_{\substack{u \in \mathrm{PSH}(\Omega) \\ 0 \leqslant u \leqslant 1}} \int_{E} -(dd^c u)^n = \frac{1}{N_{\max}(f,\Omega)^n} \operatorname{cap}(E,\Omega).
$$

Suppose then that (E,Ω) is an arbitrary condenser. If $K \subset E$ is a compact set, then $f(K) \subset f(E)$ is a compact set as f is continuous. By the first part of this proof, this yields our final result

$$
\begin{aligned}
\operatorname{cap}(E,\Omega) &= \sup_{\substack{K \subset E \\ \text{is compact}}} \operatorname{cap}(K,\Omega) \leqslant \sup_{\substack{K \subset E \\ \text{is compact}}} N_{\max}(f,\Omega)^n \operatorname{cap}(f(K),\Omega') \\
&\leqslant N_{\max}(f,\Omega)^n \sup_{\substack{K' \subset f(E) \\ \text{is compact}}} \operatorname{cap}(K',\Omega') = N_{\max}(f,\Omega)^n \operatorname{cap}(f(E),\Omega').
\end{aligned}
$$

□

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