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QUASIHYPERBOLIC GEODESICS AND UNIFORMITY IN ELEMENTARY DOMAINS

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Contents

1. Introduction and results

The class of uniform domains was introduced by O. Martio and J. Sarvas in [MaSa]. In the original definition a domain $G \subset \mathbb{R}^n$ was said to be uniform if there exists a constant $c \geq 1$ such that all $z_1, z_2 \in G$ can be joined by a curve $\alpha \subset G$ for which

$$l(\alpha) \leq c\ |z_1 - z_2|, \tag{1.1}$$

$$\min_{j=1,2} l(\alpha_j) \leq c\ \operatorname{dist}(z, \partial G) \tag{1.2}$$

for each $z \in \alpha$, where α_1, α_2 are the components of $\alpha \setminus \{z\}$. Ever since their introduction uniform domains have shown to have very useful properties, and in fact to be the "right kind" of domains to consider in many applications. For instance, it is shown in [Ge] that many classical results of function theory, originally proved for functions defined on the unit disk, hold for functions defined on uniform domains. It is well-known that a simply connected proper subdomain of the plane is a quasidisk if and only if it is a uniform domain.

In this work we shall consider another definition of uniform domains, originally stated by F. Gehring and B. Osgood in [GeOs]. This alternative definition uses comparison between the quasihyperbolic and the distance ratio metrics defined in (2.2) and (2.3), and involves also a constant $A \geq 1$ which in general is not the same as the constant c in the definition by Martio and Sarvas. As this alternative definition is better suited to the studies here, we will adhere to the uniformity concept of Definition 2.4 in this work.

The class of uniform domains is very wide, for instance it includes images of the unit ball $\mathbf{B}^n$ under a quasiconformal mapping of $\mathbb{R}^n$ into itself. It is perhaps surprising that there are very few examples of domains for which the uniformity constant A is known, and in this work we study some of the simplest cases. In most cases this task involves finding the geodesic segments of the domain studied – in general a difficult problem – which is also previously unsolved in some of the cases studied here. Often such fine-tuned analysis requires domain-specific methods, however, some of the techniques used might be of interest also in more general studies of the metrics involved.

In section 3 we prove some general lemmas concerning the quasihyperbolic metric and the distance ratio metrics. In section 4 the main subject is to derive the geodesic segments. Here the main results are the ones obtained for the planar angular domains

$$S_\varphi = \{(r, \theta) \in \mathbb{R}^2 \ :\ 0 < \theta < \varphi\},$$

and for the punctured ball $\mathbf{B}^n \setminus \{0\}$. In the following k_G denotes the quasihyperbolic metric in the domain G.

We next formulate some of our main results.

Theorem 1.3. *Let $\varphi \in (0, \pi]$, and $x, y \in S_\varphi$. Then the quasihyperbolic geodesic segment $J_{k_{S_\varphi}}[x, y]$ is a curve consisting of line segments and circular arcs orthogonal to the boundary, as explained in Theorem 4.6.*

Theorem 1.4. *Let $\varphi \in (\pi, 2\pi)$, and $x, y \in S_\varphi$. Then the quasihyperbolic geodesic segment $J_{k_{S_\varphi}}[x,y]$ is a curve consisting of line segments, logarithmic spirals and circular arcs orthogonal to the boundary, as explained in Theorem 4.10.*

Theorem 1.5. *Let $x, y \in \mathbf{B}^n_* = \mathbf{B}^n \setminus \{0\}$. Then the quasihyperbolic geodesic segment $J_{k_{\mathbf{B}^n_*}}[x,y]$ is a curve consisting of logarithmic spirals and geodesic segments of the quasihyperbolic metric in $\mathbf{B}^n$, as explained in Theorem 4.13.*

In section 5 we derive uniformity constant estimates for different domains. We study the space $\mathbb{R}^n \setminus \{0\}$, the angular domains S_φ and the punctured ball $\mathbf{B}^n \setminus \{0\}$. Finally, we include a few implications on some polygonal domains. However, for polygonal domains the geodesics are not known, which results in a lack of sharpness compared to the earlier mentioned domains, for which the estimates are best possible.

Theorem 1.6. *For the domain $\mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\}$ the uniformity constant is*

$$A_{\mathbb{R}^n_*} = \frac{\pi}{\log 3} \approx 2.8596.$$

Theorem 1.7. *For the plane domain S_φ the uniformity constant is given by*

$$A_{S_\varphi} = \frac{1}{\sin\frac{\varphi}{2}} + 1$$

when $\varphi \in (0, \pi]$.

Using the case of small angles we get bounds also for the case of large angles $\varphi \in (\pi, 2\pi)$. These, however, are not sharp.

Theorem 1.8. *Let $x, y \in S_\varphi$, where $\pi < \varphi < 2\pi$. Then we have that*

$$\max\left\{2, \frac{2\log\tan\frac{\varphi}{4} + \varphi - \pi}{\log(1 - 2\cos\frac{\varphi}{2})}\right\} \le A_{S_\varphi} \le 4\left(\frac{\varphi}{2\pi - \varphi}\right)^2 \left(\frac{1}{\sin\frac{\varphi}{2}} + 1\right).$$

For the punctured ball $\mathbf{B}^n_* = \mathbf{B}^n \setminus \{0\}$ we get, the following result for the uniformity constant.

Theorem 1.9. *For the domain $\mathbf{B}^n_*$ we have that*

$$A_{\mathbf{B}^n_*} = A_{\mathbb{R}^n_*} = \frac{\pi}{\log 3}.$$

The geometric methods in the geodesics proofs use results from [Ma] and [MaOs]. In the results regarding uniformity constant estimates, the methods are in general very elementary. The case of large angles $\varphi \in (\pi, 2\pi)$ employs a result from [GeHa].

2. Preliminaries and definitions

A *path* in $\mathbb{R}^n$ is a continuous mapping $\alpha\colon \Delta \to \mathbb{R}^n$, where Δ is an interval in $\mathbb{R}$. The *locus* of a path is the point set $\alpha\Delta \subset \mathbb{R}^n$, which will also be referred to as the *curve* α. We often talk about curves α instead of paths, because mostly the choice of path is unimportant. If $\Delta = [a,b]$, and $\alpha(a) = x$, $\alpha(b) = y$, we say that α is a *curve connecting* x *and* y. A *subpath* of α is the restriction of α to a subinterval $\Delta' \subset \Delta$. Then it is clear that by a *subcurve* we mean just $\alpha\Delta'$. If $\Delta' = [c,d]$, $c,d \in \mathbb{R}$, we write $\alpha[c,d]$ for the restriction of α to $[c,d]$.

Let $\alpha\colon [a,b] \to \mathbb{R}^n$ be a path, and let $a = t_0 \le t_1 \le \cdots \le t_k = b$ be a subdivision of $[a,b]$. Such a subdivision is called a *length sequence.* Let (G,d) be a metric space. The *d-pathlength* of α is then defined as

$$l_d(\alpha) = \sup \sum_{i=1}^{k} d\big(\alpha(t_i), \alpha(t_{i-1})\big), \tag{2.1}$$

where the supremum is over all length sequences. The length of the curve α is defined as the length of a corresponding path α. A more detailed reference to these matters, as well as integration of paths, is [Väl], chapters 1.1-1.5. When $d = |\cdot|$ is the Euclidean metric, we usually denote $l_d(\alpha) = l(\alpha)$.

For a given domain $G \subsetneq \mathbb{R}^n$ we define the *quasihyperbolic distance* between x and y in G by

$$k_G(x,y) = \inf_{\alpha \in \Gamma_{xy}} \int_\alpha \frac{|dz|}{d(z)}, \tag{2.2}$$

where $d(z)$ denotes the distance to the boundary, $\operatorname{dist}(z,\partial G)$, and Γ_{xy} stands for the family of all rectifiable curves joining x and y in G (see [GePa]). It is clear that k_G is a metric on G, and it is also well known that it is invariant under similarity mappings, i.e. translations, stretchings and orthogonal mappings. The rectifiable curve α for which the infimum in (2.2) is attained is called the *quasihyperbolic geodesic* connecting the points x and y, and is known to exist for all pairs of points ([GeOs, Theorem 2.8]).

A metric space like this, where a geodesic segment always exists, is called a *geodesic metric space.* Any subcurve of a geodesic is also a geodesic. A geodesic curve between points x and y in the metric d is denoted by $J_d[x,y]$, or $J[x,y]$ if it is clear what metric we are using.

We also define

$$j_G(x,y) = \log\Big(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\Big) \tag{2.3}$$

for every $x, y \in G$. This is also a metric, often referred to as the *distance ratio metric* or the *j-metric.* It is known to have invariance and monotonicity properties similar to the quasihyperbolic metric (see the beginning of Section 3). For instance, it is also invariant under translations, stretchings and orthogonal mappings. On the other hand, unlike the case of k_G, the metric j_G mostly fails to have geodesics (see [HäIbLi]).

Definition 2.4. A domain $G \subsetneq \mathbb{R}^n$ is said to be *uniform*, if there exists a number $A \geq 1$ such that

$$k_G(x,y) \leq A\ j_G(x,y) \tag{2.5}$$

for all $x, y \in G$. Furthermore, the number

$$A_G = \inf\{A \geq 1 \mid A \text{ satisfies } (2.5)\}$$

is called the *uniformity constant* of G.

Remarks 2.6. *i)* Using Theorems 1 and 2 in [GeOs] and [Vu1, 2.50], we see that if the constant c in the conditions (1.1) and (1.2) is known, then Definition 2.4 holds true with constant

$$A = 2\Big(c + \frac{c + c\log c + 1}{\log \frac{3}{2}}\Big).$$

Vice versa, if A is known, the original conditions hold with

$$c = 64A^2\ e^{64A^2}.$$

ii) The uniformity constant is known previously only in the cases of the disk $\mathbf{B}^n$ and the halfspace $\mathbf{H}^n$, and its value is then 2, which is shown in [Vu2, Lemma 2.41] for $\mathbf{H}^n$ and in [AnVaVu, Lemma 7.56] for $\mathbf{B}^n$.

iii) The inequality $j_G(x,y) \leq k_G(x,y)$ holds for every G and $x, y \in G$ [GePa, Lemma 2.1], and thus uniformity is equivalent to the existence of a two-sided linear estimate of the quasihyperbolic metric in terms of the j_G-metric.

iv) The quasihyperbolic geodesics are – according to the definition used here – length minimizing curves, and thus also local length minimizers. However, local length minimizers, for which the term "geodesic" is also widely used in the literature, are usually not geodesics in the sense that we speak of here.

We sometimes need to compare the quasihyperbolic metric with the *hyperbolic metric*, which is classically defined in either the disk $\mathbf{B}^n$ or in the half-space $\mathbf{H}^n$. In the case of the half-space it is known that $k_{\mathbf{H}^n}(x,y) = \rho_{\mathbf{H}^n}(x,y)$ for all $x, y \in \mathbf{H}^n$. This is convenient, since there is an explicit formula for the hyperbolic distance [Be1, p. 40]

$$\cosh \rho_{\mathbf{H}^n}(x,y) = 1 + \frac{|x-y|^2}{2d(x)d(y)}, \quad x, y \in \mathbf{H}^n. \tag{2.7}$$

In some special cases the formula is even simpler; let e_n denote the n:th unit vector in the standard basis. Then, for $r, s > 0$ we have that

$$\rho_{\mathbf{H}^n}(re_n, se_n) = \Big|\log \frac{r}{s}\Big|, \tag{2.8}$$

and if $\varphi \in (0, \frac{\pi}{2})$ and we denote $u_\varphi = (\cos\varphi)e_1 + (\sin\varphi)e_n$, we get

$$\rho_{\mathbf{H}^n}(e_n, u_\varphi) = \log\cot \tfrac{\varphi}{2}. \tag{2.9}$$

In the ball $\mathbf{B}^n$ we only know that $k_{\mathbf{B}^n}(x,y)/\rho_{\mathbf{B}^n}(x,y) \in [\frac{1}{2},1]$. Here we have the formula [Be1, p. 35]

$$\sinh^2\left(\tfrac{1}{2}\rho_{\mathbf{B}^n}(x,y)\right) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}. \tag{2.10}$$

For $0<t<1$ we have

$$\rho_{\mathbf{B}^n}(0,te_n) = \log\frac{1+t}{1-t} = 2\,\mathrm{ar}\tanh t. \tag{2.11}$$

It is also possible to define the hyperbolic metric for a subdomain G of the plane for which there exists a conformal mapping $f\colon G \to fG = \mathbf{B}^2$. Then the metric density is defined by

$$\rho_G(z) = \rho_{\mathbf{B}^2}(f(z))|f'(z)|,$$

and from the Schwarz lemma it follows that ρ_G is independent of the choice of f. Then, as for the quasihyperbolic metric, we define the hyperbolic metric by integrating the density over all curves, that is

$$h_G(x,y) = \inf_{\alpha\in\Gamma_{xy}} \int_\alpha \rho_G(z)|dz|.$$

Clearly the metric space (G,h_G) is also geodesic. Note that while in the classical cases we use the traditional notation $\rho_{\mathbf{B}^n}$ and $\rho_{\mathbf{H}^n}$ for the hyperbolic metric, in general domains we use h_G.

For future use we also mention some useful inequalities for the inverse hyperbolic functions, namely the relations

$$\log(1+x) \leq \mathrm{ar}\sinh x \leq 2\log(1+x), \tag{2.12}$$

$$2\log\left(1+\sqrt{\tfrac{1}{2}(x-1)}\right) \leq \mathrm{ar}\cosh x \leq 2\log\left(1+\sqrt{2(x-1)}\right), \tag{2.13}$$

which hold in the intervals $x \geq 0$ and $x \geq 1$, respectively. We will also make use of Bernoulli's inequalities

$$\log(1+as) \leq a\log(1+s);\ a\geq 1, s\geq 0, \tag{2.14}$$

$$a\log(1+s) \leq \log(1+as);\ a\leq 1, s\geq 0. \tag{2.15}$$

We close this section with some geometric notation and definitions. Note that some of the concepts are meaningful only for $n=2$.

Circles are usually denoted by uppercase letters, such as $C = S^1(x,r)$. The radius of the circle C is denoted by $\mathrm{rad}(C)$. Lines and points are denoted with lowercase letters, such as l or p. Given two points x and y, the segment between them is denoted by

$$[x,y] = \{(1-t)x+ty \mid 0\leq t\leq 1\}. \tag{2.16}$$

Given a vector u and a point $x \in \mathbb{R}^n$, the line passing through x with direction vector u is denoted by $L(x,u)$. The open ray emanating from x in the direction of u is denoted by $\mathrm{ray}(x,u)$. The hyperplane orthogonal to u and passing through x is denoted by $P(x,u)$. Hence

$$\begin{aligned} L(x,u) &= \{x+tu \mid t\in\mathbb{R}\},\\ \mathrm{ray}(x,u) &= \{x+tu \mid t>0\} \text{ and}\\ P(x,u) &= \{z\in\mathbb{R}^n \mid (z-x)\cdot u = 0\}. \end{aligned}$$

As the standard basis of $\mathbb{R}^n$ is denoted by $(e_1, e_2, \ldots, e_n)$, for instance $\mathrm{ray}(0,e_1)$ denotes the positive x-axis. If the lines l_1 and l_2 are parallel, i.e. if their direction vectors v_1 and v_2 satisfy $v_1 = v_2$ or $v_1 = -v_2$, this is denoted by $l_1 \parallel l_2$, and if the lines are orthogonal we denote $l_1 \perp l_2$. Orthogonality of two circles is denoted by the same symbol $\perp$. Given three points x, y and $z \in \mathbb{R}^n$, the notation $\widehat{x,z,y}$ means the angle in the range $[0,\pi]$ between the segments $[x,z]$ and $[y,z]$. Given two differentiable curves, typically a circle C and a line l, which are tangent at a point z, we denote this by $C \bigcirc_z l$. If the point z is unimportant, we just denote $C \bigcirc l$. Finally, by $\mathrm{comp}(A,x)$ we mean the component of the set A containing the point x.

3. Properties of the quasihyperbolic metric

Even though the quasihyperbolic metric has been utilised as a tool in many different contexts, very little is known regarding many natural questions. We proceed by formulating some general inequalities for the quasihyperbolic and distance ratio metrics. First, we recall that k_G has the following monotonicity property: if G and G' are domains with $G' \subset G$ and $x, y \in G'$, then $k_{G'}(x,y) \geq k_G(x,y)$. This follows directly from the definition, and the same of course can be proved also for the j-metric. Some upper and lower bounds for the k_G and j_G metrics in certain special cases can be found e.g. in [Vä2] and [Vu2, Chapter 3].

Lemma 3.1. ([Vu2, 3.7.]) *i) If $x \in G$, $y \in B_x = B^n(x, d(x))$, then*

$$k_G(x,y) \leq \log\left(1 + \frac{|x-y|}{d(x) - |x-y|}\right).$$

ii) If $s \in (0,1)$ and $|x-y| \leq s\, d(x)$, then

$$k_G(x,y) \leq \frac{1}{1-s}\, j_G(x,y).$$

□

Lemma 3.2. ([Vu2, 3.17.]) *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be an L-bilipschitz mapping, that is*

$$|x-y|/L \leq |f(x) - f(y)| \leq L|x-y|$$

for all $x, y \in \mathbb{R}^n$, and let $G \subset \mathbb{R}^n$ be uniform with constant A. Then fG is uniform and satisfies

$$\frac{k_{fG}(u,v)}{j_{fG}(u,v)} \leq AL^4$$

for all $u, v \in fG$. □

For general domains we also have the following universal lower bound.

Lemma 3.3. *Let $G \subset \mathbb{R}^2$ be a domain and let $B \subset G$ be a disk such that $\#(\overline{B} \cap \partial G) \geq 2$. If $x, y \in G$ lie in different components of $G \setminus \overline{B}$, the j-distance, and thus the k-distance, are bounded from below by the constant* $\log(1+\sqrt{2})$.

PROOF: Assume that x and y are such that $\mathrm{comp}(G \setminus \overline{B}, x) \neq \mathrm{comp}(G \setminus \overline{B}, y)$, and that $\#([x,y] \cap \partial B) = 2$. Let c be the center of B. Then there exists a point $z \in \overline{B} \cap \partial G$ such that z is also in the set $\overline{B} \setminus \mathrm{comp}(\overline{B} \setminus [x,y], c)$, namely, otherwise we would have $\mathrm{comp}(G \setminus \overline{B}, x) = \mathrm{comp}(G \setminus \overline{B}, y)$. Denoting $R = \mathbb{R}^2 \setminus \{z\}$ we can also assume that $\mathrm{dist}(x, \partial R) \leq \mathrm{dist}(y, \partial R)$. Let $y' \in [y,z]$ be such that $|y'-z| = |x-z|$. Then, letting φ be the angle $\widehat{y',x,z}$, we see that $\varphi \leq \frac{\pi}{4}$, as the maximal case occurs when x and y are antipodal on ∂B. Thus

$$\mathrm{dist}(x, \partial R) = |x-z| = \frac{|x-y'|/2}{\cos\varphi} \leq \frac{|x-y'|/2}{1/\sqrt{2}} = \frac{|x-y'|}{\sqrt{2}}.$$

Then

$$\begin{aligned} j_G(x,y) &\geq j_{\mathbb{R}^n \setminus E}(x,y) = \log\left(1 + \frac{|x-y|}{|x-z|}\right) \\ &\geq \log\left(1 + \frac{|x-y'|}{\frac{\sqrt{2}}{2}|x-y'|}\right) = \log(1+\sqrt{2}). \end{aligned}$$

If $\#([x,y] \cap \partial B) = 1$ or $[x,y] \cap \partial B = \emptyset$, we denote $E = [x,y] \cap \partial G$. Now, let $R = \mathbb{R}^n \setminus E$ and assume that $\mathrm{dist}(x,E) \leq \mathrm{dist}(y,E)$. Also, let $z \in E$ be the point for which $\mathrm{dist}(x,E) = |x-z|$. Now

$$\mathrm{dist}(x,E) = |x-z| \leq \tfrac{1}{2}|x-y|,$$

where equality may occur when E is a one-point set. Hence

$$j_G(x,y) \geq j_{\mathbb{R}^n \setminus E}(x,y) = \log\left(1 + \frac{|x-y|}{|x-z|}\right) \geq \log\left(1 + \frac{|x-y|}{\frac{1}{2}|x-y|}\right) \geq \log 3.$$

This proves the statement. □

To clarify the connection between the quasihyperbolic and the distance ratio metric further, we introduce the concept of the inner metric. Let $G \subset \mathbb{R}^n$ be a domain, and (G,d) be a metric space. Then the *inner metric* of the metric d, denoted by $\tilde{d}$ is defined by

$$\tilde{d}(x,y) = \inf_{\gamma} l_d(\gamma), \tag{3.4}$$

where the infimum is over all curves joining x and y, and d-length is defined as in (2.1).

By the triangle inequality it is immediately clear that for any metric d we have $d \leq \tilde{d}$. If also the opposite inequality holds, a metric is called *intrinsic*. All geodesic metrics, such as the quasihyperbolic metric, are trivially intrinsic. The distance ratio metric j, however, is not intrinsic. On the other hand we have the following well-known result.

Lemma 3.5. *For any proper subdomain G of $\mathbb{R}^n$ we have that $\tilde{j}_G = k_G$.*

PROOF: The part $\tilde{j}_G(x,y) \le k_G(x,y)$ follows easily from applying the inequality $j(x,y) \le k(x,y)$ on a length sequence along the quasihyperbolic geodesic. The part $\tilde{j}_G(x,y) \ge k_G(x,y)$ follows from the inequality in part ii) of Lemma 3.1, which is valid for $a \in G,\ 0 < t < 1$ and $x, y \in \overline{B}(a, t \operatorname{dist}(a, \partial G))$. $\square$

Remark 3.6. A subdomain G of a metric space (X,d) is said to be K-*quasiconvex* if there exists a constant $K \ge 1$ such that for each pair of points $x, y \in G$ one can find a connecting curve $\gamma \in \Gamma_{xy}$ such that

$$l_d(\gamma) \le K\, d(x,y).$$

Then, by the previous lemma we see that a uniform domain G is K-quasiconvex with respect to the j_G-metric, and that actually

$$A_G = \inf\{K \mid G \text{ is } K\text{−quasiconvex}\}.$$

The question whether and how much the quasihyperbolic distance grows or shrinks when points are moved, or the boundary is deformed in certain ways, depends very much on the geometry of the situation. The following quite obvious lemma is mentioned because of its convenience for the applications in the sequel. Namely, in many situations it is easy to verify the conditions of the lemma by using elementary geometry.

Lemma 3.7. *Let $G, G' \subset \mathbb{R}^n$ be domains, and let $x, y \in G,\ x', y' \in G'$. Also let $\alpha \in \Gamma_{xy}$ and $\beta \in \Gamma_{x'y'}$ be rectifiable curves with corresponding paths $\gamma\colon [a,b] \to G$ and $\gamma'\colon [a,b] \to G'$. If for every length sequence $\{t_i\}_{i=0,\dots,k}$ there is another length sequence $\{s_i\}_{i=0,\dots,k}$ such that*

$$|\alpha(t_i) - \alpha(t_{i-1})| \le |\beta(s_i) - \beta(s_{i-1})| \quad \text{for all} \quad i = 1, \dots, k \ \text{and} \tag{3.8}$$

$$d(\alpha(t_i)) \ge d(\beta(s_i)) \quad \text{for all} \quad i = 0, \dots, k, \tag{3.9}$$

then $l_{j_G}(\alpha) \le l_{j_{G'}}(\beta)$. If we have strict inequality in (3.8) or (3.9), then $l_{j_G}(\alpha) < l_{j_{G'}}(\beta)$. Furthermore, if $\beta = J_{k_{G'}}[x', y']$, then $k_G(x,y) \le k_{G'}(x', y')$.

PROOF: Let $\{t_i\}_{i=0,\dots,k}$ be an arbitrary length sequence. By (3.8) and (3.9), we have that

$$\begin{aligned}
\sum_{i=1}^{k} j_G\big(\alpha(t_i), \alpha(t_{i-1})\big) &= \sum_{i=1}^{k} \log\Big(1 + \frac{|\alpha(t_i) - \alpha(t_{i-1})|}{\min\{d(\alpha(t_i)), d(\alpha(t_{i-1}))\}}\Big) \\
&\le \sum_{i=1}^{k} \log\Big(1 + \frac{|\beta(s_i) - \beta(s_{i-1})|}{\min\{d(\beta(s_i)), d(\beta(s_{i-1}))\}}\Big) \\
&\le \sum_{i=1}^{k} j_{G'}\big(\beta(s_i), \beta(s_{i-1})\big) \\
&\le \sup_{\{u_i\}_i} \sum_{i=1}^{k} j_{G'}\big(\beta(u_i), \beta(u_{i-1})\big) = l_{j_{G'}}(\beta).
\end{aligned}$$

Since this is true for every length sequence, $l_{j_G}(\alpha) \le l_{j_{G'}}(\beta)$. The case of strict inequalities is proved in the same way. By Lemma 3.5 $\tilde{j}_G = k_G$, that is $k_G(x,y) = \inf_\alpha l_{j_G}(\alpha)$. Thus, if $\beta = J_{k_{G'}}[x', y']$, we have

$$k_G(x,y) \ \le \ l_{j_G}(\alpha) \le l_{j_{G'}}(\beta) = k_{G'}(x', y').$$

□

4. Geodesic segments

Even though it is known that the quasihyperbolic metric has geodesic segments in every domain G, there are very few cases in which the geodesics are known. In the half-plane $\mathbf{H}^n$ the quasihyperbolic metric coincides with the hyperbolic metric, and so the geodesics are circular arcs orthogonal to the boundary hyperplane. In the cases of the ball $\mathbf{B}^n$ and the punctured space $\mathbb{R}^n \setminus \{0\}$ the geodesics were computed by Martin and Osgood in [MaOs, p.38-41]. The goal in this section is to obtain the geodesic segments in the planar angular domain S_φ and in the punctured ball $\mathbf{B}^n \setminus \{0\}$. In doing this we will employ a smoothness result of geodesic segments proved in [Ma].

We start by studying the angular domain $S_\varphi \subset \mathbb{R}^2$ in the case where $\varphi \in (0, \pi]$. For the bisector of the domain S_φ we use the notation

$$\ell_\varphi = \mathrm{ray}(0, e^{\frac{\varphi}{2}i}).$$

Then we state the following useful geometric observation.

Lemma 4.1. *Assume that $x = (r, \theta) \in S_\varphi$ is a point in the lower component of $S_\varphi \setminus \ell_\varphi$, i.e $\theta < \varphi/2$. Then there exist unique circles $c(x)$ and $C(x)$ such that $x \in c(x) \cap C(x)$, $c(x) \perp L(0, e_1)$, $C(x) \perp L(0, e_1)$ and furthermore $c(x) \underline{\bigcirc}\, \ell_\varphi$ and $C(x) \underline{\bigcirc}\, \ell_\varphi$. The centers of these circles have x-coordinates*

$$\begin{aligned} x_c &= r\,\frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\frac{\varphi}{2}}}{\cos^2\frac{\varphi}{2}} \quad \textit{and} \\ x_C &= r\,\frac{\cos\theta + \sqrt{\cos^2\theta - \cos^2\frac{\varphi}{2}}}{\cos^2\frac{\varphi}{2}} \end{aligned}$$

and their radii are

$$r_c = x_c \sin\tfrac{\varphi}{2}, \quad r_C = x_C \sin\tfrac{\varphi}{2}.$$

The circles are called the small and the large tangent circle of x, respectively. The corresponding symmetric theorem holds for points in the upper component. □

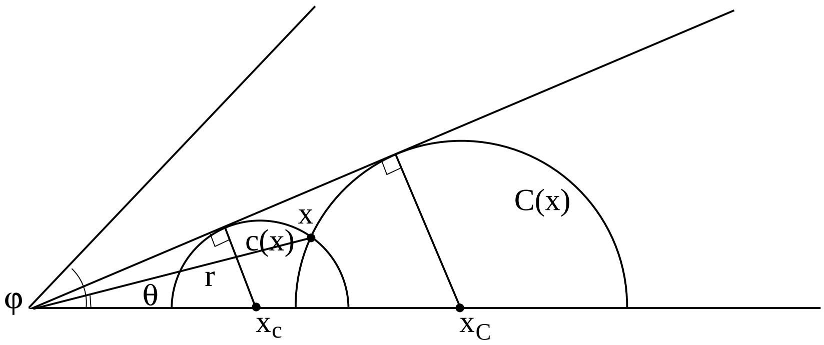

Figure 1. The small and the large tangent circles of a point x.

We first note that for some special cases the question of the geodesics is easily solved. Clearly, for two points $x, y \in \ell_\varphi$ we have $J_k[x,y] = [x,y]$, since by Lemma 3.7 we have $l_j([x,y]) \le l_j(\gamma)$ for every $\gamma \in \Gamma_{xy}$, and thus $[x,y]$ gives the infimum. Also, if the hyperbolic geodesic $J_\rho[x,y]$ is contained entirely in a component of $S_\varphi \setminus \ell_\varphi$, it is clear that $J_k[x,y] = J_\rho[x,y]$. We will now describe the idea behind the use of Lemma 4.1. Namely, assume that x and y are in the same component of $S_\varphi \setminus \ell_\varphi$, and for instance $|y| < |x|$. Also let the points z_1 and z_2 be such that

$$C(y) \,\underline{\bigcirc}_{z_1}\, \ell_\varphi \quad \text{and} \quad c(x) \,\underline{\bigcirc}_{z_2}\, \ell_\varphi. \tag{4.2}$$

Then it is clear that the above reduction occurs exactly when the intersection points satisfy the relation $|z_1| \ge |z_2|$. In the case of equality we have $C(y) = c(x)$. We will need to put somewhat more effort on the case where $|z_1| < |z_2|$.

Lemma 4.3. *Assume that $x, y \in S_\varphi$ are in the lower component of $S_\varphi \setminus \ell_\varphi$, $|y| \le |x|$, and that z_1, z_2 are points such that*

$$C(y) \,\underline{\bigcirc}_{z_1}\, \ell_\varphi \quad \textit{and} \quad c(x) \,\underline{\bigcirc}_{z_2}\, \ell_\varphi,$$

and that $|z_1| < |z_2|$. Then the quasihyperbolic geodesic connecting x and y is the curve

$$J_k[x,y] = J_\rho[x,z_2] \cup [z_2,z_1] \cup J_\rho[z_1,y].$$

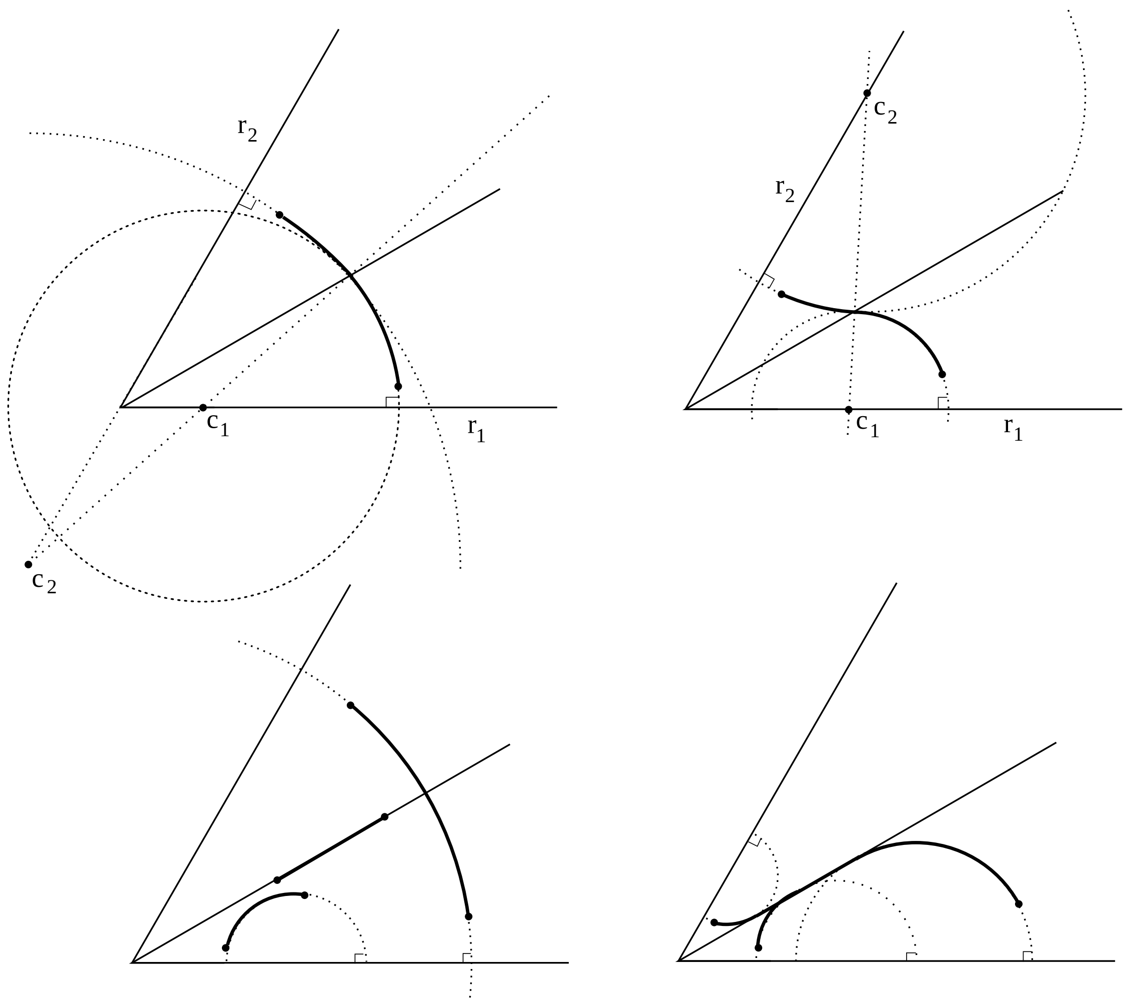

Figure 2. Different types of geodesic segments in S_φ.

PROOF: Let $l_1 = L(z_1, e_2)$ and $l_2 = L(z_2, e_2)$. Clearly, both $J_k[x,y] \cap l_1 = \{w_1\}$ and $J_k[x,y] \cap l_2 = \{w_2\}$ must consist of exactly one point. Let γ be a curve connecting x and y which is not the curve suggested in the statement of the lemma. Then, if $w_1 = z_1$ and $w_2 = z_2$, one of the subcurves $\gamma[x, z_2]$, $\gamma[z_2, z_1]$ or $\gamma[z_1, y]$ has to differ from $J_\rho[x, z_2]$, $[z_2, z_1]$ or $J_\rho[z_1, y]$, respectively, and so γ cannot be the geodesic. Then we may assume that either $w_1 \neq z_1$ or $w_2 \neq z_2$. Hence, if the subcurve $\gamma[y, w_1]$ is not the circular arc $J_\rho[y, w_1]$, replacing it with $J_\rho[y, w_1]$ gives a shorter curve, which shows that γ is not a geodesic. The same argument applies for x and w_2 at the other end of γ.

Finally, assume that $\gamma[y, w_1] = J_\rho[y, w_1]$ and $\gamma[x, w_2] = J_\rho[x, w_2]$. Then consider the subcurve $\gamma[w_1, w_2]$. First assume, that $\gamma[w_1, w_2]$ is above $[w_1, w_2]$, i.e. $\gamma[w_1, w_2] \subset C_1$, where $C_1 = \mathrm{comp}(S_\varphi \setminus L(w_1, w_2 - w_1), z_2)$. But then γ is not differentiable at either w_1 or w_2, and is thus not a geodesic by [Ma, Corollary 4.8]. Then assume that $\gamma[w_1, w_2]$ is partly below the line segment $[w_1, w_2]$, say, that

$\gamma[w_1, w_*]$, $w_* \in [w_1, w_2]$, is contained in the lower component of $S_\varphi \setminus L(w_1, w_2 - w_1)$. But then the curve $J_\rho[y, w_1] \cup [w_1, w_*] \cup \gamma[w_*, w_2] \cup J_\rho[w_2, x]$ is a shorter curve than γ by Lemma 3.7, and γ is not a geodesic. Then we have proved that any curve which is not the curve

$$J_\rho[x, z_2] \cup [z_1, z_2] \cup J_\rho[z_1, y]$$

fails to be the quasihyperbolic geodesic connecting x and y. Now the existence of the quasihyperbolic geodesic proves the statement. □

The proof for the case where x and y are on opposite sides of ℓ_φ follows the same idea as in the case $|z_1| < |z_2|$. For the case $|z_1| \geq |z_2|$ we need to prove some lemmas first. In this case we only prove the existence of the construction, since the formulas for the circular arcs involved get very complicated.

Lemma 4.4. *Assume that $x, y \in S_\varphi \subset \mathbb{R}^2$ are such that y is in the upper component of $S_\varphi \setminus \ell_\varphi$, x is in the lower component, and $|y| \leq |x|$. Let z_1 and z_2 be such that $C(y) \mathrel{\Omega}_{z_1} \ell_\varphi$ and $c(x) \mathrel{\Omega}_{z_2} \ell_\varphi$, and assume that $|z_1| \geq |z_2|$. Then there exists a unique point $z \in [z_1, z_2] \subset \ell_\varphi$ such that the arc*

$$J_{\rho_x}[x, z] \cup J_{\rho_y}[z, y]$$

is differentiable at z, where ρ_x and ρ_y stand for the hyperbolic metrics in the halfplanes defined by the lines $L(0, e_1)$ and $L(0, e^{i\phi})$, respectively.

Proof: Let $\mathcal{C}_x$ be the family of circles C such that

$$\mathcal{C}_x = \{C \ : \ x \in C, \ C \perp L(0, e_1)\},$$

and

$$\mathcal{C}_y = \{C \ : \ y \in C, \ C \perp L(0, e^{\varphi i})\}.$$

For each $C \in \mathcal{C}_x$ there is a unique circle $\widetilde{C} \in \mathcal{C}_y$ and a point $p(C, \widetilde{C})$ such that $C \mathrel{\Omega}_{p(C,\widetilde{C})} \widetilde{C}$. Now, clearly for the circle $c(x) \in \mathcal{C}_x$ defined in Lemma 4.1, we must have $\arg p(c(x), \widetilde{c(x)}) < \varphi/2$. On the other hand we can choose $C \in \mathcal{C}_x$ such that for the corresponding $\widetilde{C} \in \mathcal{C}_y$ we have $\widetilde{C} = C(y)$. In this case $\arg p(C, C(y)) > \varphi/2$.

Now each $C \in \mathcal{C}_x$ corresponds to a unique radius $r = \operatorname{rad}(C)$, and letting $a = \operatorname{rad}(c(x))$, $b = \operatorname{rad}(c)$, where c is such that $\widetilde{c} = C(y)$, we can define a function $F\colon [a, b] \to [0, \varphi]$ by

$$r \mapsto \arg p(C, \widetilde{C}).$$

This function is continuous and strictly increasing, and thus, by the Intermediate Value Theorem and monotonicity $F(R) = \varphi/2$ for a unique $R \in [a, b]$. Then the point z is the point $p(C, \widetilde{C})$ corresponding to the circle $C \in \mathcal{C}_x$ with radius R. □

Corollary 4.5. *With assumptions as in Lemma 4.4, the arc*

$$J_{\rho_x}[x, z] \cup J_{\rho_y}[z, y]$$

is the quasihyperbolic geodesic $J_{k_{S_\varphi}}[x, y]$.

PROOF: The quasihyperbolic geodesic $J_k[x,y]$ always exists, and, in this case, clearly intersects the bisector ℓ_φ at exactly one point. By Lemma 4.4 there exists a unique point $z' \in \ell_\varphi$ such that the curve $J_{\rho_x}[x,z'] \cup J_{\rho_y}[z',y]$ is differentiable. Now, let $w \in \ell_\varphi$ be the point where a candidate $\gamma \in \Gamma_{xy}$ for the geodesic $J_k[x,y]$ intersects the bisector ℓ_φ. We see then that we must have

$$\gamma|_{S_{\varphi/2}} = J_k[x,w] = J_{\rho_x}[x,w] \quad \text{and} \quad \gamma|_{S_\varphi \setminus \overline{S}_{\varphi/2}} = J_k[y,w] = J_{\rho_y}[y,w]$$

in order for γ to be the geodesic, otherwise we can easily construct a k-shorter curve replacing this part by the quasihyperbolic geodesic. But then, by Lemma 4.4, if $w \neq z'$, the curve $J_{\rho_x}[x,z] \cup J_{\rho_y}[z,y]$ is not differentiable at w, and thus not a geodesic by [Ma, Corollary 4.8]. However, since a geodesic exists, we are left with the only possibility

$$J_k[x,y] = J_{\rho_x}[x,z] \cup J_{\rho_y}[z,y].$$

□

Then the discussion preceding Lemma 4.3 together with Lemmas 4.3, 4.4 and Corollary 4.5 prove the statement of Theorem 1.3. More precisely, we get the following.

Theorem 4.6. *Let $\varphi \in (0,\pi]$, and $x,y \in S_\varphi$ be such that $|y| \le |x|$. Let ℓ_φ be the bisector, and $E = S_{\varphi/2}$, $F = S_\varphi \setminus \overline{S}_{\varphi/2}$ be the lower and upper components of $S_\varphi \setminus \ell_\varphi$, respectively. Let z_1 and z_2 be as in (4.2). Denote $k = k_{S_\varphi}$, ρ_E for the hyperbolic metric in $\mathbf{H}^2$ and ρ_F for the hyperbolic metric in $H = \{(r,\theta) \in \mathbb{R}^2 : \varphi - \pi < \theta < \varphi\}$. Then the geodesics $J_k[x,y]$ are the following curves.*

Case 1: *$x,y \in \ell_\varphi$. Then $J_k[x,y] = [x,y]$.*

Case 2: *$|x| = |y|$. Then $J_k[x,y]$ is a circular arc on the circle $S^1(0,|x|)$.*

Case 3: *$x,y \in E$ and $|z_1| \ge |z_2|$. Then $J_k[x,y] = J_{\rho_E}[x,y]$.*

Case 4: *$x \in E$, $y \in E \cup F$ and $|z_1| \le |z_2|$. Then $J_k[x,y] = J_{\rho_E}[x,z_2] \cup [z_2,z_1] \cup J_{\rho_y}[y,z_1]$, where $\rho_y := \rho_E$ if $y \in E$ and $\rho_y := \rho_F$ if $y \in F$.*

Case 5: *$x \in E$, $y \in F$ and $|z_1| \ge |z_2|$. Then $J_k[x,y] = J_{\rho_E}[x,z] \cup J_{\rho_F}[z,y]$, where z is the point given by Lemma 4.4.*

Remark 4.7. Note that in the cases where the geodesic intersects the bisector at only one point, i.e. the cases represented in Lemma 4.4, the two circular arcs building up the geodesic have a certain geometry which is drawn in Figure 2. Either, as in the upper right picture, the centers c_1 and c_2 are located on the boundary rays, r_1 and r_2, and the circles are tangential to each other at the point where the line through c_1 and c_2 intersects ℓ_φ. The other possibility, as in the upper left picture, is that one center is on a boundary ray, and the other on the extension of the other boundary ray. Again, the circles are tangential at the point where the line through c_1 and c_2 intersects ℓ_φ. Here the circle with center on the extension becomes a line when $[c_1,c_2] \parallel r_2$. In the case $|x| = |y|$ in the lower left picture the two circles are reduced to the one and the same origin-centered circle.

In the case where the angle is large, i.e. $\varphi \in (\pi, 2\pi)$, the situation is somewhat different. However, the method of finding the geodesic curves is essentially the same which we used in the case of small angles. We note that the "helplines" $r_1 = \text{ray}(0, e_2)$ and $r_2 = \text{ray}(0, e^{(\varphi - \frac{\pi}{2})i})$ define a partition of the domain S_φ, such that in the parts where points have small or large arguments, $d_{S_\varphi}(z)$ is the distance to one of the boundary rays, but in the middle part $d_{S_\varphi}(z) = |z|$. Then, using the same methods as in the case of small angles and properties of the logarithmic spiral (see Lemma 4.11), one can show that there exists a unique differentiable curve γ connecting two points $x, y \in S_\varphi$ which is a circular arc orthogonal to the boundary lines on the sides, and a logarithmic spiral in the middle part.

Lemma 4.8. *Let $\pi < \varphi < 2\pi$, and denote $E_1 = S_{\pi/2}$ and $E_2 = S_\varphi \setminus \overline{S_{\varphi-\pi/2}}$. Furthermore, let $r_1 = \text{ray}(0, e_2)$ and $r_2 = \text{ray}(0, e^{(\varphi-\frac{\pi}{2})i})$, and assume that $x \in E_1$ and $y \in E_2$. Then there exist points $z_1 \in r_1$ and $z_2 \in r_2$ such that the curve*

$$J_{\rho_1}[x, z_1] \cup J_k[z_1, z_2] \cup J_{\rho_2}[z_2, y],$$

is differentiable at z_1 and z_2. Here $\rho_1 = \rho_{\mathbf{H}^2}$, $\rho_2 = \rho_H$, where $H = \{(r, \theta) \in \mathbb{R}^2 : \varphi - \pi < \theta < \varphi\}$ and $k = k_{\mathbb{R}^2_*}$.

PROOF: Define circle families

$$\mathcal{C}_x = \{C \mid x \in C \text{ and } C \perp L(0, e_1)\}$$

and

$$\mathcal{C}_y = \{C \mid y \in C \text{ and } C \perp L(0, e^{\varphi i})\}.$$

Assume that $|x| < |y|$ (The case $|x| = |y|$ is obvious). Denote by C_x and C_y the members of the above circle families, which also satisfy $C_x \perp r_1$ and $C_y \perp r_2$, respectively, and further denote $w_1 = C_x \cap r_1$, $w_2 = C_y \cap r_2$. Then clearly $|w_1| < |w_2|$, and thus $C_x \cap C_y = \emptyset$. For any circle $C \in \mathcal{C}_x$, let $\widehat{C}$ denote the larger angle of intersection $C \cap r_1$, and similarly for $C \in \mathcal{C}_y$ let $\widehat{C}$ denote the smaller angle at $C \cap r_2$. Obviously $\widehat{C_x} = \widehat{C_y} = \frac{\pi}{2}$. Now, for any $\omega > \frac{\pi}{2}$ there exists circles $C_1 \in \mathcal{C}_x$ and $C_2 \in \mathcal{C}_y$ such that $\widehat{C_1} = \widehat{C_2} = \omega$. Denote the points of intersection $C_1 \cap r_1$ and $C_2 \cap r_2$ by $\tilde{w_1}$ and $\tilde{w_2}$, and the logarithmic spirals through $\tilde{w_1}$ and $\tilde{w_2}$ with radial angle ω by $\gamma_{\omega,1}$ and $\gamma_{\omega,2}$. Then the curve $C_1 \cup \gamma_{\beta,1} \cup C_2 = C_1 \cup \gamma_{\beta,2} \cup C_2$ where $\widehat{C_1} = \widehat{C_2} = \beta$, and β is the angle such that $\gamma_{\omega,1} \cap \gamma_{\omega,2} \neq \emptyset$, is the uniquely determined curve of the above form. The angle β can be shown to exist using continuity and the Intermediate Value Theorem in the same way as in Lemma 4.4. □

Lemma 4.9. *Let $\pi < \varphi < 2\pi$, and let E_1, E_2 be as in Lemma 4.8. Let $E_c = S_\varphi \setminus (\overline{E_1 \cup E_2})$, and assume that $x \in E_1$ and $y \in E_c$. Then there exists a point $z \in r_1$ such that the curve*

$$J_{\rho_1}[x, z] \cup J_k[z, y]$$

is differentiable at z.

PROOF: Let $\mathcal{C}_x$ be a family of curves as in Lemma 4.8. For each $C \in \mathcal{C}_x$ there is a unique logarithmic spiral γ_C in E_c such that $C \cup \gamma_C$ is differentiable. Also, it is easy to see that

$$\bigcup \{\gamma_C \mid C \in \mathcal{C}_x\} \supset E_c$$

and that each point of E_c lies on exactly one curve γ_C in this family. □

In fact the curves given by Lemmas 4.8 and 4.9 are quasihyperbolic geodesics in S_φ. This is proved by following the methods used in Lemma 4.3 and Corollary 4.5, and the proofs will be omitted here. Summarizing the results on geodesics for large angles we get Theorem 1.4. More precisely, we have the following.

Theorem 4.10. *Let $\pi < \varphi < 2\pi$, and let $E_1 = S_{\pi/2}$, $E_2 = S_\varphi \setminus \overline{S_{\varphi-\pi/2}}$ and $E_c = S_\varphi \setminus (\overline{E_1 \cup E_2})$. Also, let $r_1 = \mathrm{ray}(0, e_2)$ and $r_2 = \mathrm{ray}(0, e^{(\varphi - \frac{\pi}{2})i})$. Then, denoting $\rho_1 = \rho_{\mathbf{H}^2}$, $\rho_2 = \rho_H$, where $H = \{(r,\theta) \in \mathbb{R}^2 \ : \ \varphi - \pi < \theta < \varphi\}$, $k = k_{S_\varphi}$ and $k_* = k_{\mathbb{R}^2_*}$, the quasihyperbolic geodesics of S_φ are the following curves;*

Case 1: $x, y \in r_1$ *or* $x, y \in r_2$. *Then* $J_k[x,y] = [x,y]$.
Case 2: $x, y \in E_1$. *Then* $J_k[x,y] = J_{\rho_1}[x,y]$.
Case 3: $x, y \in E_c$. *Then* $J_k[x,y] = J_{k_*}[x,y]$.
Case 4: $x \in E_1$ *and* $y \in E_c$. *Then* $J_k[x,y] = J_{\rho_1}[x,z] \cup J_{k_*}[z,y]$, *where z is as in Lemma 4.9.*
Case 5: $x \in E_1$ *and* $y \in E_2$. *Then* $J_k[x,y] = J_{\rho_1}[x,z_1] \cup J_{k_*}[z_1,z_2] \cup J_{\rho_2}[z_2,y]$, *where z_1 and z_2 are as in Lemma 4.8.*

We conclude this section with a discussion on the proof of Theorem 1.5. Again, the actual proofs for that the proposed curves are geodesic segments follow from the known cases $\mathbf{B}^n$ and $\mathbb{R}^n \setminus \{0\}$, which were studied by Martin and Osgood in [MaOs]. The idea is then the same as previously, that is, we use $S^1(\frac{1}{2})$ as a "helpline", and apply the methods used for the angular domain S_φ.

For $x, y \in \mathbb{R}^n \setminus \{0\}$, it clearly suffices to consider the 2-dimensional plane Σ determined by the points $0, x, y$, because of the domain symmetry. In this case there is even a formula for the quasihyperbolic distance. In the sequel we denote $\mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\}$.

Lemma 4.11. ([MaOs, Ch.2]) *Let $x, y \in \mathbb{R}^n_*$, and $\varphi = \widehat{x, 0, y} > 0$, and denote by Σ the 2-dimensional plane determined by the triple $0, x, y$. Then the quasihyperbolic geodesic in $\mathbb{R}^n_*$ is the logarithmic spiral in Σ with polar equation*

$$r(\omega) = |x| \ \exp\Big(\frac{\omega}{\varphi} \log \frac{|y|}{|x|}\Big),$$

and the quasihyperbolic distance is given by the formula

$$k_{\mathbb{R}^n_*}(x,y) = \sqrt{\varphi^2 + \log^2 \frac{|y|}{|x|}}.$$

□

The geodesics of $\mathbf{B}^n$ can also be determined by reducing to the planar case. We have the following result.

Lemma 4.12. ([MaOs, Ch.2]) *Let γ be the quasihyperbolic geodesic in $\mathbf{B}^n$ perpendicular to the e_2-axis, which intersects the e_2-axis at the point ae_2, $a > 0$. Then the equation of the geodesic in polar form is*

$$\theta - \tfrac{\pi}{2} = k \int_a^r \Big(\frac{t^4}{(1-t)^2} - t^2 k^2 \Big)^{-1/2} \, dt,$$

where $k = a/(1-a)$. □

Then we get the following refinement of the statement in Theorem 1.5.

Theorem 4.13. *Let $x, y \in \mathbf{B}^n_*$ be such that $x = te_1$ and $|y| = s$. Then, denoting $k = k_{\mathbf{B}^n_*}$ the geodesics $J_k[x, y]$ are the following curves;*

Case 1: *$s \le 1/2$ and $t \le 1/2$. Then $J_k[x, y]$ is the logarithmic spiral $J_{\mathbb{R}^n_*}[x, y]$.*

Case 2: *$s \le 1/2$ and $t \ge 1/2$. Then $J_k[x, y]$ is the curve $\gamma = J_{\mathbb{R}^n_*}[y, z] \cup J_{\mathbf{B}^n}[z, y]$, where $z \in S^{n-1}(1/2)$ is the unique point giving a differentiable curve γ.*

Case 3: *$s \ge 1/2$, $t \ge 1/2$ and*

$$\theta \le \arccos\big(\frac{1-t}{t}\big) + \arccos\big(\frac{1-s}{s}\big) - \sqrt{2t-1} - \sqrt{2s-1},$$

where θ is the angle $\widehat{x, 0, y}$. Then $J_k[x, y]$ coincides with the quasihyperbolic geodesic in $\mathbf{B}^n$, see Lemma 4.12.

Case 4: *$s \ge 1/2$, $t \ge 1/2$ and*

$$\theta \ge \arccos\big(\frac{1-t}{t}\big) + \arccos\big(\frac{1-s}{s}\big) - \sqrt{2t-1} - \sqrt{2s-1}.$$

Then $J_k[x, y]$ is a curve $J_{k^n_{\mathbf{B}}}[x, z_1] \cup C(z_1, z_2) \cup J_{k^n_{\mathbf{B}}}[z_2, y]$, where z_1 and z_2 are tangent points with the circle $S^{n-1}(1/2)$ as given by Lemma 5.33, and $C(z_1, z_2) \subset S^{n-1}(1/2)$ is the circular arc connecting z_1 and z_2.

□

5. Uniformity constant estimates

It seems natural to expect that the less complicated the boundary of the domain is, the easier the uniformity constant is to establish. In the one-point case $G = \mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\}$ things are made even simpler by the fact that we have the explicit formula in Lemma 4.11 for the quasihyperbolic distance.

Some upper bounds for the uniformity constant $A_{\mathbb{R}^n_*}$ have been proved previously. The estimate $A_{\mathbb{R}^n_*} \le \sqrt{1 + (\pi/\log 2)^2} \approx 4.6414$ was suggested by M. Vuorinen in [Vu2, 3.13], who has later shown to the author that it is possible to reach the refined estimate $A_{\mathbb{R}^n_*} \le \sqrt{1 + (\pi/\log 3)^2} \approx 3.0294$ by using an argument based on l'Hôspital's monotone rule (cf. [AnVaVu, Theorem 1.25]). We now want to establish the improvement stated in Theorem 1.6.

As seen previously, it is essentially enough to solve the problem for $\mathbb{R}^2 \setminus \{0\}$. Moreover, since for any point $x \in \mathbb{R}^n_*$

$$j_{\mathbb{R}^n_*}(x,-x) = \log 3 \quad \text{and} \quad k_{\mathbb{R}^n_*}(x,-x) = \pi,$$

it is immediate that the constant $\pi/\log 3$ is attained, so what we in fact need to do is to prove the inequality (2.5) of the uniformity definition with this constant. To do this we will in fact solve the problem of the maximal ratio between k and j, by proving that the pair of points x and $-x$ represents the extremal case.

Lemma 5.1. *The inequalities*

$$(1+x)\log(1+x) - x \ \geq\ 0, \qquad x \geq 0, \tag{5.2}$$

$$(x-1)^2 - x\log^2 x \ \geq\ 0, \qquad x \geq 0, \tag{5.3}$$

$$x - 1 - \log^2 x \ \geq\ 0, \qquad x \geq 1, \tag{5.4}$$

hold for the intervals indicated. □

PROOF: In [AnVaVu, 1.58(3)] it is shown that for $x \geq 1$ we have

$$\frac{x-1}{x} \leq \log x \leq \frac{x-1}{\sqrt{x}}. \tag{5.5}$$

Then (5.2) follows directly from the left inequality in (5.5), and (5.4) follows from the right inequality, since for $x \geq 1$ we have $(x-1)/\sqrt{x} \leq \sqrt{x-1}$. Finally, (5.3) follows for $x \geq 1$ from the right inequality in (5.5) by squaring. If $x \leq 1$, the opposite inequality holds, i.e. we have that $\log x \geq (x-1)/\sqrt{x}$. Here, however, both sides are negative, so (5.3) follows again by squaring. □

PROOF OF THEOREM 1.6: Assume that $x, y \in \mathbb{R}^2 \setminus \{0\}$, are such that $\theta = \widehat{x,0,y} \in (0, \pi/4]$. Using similarity invariance of $j_{\mathbb{R}^2_*}$ and $k_{\mathbb{R}^2_*}$, and symmetricity of metrics, without loss of generality we may assume that $x = e_1$ and that $y = te^{i\theta}$, where $t \geq 1$. From the definition of the metric $j_{\mathbb{R}^n_*}$, we obtain

$$j_{\mathbb{R}^n_*}(x,y) = \log\Big(1 + \frac{\sqrt{|x|^2 - 2x\cdot y + |y|^2}}{|x|}\Big).$$

Also, $|x|^2 + |y|^2 - 2|x||y|\cos\varphi = |x-y|^2$, and thus

$$\varphi = \arccos\Big(\frac{|x|^2 + |y|^2 - |x-y|^2}{2|x||y|}\Big).$$

Using Lemma 4.11, we see that if A is as in the definition (2.5) of a uniform domain, then A must in fact satisfy the inequality

$$\arccos^2\Big(\frac{|x|^2+|y|^2-|x-y|^2}{2|x||y|}\Big) + \log^2\frac{|x|}{|y|} \leq A^2 \log^2\Big(1 + \frac{\sqrt{|x|^2 - 2x\cdot y + |y|^2}}{|x|}\Big). \tag{5.6}$$

Then, considering the right half-plane

$$H = \{(r,\omega) \in \mathbb{R}^2 \ : \ -\tfrac{\pi}{2} < \omega < \tfrac{\pi}{2}\},$$

we see directly by monotonicity that $k_{\mathbb{R}^2_*}(x,y) \le k_H(x,y)$. Also, denoting $d(z) = \mathrm{dist}(z, L(0,e_2))$ we get by Bernoulli's inequality (2.15)

$$\begin{aligned} j_{\mathbb{R}^2_*}(x,y) &\ge \log\Big(1+\frac{|x-y|}{|y|}\Big) = \log\Big(1+\frac{|x-y|}{d(y)/\cos\theta}\Big) \\ &= \log\Big(1+\frac{\cos\theta\ |x-y|}{d(y)}\Big) \ge \cos\theta\ \log\Big(1+\frac{|x-y|}{d(y)}\Big) \\ &\ge \frac{1}{\sqrt{2}} j_H(x,y), \end{aligned}$$

in case $d(y) \le d(x)$. On the other hand, if $d(y) \ge d(x)$, then obviously $j_{\mathbb{R}^2_*}(x,y) \ge j_H(x,y)$. We now estimate

$$\frac{k_{\mathbb{R}^2_*}(x,y)}{j_{\mathbb{R}^2_*}(x,y)} \le \sqrt{2}\ \frac{k_H(x,y)}{j_H(x,y)} \le 2\sqrt{2} < \frac{\pi}{\log 3}.$$

After this we consider points for which $\pi/4 \le \theta \le 3\pi/8$. Then we can apply the same trick as above, only using the halfplane

$$H = \{(r,\theta) \in \mathbb{R}^2\ :\ -\tfrac{5\pi}{16} < \theta < \tfrac{11\pi}{8}\}$$

instead. Namely, for all such points $d(y) \ge d(x)$, and as above we have that $j_{\mathbb{R}^2_*}(x,y) \ge j_H(x,y)$.

Finally, we are left with the case $x = e_1$ and $y = te^{i\theta}$, where $t \ge 1$ and $3\pi/8 \le \theta \le \pi$. The goal is now to find the maximum of the expression

$$\frac{\arccos^2\Big(\frac{|x|^2+|y|^2-|x-y|^2}{2|x||y|}\Big) + \log^2\frac{|x|}{|y|}}{\log^2\Big(1+\frac{\sqrt{|x|^2-2x\cdot y+|y|^2}}{|x|}\Big)}. \tag{5.7}$$

By the choice of the points x and y we see that

$$|x-y|^2 = 1 + t^2 - 2t\cos\theta.$$

Then the expression (5.7) can be written as a function of t and θ;

$$F(\theta,t) \;=\; \frac{\theta^2 + \log^2 t}{\log^2(1+\sqrt{1+t^2-2t\cos\theta})}, \tag{5.8}$$

and we have reduced the problem to maximizing the function F for $t \ge 1$ and $3\pi/8 \le \theta \le \pi$.

Denoting $w(\theta) = \sqrt{1+t^2-2t\cos\theta}$, and differentiating with respect to θ, we obtain

$$\frac{\partial F}{\partial\theta} = \frac{2\Big[\theta\log(1+w(\theta))\ [1+w(\theta)]\ w(\theta) - (\log^2 t + \theta^2)t\sin\theta)\Big]}{\log^3(1+w(\theta))\ [1+w(\theta)]\ w(\theta)}.$$

To determine the sign of this derivative, it now suffices to examine the sign of the numerator. However, because $2\cos x + x\sin x$ is decreasing in $[3\pi/8, \pi]$, we have by (5.2) and (5.3) that

$$
\begin{aligned}
& \theta \log(1+w(\theta))\ [1+w(\theta)]\ w(\theta) - (\log^2 t + \theta^2) t \sin\theta \\
\geq\ & \theta\ w(\theta)^2 - t\log^2 t - \theta^2 t \sin\theta \\
=\ & \theta + \theta t^2 - 2\theta t\cos\theta - t\log^2 t - \theta^2 t\sin\theta \\
=\ & \theta + \theta t^2 - t\theta(2\cos\theta + \theta\sin\theta) - t\log^2 t \\
\geq\ & \theta\left[1 + t^2 - t\Big(\sqrt{2-\sqrt{2}} + \frac{3\pi}{16}\sqrt{2+\sqrt{2}}\Big)\right] - t\log^2 t \\
\geq\ & \frac{3\pi}{8}\Big(1 + t^2 - 2t\Big) - t\log^2 t \geq (1-t)^2 - t\log^2 t \geq 0.
\end{aligned}
$$

Then the function F is increasing with respect to θ. The maximum is consequently obtained when $\theta = \pi$. Inserting this in (5.8) yields a function of t,

$$f(t) = \frac{\pi^2 + \log^2 t}{\log^2(t+2)}.$$

Next we show that this function has a maximum at $t = 1$. First, let

$$
f^*(t) = \begin{cases} \dfrac{\pi^2 + (t-1)}{\log^2(t+2)}, & 1 \leq t \leq 6 \\[2ex] \dfrac{\pi^2+5}{\log^2 8}, & 6 \leq t < \infty \end{cases},
$$

which by construction is continuous. Also, it is clear that $f(1) = f^*(1)$.

By differentiation one shows that f^* is strictly decreasing for $1 \leq t \leq 6$. By definition f^* is constant for $t \geq 6$. Since by (5.4) $\log^2 t \leq t-1$, it is obvious that $f(t) \leq f^*(t)$ for all $1 \leq t \leq 6$. It remains to show that $f(t) \leq f^*(t)$ also for $t \geq 6$. This follows, since

$$
\begin{aligned}
\frac{\pi^2 + \log^2 t}{\log^2(t+2)} &= \frac{\pi^2}{\log^2(t+2)} + \frac{\log^2 t}{\log^2(t+2)} \\
&\leq \frac{\pi^2}{\log^2 8} + 1 \approx 3.28 < 3.44 \approx \frac{\pi^2+5}{\log^2 8}.
\end{aligned}
$$

Then $f(t) \leq f^*(t)$ for all $t \geq 1$. Furthermore, equality holds only at the point $t = 1$, where f^* attains its maximum value. This means that also f attains its maximum value in the point $t = 1$, and that this value is

$$\Big(\frac{\pi}{\log 3}\Big)^2.$$

This number is then the maximum of the function $F(\theta, t)$, and attained at $(\pi, 1)$.

□

The next domain we consider is the angular domain S_φ in the case where $\varphi \in (0, \pi]$. Some of the results are true also in higher dimensions, where we will use the notation

$$C^n_\varphi = \{z \in \mathbb{R}^n \mid z \cdot e_n = |z| \cos\varphi\}$$

for the n-dimensional φ-cone. In this case of course $\ell^n_\varphi = \ell_\varphi := \{te_n \mid t > 0\}$. This definition is perhaps the most convenient in the case $n \geq 3$; note however, that as sets $S_\varphi \neq C^2_\varphi$, but rotating S_φ counterclockwise by an angle $\frac{\pi-\varphi}{2}$ gives the set C^2_φ. We start by proving some inequalities for the metrics involved in certain special cases. The following lemma gives a lower bound for the uniformity constant, using only points at the bisector. Namely, since in the case $x, y \in \ell_\varphi$, the k-geodesic $J_k[x, y]$ is trivially the line segment $[x, y]$, we get the following.

Lemma 5.9. *For the domain C^n_φ, $\varphi \in (0, \pi], n \geq 2$ and for all points $x, y \in \ell_\varphi$*

$$k_{C^n_\varphi}(x, y) \leq \frac{1}{\sin\frac{\varphi}{2}} j_{C^n_\varphi}(x, y).$$

Furthermore, if $z_t = te_n$, $t \geq 1$, then

$$\lim_{t\to\infty} \frac{k_{C^n_\varphi}(e_n, z_t)}{j_{C^n_\varphi}(e_n, z_t)} = \frac{1}{\sin\frac{\varphi}{2}}.$$

In the case $\varphi \in (\pi, 2\pi)$, we have that $k_{C^n_\varphi}(x, y) = j_{C^n_\varphi}(x, y)$ for all $x, y \in \ell_\varphi$.

PROOF: We prove the case $n = 2$, since by symmetry the proof is essentially 2-dimensional. By similarity invariance, we may replace C^2_φ by S_φ, and assume that $|y| \geq |x| = 1$. Then we see that

$$\begin{aligned} k_{S_\varphi}(x, y) &= \int_{[x,y]} \frac{|dt|}{d(t)} = \int_{[x,y]} \frac{|dt|}{|t| \sin\frac{\varphi}{2}} \\ &= \left| \int_1^{|y|} \frac{dt}{t\ \sin\frac{\varphi}{2}} \right| = \frac{1}{\sin\frac{\varphi}{2}} \log|y|. \end{aligned} \tag{5.10}$$

Since $d(x) \leq 1$, we get

$$j_{S_\varphi}(x, y) = \log\left(1 + \frac{|y| - 1}{d(x)}\right) \geq \log\left(1 + \frac{|y| - 1}{1}\right) = \log|y|,$$

and the first statement follows. Actually,

$$j_{S_\varphi}(z, z_t) = \log\left(1 + \frac{t-1}{\sin\frac{\varphi}{2}}\right),$$

and thus

$$\lim_{t\to\infty} \frac{k_{S_\varphi}(z, z_t)}{j_{S_\varphi}(z, z_t)} = \frac{1}{\sin\frac{\varphi}{2}} \lim_{t\to\infty} \frac{\log t}{\log\left(1 + \frac{t-1}{\sin\frac{\varphi}{2}}\right)} = \frac{1}{\sin\frac{\varphi}{2}}.$$

In the case $\varphi \in (\pi, 2\pi)$, let H be the halfplane with respect to the boundary line $L(0, e^{\frac{\varphi-\pi}{2}i})$, we see that

$$k_{S_\varphi}(x, y) = \rho_H(x, y) = \log|y| = \log\left(1 + \frac{|y| - 1}{1}\right) = j_{S_\varphi}(x, y).$$

□

As $\varphi \to \pi^-$ we see that the expression $1/\sin\frac{\varphi}{2}$ approaches 1. However, the case $\varphi = \pi$ represents the halfplane, for which the best constant is known to be 2. In fact the number 2 is attained for every angle. Namely, let B be an arbitrary ball such that $\partial B \bigcirc \ell_\varphi$ and $\partial B \perp L(0, e_1)$. For points within $B \cap S_\varphi$, the situation is then reduced to the half-plane hyperbolic metric. Now, choosing the two points at the boundary $\partial B \cap S_\varphi$ on opposite sides of B, and letting them approach the boundary line $L(0, e_1)$, gives the extremal case of $\mathbf{H}^2$. Since the same construction is obviously possible in C^n_φ, from this and Lemma 5.9 we obtain a lower bound

$$A_{C^n_\varphi} \geq \max\left\{2, \frac{1}{\sin\frac{\varphi}{2}}\right\} \tag{5.11}$$

for the domain C^n_φ, $\varphi \in (0, \pi]$, $n \geq 2$.

For points having the same distance to the origin, or points having the same argument, we derive the next lemma.

Lemma 5.12. *Let $x, y \in S_\varphi$.*

i) Let $\arg(x) = \arg(y) = \theta$, $\varphi \in (0, 2\pi)$, and denote $\tilde\theta = \min\{\theta, \varphi - \theta, \frac{\pi}{2}\}$. Then

$$k_{S_\varphi}(x, y) \leq \frac{1}{\sin\tilde\theta}\, j_{S_\varphi}(x, y).$$

ii) If $|x| = |y|$, and $\varphi \in (0, \pi]$, then

$$k_{S_\varphi}(x, y) \leq 2\, j_{S_\varphi}(x, y).$$

PROOF: *i)* Without loss of generality we may, by symmetry, assume that $\theta < \varphi/2$. Now

$$S_{2\theta} = \big\{(r, \omega) \in \mathbb{R}^2 \ :\ 0 < \omega < 2\theta\big\} \subset \big\{(r, \omega) \in \mathbb{R}^2 \ :\ 0 < \omega < \varphi\big\} = S_\varphi.$$

Then by the monotonicity property of k we have that $k_{S_\varphi}(x, y) \leq k_{S_{2\theta}}(x, y)$, but clearly $j_{S_\varphi}(x, y) = j_{S_{2\theta}}(x, y)$. Since x and y are contained in the bisector of the domain $S_{2\theta}$, we see that

$$k_{S_\varphi}(x, y) \leq k_{S_{2\theta}}(x, y) \leq \int_{[x,y]} \frac{|dt|}{d(t)},$$

and then the same calculations as in the proof of Lemma 5.9 show that

$$\frac{k_{S_\varphi}(x, y)}{j_{S_\varphi}(x, y)} \leq \frac{k_{S_{2\theta}}(x, y)}{j_{S_{2\theta}}(x, y)} \leq \frac{1}{\sin\theta}.$$

The case $\tilde\theta = \frac{\pi}{2}$ only occurs for angles $\varphi > \pi$, and the proof is then clear, following Lemma 5.9. In fact $k_{S_\varphi}(x, y) \leq\ j_{S_\varphi}(x, y)$ holds with equality.

ii) The case where x and y are on the same side of ℓ_φ follows by the result for the halfplane. Thus we assume that $\arg(x) < \varphi/2 < \arg(y)$, that $|x| = |y| = 1$ and that $\arg(x) \leq \varphi - \arg(y)$. Letting $B_x = B(x, d(x))$ and $B_y = B(y, d(x))$, and defining z to be the intersection point $\ell_\varphi \cap \big\{(w_1, w_2) \in S_\varphi \mid w_1 = x_1\big\}$, it is easy to see that the ball $B = B(z, d(z)) \subset S_\varphi$ contains both B_x and B_y, and that $B \bigcirc B_x$. Thus $j_B(x, y) = j_{S_\varphi}(x, y)$, and furthermore by monotonicity

$$\frac{k_{S_\varphi}(x, y)}{j_{S_\varphi}(x, y)} \leq \frac{k_B(x, y)}{j_{S_\varphi}(x, y)} = \frac{k_B(x, y)}{j_B(x, y)} \leq 2.$$

□

For points with equal radial distance to the origin, the case $\varphi \in (\pi, 2\pi)$ is somewhat different, and will be discussed further later in this section.

We continue by deriving a universal lower bound for the k- and j-distances in a special case where the points x and y are separated by an "inscribed ball" in the angle domain, that is, we improve Lemma 3.3 for the case S_φ.

Lemma 5.13. *Let $B \subset S_\varphi \subset \mathbb{R}^2$ be a ball tangent to ∂S_φ and centered at some point of ℓ_φ. If $x, y \in S_\varphi$ are in different components of $S_\varphi \setminus B$, the j-distance and the k-distance are bounded from below by the constant* $\log 3$.

Proof: By Remark 2.6(3) it suffices to prove the statement for the j-metric. By stretching we can assume that $B = B(z, 1)$, where $z = (\cot \frac{\varphi}{2}, 1)$. We may also assume that $|y| \geq |x|$. Furthermore, we may assume that the points are on the same side of the bisector, namely if not, denote the reflection point of x in ℓ_φ by x^*. Then $d(x) = d(x^*)$ and $|x - y| \geq |x^* - y|$, and consequently $j_{S_\varphi}(x, y) \geq j_{S_\varphi}(x^*, y)$.

Now, assume x and y are below the bisector, and denote $l_x = L(x, e_1)$, $l_y = L(y, e_1)$. Let x_1 be the intersection point $\partial B \cap l_x$ closest to x. Similarly, let y_1 be the intersection point $\partial B \cap l_y$ closest to y, however, if either $\partial B \cap l_y \subset S_\varphi \setminus \overline{S_{\frac{\varphi}{2}}}$ or $\partial B \cap l_y = \emptyset$, let y_1 be the point of $B \cap \ell_\varphi$ farther away from the origin. Then $|x - y| \geq |x_1 - y_1|$, but $\min\{d(x), d(y)\} = \min\{d(x_1), d(y_1)\}$. Thus $j_{S_\varphi}(x, y) \geq j_{S_\varphi}(x_1, y_1)$, and we have reduced the situation to the case where both points are on ∂B.

Next, denote $\theta_x = \arg(x)$, $\theta_y = \arg(y)$, and assume $\theta_x \geq \theta_y$. If $d(x) \geq d(y)$, and we denote the intersection point of ∂B and the e_1-axis by w, we can reflect the points x and y in the line segment $[z, w]$. Then we get points x^*, y^* for which $d(x^*) \leq d(y^*)$ and $j_{S_\varphi}(x, y) = j_{S_\varphi}(x^*, y^*)$. Thus we may as well assume that $d(x) \leq d(y)$. In this case, clearly $|x - y| \geq 2d(x)$, and since $d(x) \leq d(y)$, we have that

$$j_{S_\varphi}(x, y) = \log\left(1 + \frac{|x - y|}{d(x)}\right) \geq \log 3.$$

Finally we have the case where $\theta_x \leq \theta_y$. Then it is immediate that $d(x) \leq d(y)$, and that $|x - y| \geq 2d(x)$. As above we see that $j_{S_\varphi}(x, y) \geq \log 3$. This covers all the different cases. □

In the following we summarize some useful results on how the quasihyperbolic metric in S_φ changes when points are moved in certain ways. This also demonstrates the typical use of Lemma 3.7 in estimations.

Lemma 5.14. *As before, let $S_\varphi \subset \mathbb{R}^2$ be the angular domain, let ℓ_φ denote the bisector, and let $S_1 = S_{\varphi/2}$ and $S_2 = S_\varphi \setminus \overline{S_{\varphi/2}}$ be the upper and lower components. Then the following hold:*

i) *If $x, y \in S_\varphi$, and x', y' are the orthogonal projections on ℓ_φ, respectively, then $k_{S_\varphi}(x, y) \geq k_{S_\varphi}(x', y')$.*
ii) *Let $x, y \in S_\varphi$ be such that $|y| \leq |x|$, and let $z \in \ell_\varphi$. Also, let $B = B(z, |z-y|)$ and assume that $J_k[y, x] \cap B = \{y\}$. Let x' and y' be the points on ℓ_φ obtained by rotating x and y about z towards ℓ_φ. Then $k_{S_\varphi}(x, y) \geq k_{S_\varphi}(x', y')$.*
iii) *Let $x, y \in S_1$ (or $x, y \in S_2$) be such that $|y| \leq |x|$ and $\operatorname{dist}(y, \partial S_\varphi) \leq \operatorname{dist}(x, \partial S_\varphi)$. Also, let y' be the point of ℓ_φ obtained by rotating about x. Then $k_{S_\varphi}(x, y) \geq k_{S_\varphi}(x, y')$.*
iv) *Let $x, y \in S_1$ (or $x, y \in S_2$) be such that $|y| \leq |x|$ and $\operatorname{dist}(y, \partial S_\varphi) \leq \operatorname{dist}(x, \partial S_\varphi)$. Let y' be the point of ℓ_φ such that $|y| = |y'|$. Then $k_{S_\varphi}(x, y) \geq k_{S_\varphi}(x, y')$.*

PROOF: i) Let $\gamma : [0, 1] \to S_\varphi$ be the geodesic segment $J_k[x, y]$, and let $\alpha : [0, 1] \to S_\varphi$ be the geodesic segment $J_k[x', y'] = [x', y']$. Let $\{t_i\}$ be an arbitrary length sequence. Form another length sequence $\{s_i\}$ by setting $s_i = \alpha^{-1}(p_i)$, where p_i is the orthogonal projection of the point $\gamma(t_i)$ onto ℓ_φ. Then it is clear that the conditions of Lemma 3.7 hold, and the claim follows.
ii) As in case i), only choose the points p_i by rotating in z-centered circles.
iii) As in case i), choose the points p_i by rotating in x-centered circles.
iv) As in case i), choose the points p_i by rotating in circles centered at the origin. □

The results of Corollary 5.14 are easily seen to hold for the j-metric also. For the j-metric, however, we can also derive the following.

Lemma 5.15. *Let $x, y \in S_\varphi$, and assume that $r = |y| \leq |x| = 1$. Let y' be such that $\arg(y') = \arg(y)$ and $|y'| = 1$. Then*

$$j_{S_\varphi}(x, y') \leq A(r, \theta)\, j_{S_\varphi}(x, y) \leq 2\, j_{S_\varphi}(x, y),$$

where

$$A = \sqrt{\frac{2 - 2\cos\theta}{1 + r^2 - 2r\cos\theta}},$$

and $\theta = \varphi - \arg(x) - \arg(y)$ is the angle $\widehat{x, 0, y}$. In particular, if $\arg(x) = \varphi - \arg(y)$, then $j_{S_\varphi}(x, y') \leq j_{S_\varphi}(x, y)$.

PROOF: First assume that $d(x) \leq d(y)$. Then $|x - y'| = \sqrt{2 - 2\cos\theta}$ and $|x - y| = \sqrt{1 + r^2 - 2r\cos\theta}$ and $d(x)$ is the minimum distance to the boundary in both cases. By Bernoulli's inequality (2.14), we get $j(x, y') \leq A\, j(x, y)$.

Then assume that $d(x) \geq d(y)$ and $d(x) \leq d(y')$. In this case $|x - y|$ and $|x - y|$ are as above, but $\min\{d(x), d(y')\} = d(x)$ and $\min\{d(x), d(y)\} = d(y)$. But then we get

$$\frac{|x - y'|}{|x - y|} \frac{d(y)}{d(x)} \leq A\, \frac{d(y)}{d(x)} \leq A,$$

and again we get $j(x,y') \le A\, j(x,y)$.

Finally, if $d(x) \ge d(y)$ and $d(x) \ge d(y')$, we have that $\min\{d(x), d(y')\} = d(y')$ and $\min\{d(x), d(y)\} = d(y)$. Now, by similar triangles, $d(y)/d(y') = r$, so that

$$\frac{|x-y'|}{|x-y|}\frac{d(y)}{d(y')} \le rA \le A,$$

and the first inequality follows. It is easy to show that $A \le 2$ using elementary calculus. Also, one can show that $rA \le 1$, which proves the last claim, since if $\arg(x) = \varphi - \arg(y)$, we have that $d(x) = d(y') \ge d(y)$. □

In the previous lemmas we have considered several special cases of the uniformity inequality (2.5), where x and y are chosen in some specific way. In the general case the uniformity of the domain S_φ actually follows from the well known fact that S_φ is a quasidisk (For the sharp result see [GeHa, 4.1]), and also quite easily from the original uniformity definition by Martio and Sarvas. Here however, we are mostly interested in versions of the uniformity inequality (2.5), as in the following lemma. The constant obtained here is not optimal, but on the other hand it is valid also for the higher dimensional cone C^n_φ.

Lemma 5.16. *Let $\varphi \in (0,\pi]$. Then, for all $x, y \in C^n_\varphi$ and $n \ge 2$,*

$$k_{C_\varphi}(x,y) \le \Big(4 + \frac{1}{\sin\frac{\varphi}{2}}\Big) j_{C_\varphi}(x,y).$$

PROOF: Let $x, y \in C^n_\varphi$ be arbitrary, and choose points x', y' at the middle axis $\ell_\varphi = \{t z_n \mid t > 0\}$ such that $[x,x'] \perp \ell_\varphi \perp [y,y']$. By the triangle inequality we have that

$$k_{C_\varphi}(x,y) \le k_{C_\varphi}(x,x') + k_{C_\varphi}(x',y') + k_{C_\varphi}(y',y).$$

Let y^* be the point such that $x - x' = t\cdot(y^* - y')$ for some $t > 0$, and $|y'-y^*| = |y'-y|$. In other words, x, y^*, x' and y' lie in the same 2-dimensional plane Σ. By symmetry $k_{C_\varphi}(y,y') = k_{C_\varphi}(y^*,y')$. Hence

$$k_{C_\varphi}(x,y) \le k_{C_\varphi}(x,x') + k_{C_\varphi}(x',y') + k_{C_\varphi}(y',y^*).$$

By Lemma 5.9, for the middle term we get an estimate

$$k_{C_\varphi}(x',y') \le \frac{1}{\sin\frac{\varphi}{2}}\left|\log\frac{|x'|}{|y'|}\right| \le \frac{1}{\sin\frac{\varphi}{2}} j_{C_\varphi}(x',y').$$

We now show that $j_{C_\varphi}(x',y') \le j_{C_\varphi}(x,y)$. This is true exactly when

$$\frac{|x-y|}{\min\{d(x),d(y)\}} \ge \frac{|x'-y'|}{\min\{d(x'),d(y')\}}.$$

Obviously $\min\{d(x),d(y)\} \le \min\{d(x'),d(y')\}$, and it is also clear that $|x'-y'| \le |x-y|$. Thus

$$k_{C_\varphi}(x',y') \le \frac{1}{\sin\frac{\varphi}{2}} j_{C_\varphi}(x,y). \tag{5.17}$$

After this we concentrate on the remaining terms, i.e. we want to show that there are constants $c_x, c_y \geq 1$ such that

$$k_{C_\varphi}(x,x') \leq c_x\, j_{C_\varphi}(x,y^*) \quad \text{and} \quad k_{C_\varphi}(y,y') \leq c_y\, j_{C_\varphi}(x,y^*).$$

For this we may, without loss of generality, assume that $d(x) \leq d(y) = d(y^*)$, and also that both

$$y^* \notin B(x,|x-x'|) \cap C_\varphi,$$
$$x \notin B(y^*,|y^*-y'|) \cap C_\varphi$$

hold. Namely, if this would not hold, the geodesic $J_{C_\varphi}[x,y^*]$ would be a hyperbolic $\mathbf{H}^2$-geodesic which lies within component of $(\Sigma \cap C_\varphi) \setminus \ell_\varphi$, and we would have $k_{C_\varphi}(x,y) \leq 2j_{S_\varphi}(x,y)$.

Now, by the choice of the points x' and y' it is obvious that there exists a hyperbolic geodesic connecting x and x' within a component of $(\Sigma \cap C_\varphi) \setminus \ell_\varphi$, and similarly for y^* and y'. Thus we see that

$$k_{S_\varphi}(x,x') \leq 2j_{S_\varphi}(x,x') \quad \text{and} \quad k_{S_\varphi}(y,y') \leq 2j_{S_\varphi}(y,y'). \tag{5.18}$$

Finally, it is easy to show that

$$j_{S_\varphi}(x,x') \leq j_{S_\varphi}(x,y^*) \qquad \text{and} \qquad j_{S_\varphi}(y,y') \leq j_{S_\varphi}(x,y^*). \tag{5.19}$$

Then, using (5.17),(5.18),(5.19) and the obvious inequality $k_{C_\varphi}(x,y^*) \leq k_{C_\varphi}(x,y)$, we get an upper bound for the uniformity constant;

$$k_{C_\varphi}(x,y) \leq \left(4 + \frac{1}{\sin\frac{\varphi}{2}}\right) j_{C_\varphi}(x,y).$$

□

Next we will improve Lemma 5.16 in the case $n=2$ by determining the points $x,y \in S_\varphi$ which give the maximal value of the ratio $k_{S_\varphi}/j_{S_\varphi}$.

Definition 5.20. Let $G \subset \mathbb{R}^n$. A curve $|\gamma| \subset G$ with a representing path $\gamma\colon [0,1] \to G$ is said to be *convex*, if for all $t,s \in [0,1]$, $t \leq s$ we have that

$$\gamma|(t,s) \cap (\gamma(t),\gamma(s))$$

is either $(\gamma(t),\gamma(s))$ or empty. Here (a,b) denotes the open segment $[a,b] \setminus \{a,b\}$. A curve which is not convex is called *nonconvex.*

Lemma 5.21. *Let $\varphi \in (0,\pi]$ and let $x,y \in S_\varphi$ be such that x and y are in different components of $S_\varphi \setminus \ell_\varphi$, $d(x)=d(y)$ and the quasihyperbolic geodesic $J_k[x,y]$ is a convex curve which intersects ℓ_φ at exactly one point. Then*

$$\frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} \leq 2.$$

PROOF: Assume that $y \in S_{\varphi/2}$ and $x = x_\theta$ in $S_\varphi \setminus \overline{S_{\varphi/2}}$ are such that $d(y) = d(x_\theta) = t$. Let $y^* \in S_\varphi \setminus \overline{S_{\varphi/2}}$ be the point such that $d(y) = d(y^*)$ and $|y| = |y^*|$, and let y_0 be such that $d(y) = d(y_0)$ and $J[y, y_0] \perp \partial S_\varphi$. In the picture below the requirement that $J_k[x_\theta, y]$ is convex means that $x_\theta \in [y^*, y_0]$. Now we see that the situation can be studied as a function of the angle $\theta = \widehat{y_c + e_1, y_c}, z_\theta$, where z_θ is the intersection point of the domain bisector ℓ_φ and $J_k[y, x_\theta]$. Clearly $\theta \in (\varphi/2, \varphi)$, and as $x_\theta \to y_0$, the angle θ increases. We now want to show that $\theta \mapsto k(x_\theta, y)$ is increasing.

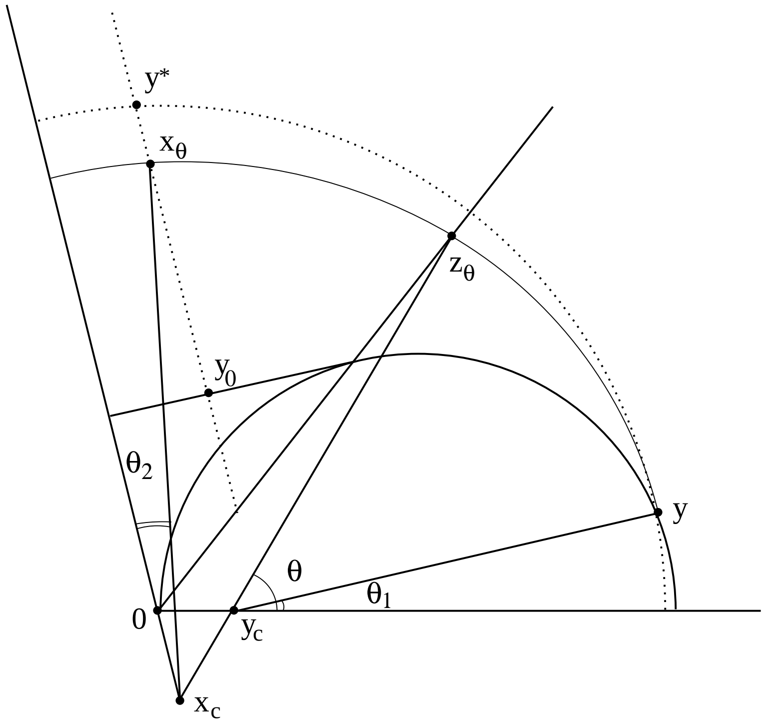

Figure 3. The case of convex geodesics.

Now we see that

$$\begin{aligned} k(x_\theta, y) &= k(x_\theta, z_\theta) + k(z_\theta, y) = \\ &= \int_{\theta_1}^{\theta} \frac{d\omega}{\sin\omega} + \int_{\theta_2}^{\varphi-\theta} \frac{d\omega}{\sin\omega}, \end{aligned}$$

where also θ_1 and θ_2 can be regarded as functions of the angle θ. More specifically, since

$$\frac{d(y)}{|y - y_c|} = \sin\theta_1 \quad \text{and} \quad \frac{d(z_\theta)}{|z_\theta - y_c|} = \sin\theta_2,$$

we have that

$$\theta_1(\theta) = \arcsin(g(\theta)\sin\theta) \quad \text{and} \quad \theta_2 = \arcsin(g(\theta)\sin(\varphi - \theta))$$

where $g(\theta) = d(y)/d(z_\theta)$ is an increasing function of θ. Now we may assume $d(y)$ to be constant, and since we are restricting to the convex case we see that g is bounded, i.e. there exists a number $a \in (0,1)$ such that

$$a \le g(\theta) \le 1.$$

Then

$$k(x_\theta, y) \;\le\; \int_{\arcsin(a\sin\theta)}^{\theta} \frac{d\omega}{\sin\omega} + \int_{\arcsin(a\sin(\varphi-\theta))}^{\varphi-\theta} \frac{d\omega}{\sin\omega}.$$

We now show that this expression obtains its maximum for $\theta = \varphi$, for every constant a. We see that

$$\frac{d}{d\theta}\Big(\int_{\arcsin(a\sin\theta)}^{\theta} \frac{d\omega}{\sin\omega} + \int_{\arcsin(a\sin(\varphi-\theta))}^{\varphi-\theta} \frac{d\omega}{\sin\omega}\Big)$$
$$= \frac{1}{\sin\theta} - \frac{1}{\sin(\varphi-\theta)} - \frac{\cot\theta}{\sqrt{1-a\sin^2\theta}} + \frac{\cot(\varphi-\theta)}{\sqrt{1-a\sin^2(\varphi-\theta)}}.$$

This is ≥ 0 if

$$\frac{1}{\sin\theta} - \frac{\cot\theta}{\sqrt{1-a\sin^2\theta}} \ge \frac{1}{\sin(\varphi-\theta)} - \frac{\cot(\varphi-\theta)}{\sqrt{1-a\sin^2(\varphi-\theta)}}.$$

But this is true, since $\theta \ge \varphi - \theta$, and the function

$$x \mapsto \frac{1}{\sin x} - \frac{\cot x}{\sqrt{1-a\sin^2 x}}$$

can be shown to be increasing for all $a \in (0,1)$. Also, in the convex case it is clear that $j(x_\theta, y) \ge j(x_\varphi, y)$, and thus the case $\theta = \varphi$ gives the maximal situation.

Finally, we want to show that in the maximal situation where x_θ is such that $[x_\theta, z_\theta] \perp \partial S_\varphi$ we have that

$$\frac{k(x,y)}{j(x,y)} \le 2.$$

In this special case the ratio can be calculated explicitly, so actually we want to show that

$$\frac{k(x,y)}{j(x,y)} = \frac{\log\Big(\frac{1}{t}\big(\sin\varphi\tan\frac{\varphi}{2}\cot(\frac{1}{2}\arcsin t)\big)\Big)}{\log\big(1+\sqrt{\frac{1}{t^2}-1}+\frac{1}{t}-\cot\frac{\varphi}{2}\big)} \le 2$$

Using the equality $\cot(\frac{1}{2}\arcsin x) = \frac{1+\sqrt{1-x^2}}{x}$, we see that it suffices to prove the inequality

$$2\sin^2\tfrac{\varphi}{2}\big(1+\sqrt{1-t^2}\big) \le \big(1+\sqrt{1-t^2}+t(1-\cot\tfrac{\varphi}{2})\big)^2 \tag{5.22}$$

for values $t \in (0, \sin\varphi]$.

Let

$$f(\varphi, t) = \big(1+\sqrt{1-t^2}+t(1-\cot\tfrac{\varphi}{2})\big)^2 - 2\sin^2\tfrac{\varphi}{2}(1+\sqrt{1-t^2}).$$

In the case $\frac{\pi}{2} \le \varphi \le \pi$, we see from the inequality

$$at + b\sqrt{1-t^2} \ge \min\{a,b\}, \quad t \in (0,1), \;\; a,b \in (0,\infty)$$

([AnVaVu, 1.58(18)]), that

$$\begin{aligned} & f(\varphi,t) \\ = \; & (1+\sqrt{1-t^2})(1-2\sin^2\tfrac{\varphi}{2}) + (1+\sqrt{1-t^2})\big(\sqrt{1-t^2}+2t(1-\cot\tfrac{\varphi}{2})\big) \\ & + t^2(1-\cot\tfrac{\varphi}{2})^2 \\ \geq \; & (1+\sqrt{1-t^2})\big(1-2\sin^2\tfrac{\varphi}{2}+\min\{1,2(1-\cot\tfrac{\varphi}{2})\}\big) + t^2(1-\cot\tfrac{\varphi}{2})^2 \geq 0. \end{aligned}$$

In the case $0<\varphi\leq\frac{\pi}{2}$ it is easy to show the inequality

$$1+\sqrt{1-t^2}+t(1-\cot\tfrac{\varphi}{2}) \geq \sin^2\tfrac{\varphi}{2} \tag{5.23}$$

to hold for all $t\in(0,\sin\varphi]$ by studying partial derivatives. Then

$$\frac{\partial f}{\partial t} = 2\frac{t}{\sqrt{1-t^2}}\left(\big(1+\sqrt{1-t^2}+t(1-\cot\tfrac{\varphi}{2})\big)\big(-1+\frac{\sqrt{1-t^2}}{t}(1-\cot\tfrac{\varphi}{2})\big)+\sin^2\tfrac{\varphi}{2}\right),$$

and using (5.23), one finds that $\frac{\partial f}{\partial t}\leq 0$. Thus f is decreasing with respect to t, and keeping φ constant, the smallest value of φ is found at the boundary $t=\sin\varphi$. But then we see that for every $\varphi\in(0,\frac{\pi}{2}]$ we have

$$f(\varphi,t)\geq f(\varphi,\sin\varphi) = -4+12\cos^2\tfrac{\varphi}{2}-8\cos^4\tfrac{\varphi}{2}\geq 0.$$

□

Lemma 5.24. *Let $\varphi\in(0,\pi]$ and let $x,y\in S_\varphi$ be such that x and y are in different components of $S_\varphi\setminus\ell_\varphi$, $d(x)=d(y)$ and the quasihyperbolic geodesic $J_k[x,y]$ is a nonconvex curve which intersects ℓ_φ at exactly one point. Then*

$$\frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)}\leq 2.$$

PROOF: The idea is to reduce to the convex case and use the previous theorem. Assume that $y\in S_{\varphi/2}$ and that x in $S_\varphi\backslash\overline{S_{\varphi/2}}$ are such that $d(y)=d(x)=t$, and the geodesic $J_k[x,y]$ is nonconvex. Let z be the intersection point of ℓ_φ and $J_k[x,y]$, as in the picture below, and let l be the line such that $l \bigcirc_z J_k[x,y]$. We now denote the point $l\cap l(0,e^{\varphi i})$ by x_1.

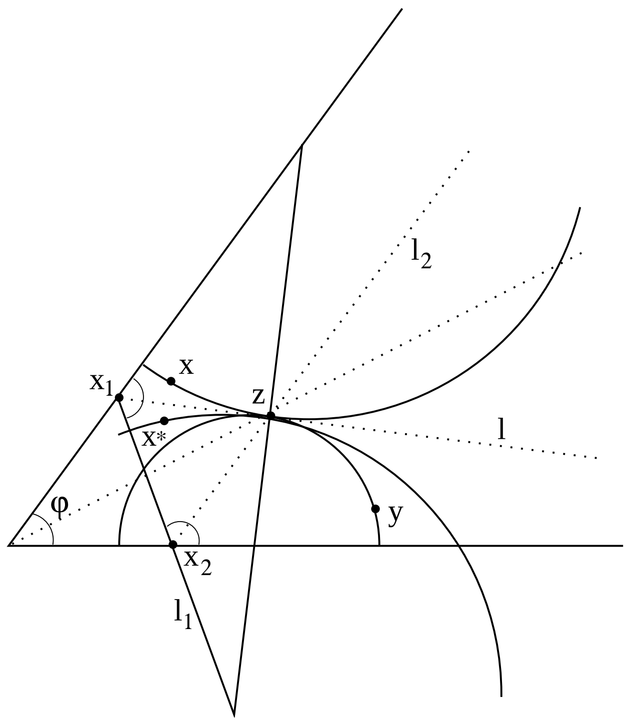

Figure 4. Reducing the nonconvex case to the convex case.

Now let $x_2 \in l(0,e_1)$ be the point such that $\widehat{z,x_1,x_2}$ is the same size as the angle between l and $l(0,e^{\varphi i})$, and denote $l_1 = l(x_1, x_2 - x_1)$. Denote by J^* the mirror image of $J[x,z]$ in the line l. Letting l_2 be the bisector of the angle $x_1, \widehat{x_2, x_2} + e_1$, we see that actually l and l_2 are exterior angles of the triangle $\Delta(0,x_1,x_2)$ at x_1 and x_2, whereas ℓ_φ is the interior angle of $\Delta(0,x_1,x_2)$ at 0. Therefore the bisectors meet in a common point, which means that $z \in l_2$. Also, since y is below the line l, we see that $x^* \in B(y,|x-y|)$, and thus $|x-y| \geq |x^*-y|$. Then, letting S be the domain between $\text{ray}(x_2,e_1)$ and $\text{ray}(x_2,x_1-x_2)$, we have that

$$\frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} = \frac{k_{S_\varphi}(x,z)+k_{S_\varphi}(z,y)}{j_{S_\varphi}(x,y)} = \frac{k_S(x^*,z)+k_S(z,y)}{j_{S_\varphi}(x,y)} \leq \frac{k_S(x^*,y)}{j_{S_\varphi}(x^*,y)},$$

where $k_S(x^*,y)$ is obtained along the convex geodesic $J^* \cup J[z,y]$ with respect to the domain S. But as we proved in Lemma 5.21, $\frac{k_S(x^*,y)}{j_{S_\varphi}(x^*,y)} \leq 2$. □

Lemma 5.25. *Let $\varphi \in (0,\pi]$ and let $x,y \in S_\varphi$. Then, either there exists points $x',y' \in S_\varphi$ such that $d(x') = d(y')$, $y' \in \ell_\varphi$ and*

$$\frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} \leq \frac{k_{S_\varphi}(x',y')}{j_{S_\varphi}(x',y')},$$

or otherwise

$$\frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} \leq 2.$$

PROOF: Without loss of generality, assume that $x, y \in S_\varphi$ are such that $|y| \leq |x| = 1$. Also, let C_1 and C_2 be the components of the domain $S_\varphi \setminus \ell_\varphi$, that is $C_1 = S_{\varphi/2}$ and $C_2 = S_\varphi \setminus \overline{S_{\varphi/2}}$. Now we may split up the problem into different cases:

Case 1. $x, y \in \overline{C}_1$ and $d(x) \leq d(y)$. Let l be the line such that $l \parallel L(0, e^{\varphi i})$, $l \bigcirc B(y, d(x))$ and $y \in \mathrm{comp}(S_\varphi \setminus l, x) =: E$. By monotonicity

$$k_{S_\varphi}(x, y) \leq k_E(x, y).$$

Using a translation mapping we find points x' and y' such that $k_E(x, y) = k_{S_\varphi}(x', y')$. Also, it is clear that $j_{S_\varphi}(x, y) = j_{S_\varphi}(x', y')$, so the situation reduces to Case 5.

Case 2. $x, y \in \overline{C}_1$ and $d(x) > d(y)$. Let l be the line such that $l \parallel L(0, e^{\varphi i})$, $l \bigcirc B(y, d(y))$ and $y \in \mathrm{comp}(S_\varphi \setminus l, x)$. Again, by using a translation we obtain points $\tilde{x}$ and $\tilde{y}$ such that

$$\frac{k_{S_\varphi}(x, y)}{j_{S_\varphi}(x, y)} \leq \frac{k_{S_\varphi}(\tilde{x}, \tilde{y})}{j_{S_\varphi}(\tilde{x}, \tilde{y})}.$$

Denote by C_t the circle $S^1(\tilde{y}, t)$, let γ be the geodesic $J_{S_\varphi}[\tilde{y}, \tilde{x}]$, and γ' be the geodesic $J_{S_\varphi}[\tilde{y}, x']$, where x' is the point such that $d(x') = d(\tilde{y})$ and $|\tilde{y} - \tilde{x}| = |\tilde{y} - x'|$. It is clear that the small tangent circles $c(\tilde{x})$ and $c(x')$ (as given by Lemma 4.1) are such that $\mathrm{rad}(c(\tilde{x})) \geq \mathrm{rad}(c(x'))$, and for the intersection points $\{\tilde{z}\} = \ell_\varphi \cap c(\tilde{x})$ and $\{z'\} = \ell_\varphi \cap c(x')$ we have that $|z'| \leq |\tilde{z}|$. Clearly, every circle C_t intersects both γ and γ' at exactly one point for $t \in [0, |x - y|]$. By geometry we see that for $t > s$ we have

$$\mathrm{dist}(c(s) \cap \gamma, c(t) \cap \gamma) \leq \mathrm{dist}(c(s) \cap \gamma', c(t) \cap \gamma'),$$

and that

$$d(c(s) \cap \gamma) \geq d(c(s) \cap \gamma')$$

for all s. Then the conditions of Lemma 3.7 are fulfilled for every length sequence, and since γ and γ' were geodesics, we see that $k_{S_\varphi}(\tilde{x}, \tilde{y}) \leq k_{S_\varphi}(x', \tilde{y})$. On the other hand it is again clear that $j_{S_\varphi}(\tilde{x}, \tilde{y}) = j_{S_\varphi}(x', \tilde{y})$.

Case 3. $x \in \overline{C}_1,\ y \in \overline{C}_2,\ d(x) \neq d(y)$ and $\varphi \leq 3\pi/4$. First assume $d(x) < d(y)$. Use a line l parallel to $L(0, e^{\varphi i})$ such that $l \bigcirc B(y, d(x))$, and repeat the procedure in Case 1. This reduces the case to Case 5. The case $d(x) > d(y)$ is handled symmetrically by using a line parallel to $L(0, e_1)$ instead.

Case 4. $x \in \overline{C}_1,\ y \in \overline{C}_2,\ d(x) \neq d(y)$ and $\varphi \geq 3\pi/4$. If possible, use the same procedure as in Case 3, when the situation is reduced to Case 5. However, if no line l exists, use a line l_1 parallel to $L(0, e^{\varphi i})$ such that $l \bigcirc B(x, d(x))$ instead. Then the situation is reduced to the situation in Case 2.

Case 5. $x \in \overline{C}_1,\ y \in \overline{C}_2$, and that $d(x) = d(y)$. First we consider the case where the geodesic $\gamma = J_{k_{S_\varphi}}[x, y]$ intersects ℓ_φ in more than one point, like in the picture in the lower right corner of Figure 2. Let $C(y)$ be the big tangent circle of y, and denote by y' the reflection of y in the line ℓ_φ. Then $C(y')$ is the reflection of $C(y)$ in ℓ_φ, and clearly $k(x, y) = k(x, y')$. Also, for every angle $\varphi \in (0,\pi)$ it is easy to see that $|x-y| \geq |x'-y|$, and thus $j(x,y) \geq j(x',y)$. Then this situation is reduced to Case 2 above.

Finally, if the geodesic $\gamma = J_{k_{S_\varphi}}[x,y]$ intersects ℓ_φ in only one point, by Lemmas 5.21 and 5.24 we immediately obtain

$$\frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} \leq 2.$$

Finally, since all cases reduce to either Case 2, which corresponds to the first statement of the lemma, or Case 5, which corresponds to the second statement of the lemma, we are done. □

Now we are ready for proving the main result for angular domains.

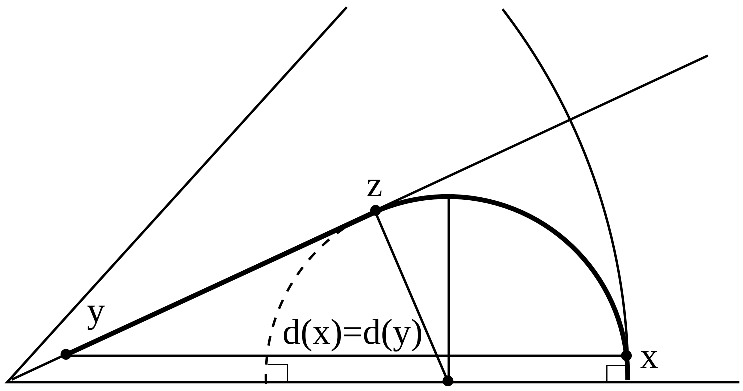

Figure 5. The geodesic of the extremal case.

Proof of Theorem 1.7: By Lemma 5.25 it suffices to consider pairs of points x,y such that $d(x)=d(y)$ and $y \in \ell_\varphi$, such as in Figure 5. Assume, without loss of generality, that $\operatorname{rad}(c(x)) = 1$. Denote the point $\ell_\varphi \cap c(x)$ by z. Letting w be the center of $c(x)$, denote the angle $\widehat{x,w,w+e_1}$ by θ. Letting $x_\theta \in c(x)$ and $y_\theta \in \ell_\varphi$ be the points such that $d(x_\theta) = d(y_\theta) = \sin\theta$, we want to study the ratio $k(x_\theta,y_\theta)/j(x_\theta,y_\theta)$ as a function of θ. Then, by elementary geometry, Lemma 5.9 and the formula (2.9), we obtain

$$\frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} = \frac{\frac{1}{\sin\frac{\varphi}{2}}\log\frac{\cos\frac{\varphi}{2}}{\sin\theta} + \log\cot\left(\frac{\pi-\varphi}{4}\right) + \log\cot\frac{\theta}{2}}{\log\left(1+\frac{\cos\theta + \frac{1}{\sin\frac{\varphi}{2}} - \frac{\sin\theta}{\tan\frac{\varphi}{2}}}{\sin\theta}\right)}.$$

For convenience, denote $x = x_\theta,\ y = y_\theta$. We first show that

$$\frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} \leq \frac{1}{\sin\frac{\varphi}{2}} + 1.$$

Clearly we may restrict to angles $\theta \in (0, \frac{\pi-\varphi}{2})$, as otherwise the geodesic segment $J_k[x,y]$ is contained in a component of $S_\varphi \setminus \ell_\varphi$, and the ratio is $\leq 2 \leq 1/\sin\frac{\varphi}{2} + 1$.

For these angles, however

$$\frac{1}{\sin\frac{\varphi}{2}} - \frac{\sin\theta}{\tan\frac{\varphi}{2}} \geq \frac{1}{\sin\frac{\varphi}{2}} - \cot\tfrac{\varphi}{2} \in (0,1),$$

and for $\theta \in (0, \frac{\pi}{2})$ we have that $\sin\theta + \cos\theta \geq 1$. Hence

$$\cos\tfrac{\varphi}{2} \leq 1 \leq \sin\theta + \cos\theta + \frac{1}{\sin\frac{\varphi}{2}} - \frac{\sin\theta}{\tan\frac{\varphi}{2}},$$

and thus

$$\log\left(\frac{\cos\frac{\varphi}{2}}{\sin\theta}\right) \leq \log\left(1 + \frac{\cos\theta + \frac{1}{\sin\frac{\varphi}{2}} - \frac{\sin\theta}{\tan\frac{\varphi}{2}}}{\sin\theta}\right) = j(x,y).$$

Also, one can show that the inequality

$$\sqrt{2}\cos\tfrac{\theta}{2}\sqrt{1+\sin\tfrac{\varphi}{2}} \leq \sin\theta + \cos\theta + \frac{1}{\sin\frac{\varphi}{2}} - \frac{\sin\theta}{\tan\frac{\varphi}{2}} \tag{5.26}$$

holds. The proof is lengthy but elementary; define a function f by subtracting the right-hand side of the inequality from the left-hand side. Studying the subcase $\varphi \in [\frac{2\pi}{3}, \pi]$, $\theta \in (0, \frac{\pi}{6}]$ one shows that the partial derivative $\partial_\theta f < 0$, and thus no stationary points exist. Then, studying the boundary, the inequality follows. In the subcase $\varphi \in (0, \frac{2\pi}{3}]$, $\theta \in (0, \frac{\pi-\varphi}{2}]$ a similar approach works, only here it is easier to show that $\partial_\varphi f > 0$.

Using trigonometric formulas, (5.26) is seen to imply the inequality

$$\frac{\cos\frac{\varphi}{2}}{\sin\theta}\cot\left(\tfrac{\pi-\varphi}{4}\right)\cot\tfrac{\theta}{2} \leq \left(\sin\theta + \cos\theta + \frac{1}{\sin\frac{\varphi}{2}} - \frac{\sin\theta}{\tan\frac{\varphi}{2}}\right)^2,$$

and thus

$$\log\frac{\cos\frac{\varphi}{2}}{\sin\theta} + \log\cot\left(\tfrac{\pi-\varphi}{4}\right) + \log\cot\tfrac{\theta}{2} \leq 2\, j(x,y).$$

Then

$$\begin{aligned}\frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} &= \frac{\left(\frac{1}{\sin\frac{\varphi}{2}} - 1\right)\log\frac{\cos\frac{\varphi}{2}}{\sin\theta} + \log\frac{\cos\frac{\varphi}{2}}{\sin\theta} + \log\cot\left(\frac{\pi-\varphi}{4}\right) + \log\cot\frac{\theta}{2}}{j_{S_\varphi}(x,y)} \\ &\leq \frac{\left(\frac{1}{\sin\frac{\varphi}{2}} - 1\right)j_{S_\varphi}(x,y) + 2\, j_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} = \frac{1}{\sin\frac{\varphi}{2}} + 1.\end{aligned}$$

Now, it is easy to compute the limit as $\theta \to 0$, when one in fact sees that

$$\lim_{\theta\to 0}\frac{k_{S_\varphi}(x_\theta,y_\theta)}{j_{S_\varphi}(x_\theta,y_\theta)} = \frac{1}{\sin\frac{\varphi}{2}} + 1.$$

But this proves that

$$A_{S_\varphi} = \frac{1}{\sin\frac{\varphi}{2}} + 1$$

cannot be replaced by a smaller constant, and that the extremal case occurs for points x and y when $x \to 0$ and $y \to z \in \partial S_\varphi$, $z \neq 0$. □

As we have seen, in the case of the angular domain S_φ it was easy to obtain the uniformity constant after we had found out which case represents the maximal case. However, in a situation where the maximal case is already known, or if one just wants an estimate for some special symmetric pair of points, the technique presented next might be easier to use, as it doesn't require the geometric machinery used in the previous proofs. On the other hand, the results are not sharp, although they seem to be good approximations (see Figure 6).

The idea is to pull back the quasihyperbolic metric to the known case of the halfplane, and to use integral inequalities. In the case of the domain S_φ we can use a modification by H. Brunn of the classical Chebyshev inequality (see e.g. [MiPeFi, IX.13]). Namely, the result by Brunn gives us a Chebyshev type inequality in the case where one of the functions is symmetric with respect to the midpoint of its definition interval, and the other one is concave.

Lemma 5.27 (Brunn). *Let $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ be continuous and integrable functions defined on an interval (a,b). Let $f_m = \fint_a^b f(x)\,dx$ and let x_m be a point determined by the condition $f(x_m) = f_m$. Then, if $\operatorname{sgn}(f(x) - f_m) = -\operatorname{sgn}(g(x) - g(x_m))$ the inequality*

$$\int_a^b f(x)g(x)\,dx \leq \fint_a^b f(x)\,dx \int_a^b g(x)\,dx$$

holds. In particular, the result is valid if the function g is concave, and f is symmetric with respect to the midpoint of the interval, $\frac{a+b}{2}$, and decreasing in $[a, \frac{a+b}{2}]$. □

Theorem 5.28. *Let $G \subset \mathbb{R}^2$ be a domain such that there exists a conformal mapping $f\colon \mathbf{H}^2 \to G$ for which*

$$\frac{|f'(z)||z|}{|f(z)|} \leq c$$

for some constant c and all $z \in \mathbf{H}^2$. Let $\gamma = J_\rho[f^{-1}(x), f^{-1}(y)]$, and let $\theta_1 = \arg f^{-1}(x)$, $\theta_2 = \arg f^{-1}(y)$. Assume that $\operatorname{dist}(f(z), \partial G) = |f(z)|\beta(\theta_z)$ for some continuous and positive function $\beta : [\theta_1, \theta_2] \to \mathbb{R}$, where $\theta_z = \arg z$. Also assume that the function $\sin\theta/\beta(\theta)$ is concave, and that for all $z \in \gamma$ the function $|dz|/z_2$ is even with respect to the midpoint of the interval $[\theta_1, \theta_2]$ and decreasing in $[\theta_1, \frac{\theta_1+\theta_2}{2}]$. Then the estimate

$$k_G(x,y) \leq c\ \rho_{\mathbf{H}^2}(f^{-1}(x), f^{-1}(y)) \fint_{\theta_1}^{\theta_2} \frac{\sin\theta}{\beta(\theta)}\,d\theta \tag{5.29}$$

holds. Here z_2 denotes the second coordinate of z.

Proof: Denote $\sigma(z) = \operatorname{dist}(z, \partial G)^{-1}$, and let $\gamma\colon \Delta \to \mathbf{H}^2$ be a path representing the hyperbolic geodesic $J_\rho[f^{-1}(x), f^{-1}(y)]$. Then (see e.g. [Väl, 5.6]) we have that

$$\int_{f(\gamma)} \sigma(z)|dz| = \int_\gamma \sigma(f(z))|f'(z)||dz|.$$

Now, by our assumptions

$$\begin{aligned}\frac{|f'(z)||dz|}{\operatorname{dist}(f(z),\partial fG)} &= \frac{|f'(z)||dz|}{|f(z)|\beta(\theta_z)} = \frac{|z||f'(z)|}{|f(z)|}\,\frac{|dz|}{z_2}\,\frac{\sin\theta_z}{\beta(\theta_z)}\\ &\leq c\,\frac{|dz|}{z_2}\,\frac{\sin\theta_z}{\beta(\theta_z)}.\end{aligned}$$

Because of the assumed properties of the function $|dz|/z_2$, it can be written as a function of the argument θ_z. Then by concavity of $\sin(\theta_z)/\beta(\theta_z)$ and Theorem 5.27 it follows that

$$\begin{aligned}\int_{f(\gamma)}\frac{|f'(z)||dz|}{\operatorname{dist}(f(z),\partial fG)} &\leq c\int_\gamma \frac{|dz|}{z_2}\,\frac{\sin\theta_z}{\beta(\theta_z)} = c\int_{\theta_1}^{\theta_2}\frac{|dz|}{z_2}\,\frac{\sin\theta_z}{\beta(\theta_z)}\,d\theta_z\\ &\leq c\int_{\theta_1}^{\theta_2}\frac{|dz|}{z_2}\,d\theta_z \fint_{\theta_1}^{\theta_2}\frac{\sin\theta_z}{\beta(\theta_z)}\,d\theta_z\\ &= c\,\rho_{\mathbf{H}^2}(f^{-1}(x),f^{-1}(y))\fint_{\theta_1}^{\theta_2}\frac{\sin\theta_z}{\beta(\theta_z)}\,d\theta_z.\end{aligned}$$

□

Corollary 5.30. *The uniformity constant A_{S_φ} satisfies*

$$A_{S_\varphi} \leq \frac{2\varphi}{\pi^2}\left(\int_0^{\pi/2}\frac{\sin\theta}{\sin\frac{\varphi}{\pi}\theta}\,d\theta\right)\left(\lim_{t\to 0}\frac{\Phi_\varphi(t)}{\Psi_\varphi(t)}\right),$$

where

$$\Phi_\varphi(t) = \operatorname{ar\,cosh}\left[\frac{1}{2\sin(\frac{\pi}{\varphi}\arcsin t)}\left(\left(\frac{t}{\sin\frac{\varphi}{2}}\right)^{-\frac{\pi}{\varphi}} + \left(\frac{t}{\sin\frac{\varphi}{2}}\right)^{\frac{\pi}{\varphi}}\right)\right]$$

and

$$\Psi_\varphi(t) = \log\left(1-\cot\tfrac{\varphi}{2}+\tfrac{1}{t}\sqrt{1-t^2}\right).$$

PROOF: Let $\varphi \in (0,\pi]$ and let $f\colon \mathbf{H}^2 \to S_\varphi$ be the conformal mapping $z \mapsto z^{\varphi/\pi}$. Then

$$\frac{|f'(z)||z|}{|f(z)|} = \tfrac{\varphi}{\pi}$$

for every $z \in \mathbf{H}^2$. Also, since $\arg(f(z)) = \frac{\varphi}{\pi}\arg(z)$, we have $\operatorname{dist}(f(z),\partial S_\varphi) = |f(z)|\sin(\frac{\varphi}{\pi}\arg z)$, so we may choose $\beta\colon [0,\varphi/2] \to \mathbb{R}$, $\beta(\theta) = \sin\frac{\varphi}{\pi}\theta$. Furthermore, the function

$$\theta \mapsto \frac{\sin\theta}{\sin\frac{\varphi}{\pi}\theta}$$

is concave for $\theta \geq 0$. Following Lemma 5.25 we choose the points $x_t, y_t \in S_\varphi$ by setting, in complex polar notation,

$$\begin{cases} x_t = e^{\theta_x i}\\ y_t = \left(\frac{t}{\sin\frac{\varphi}{2}}\right)e^{\frac{\varphi}{2}i}\end{cases},$$

where $\theta_x = \arg x = \arcsin t$. Then the corresponding points in $\mathbf{H}^2$ are

$$\begin{cases} f^{-1}(x_t) = e^{\frac{\pi}{\varphi}\theta_x i} \\ f^{-1}(y_t) = \left(\frac{t}{\sin\frac{\varphi}{2}}\right)^{\frac{\pi}{\varphi}} e^{\frac{\pi}{2}i} \end{cases}.$$

Now, as $t \to 0$, $\arg f^{-1}(x_t) \to 0$ and $\arg f^{-1}(y_t) \to \frac{\pi}{2}$. Then the points $f^{-1}(x_t)$ and $f^{-1}(y_t)$ themselves approach e_1 and 0, respectively. Clearly $|dz|/z_2 = d\theta/\sin\theta\cos\theta$ is even with respect to $[0, \frac{\pi}{2}]$, and decreasing in $[0, \frac{\pi}{4}]$, so by Lemma 5.25 and Corollary 5.28 we have that

$$\begin{aligned} A_{S_\varphi} &= \lim_{\substack{x\to e_1 \\ y\to 0}} \frac{k_{S_\varphi}(x,y)}{j_{S_\varphi}(x,y)} \le \lim_{t\to 0} \frac{\rho_{\mathbf{H}^2}(f^{-1}(x_t), f^{-1}(y_t)) \displaystyle\fint_{\arcsin t}^{\pi/2} \frac{\frac{\varphi}{\pi}\sin\theta}{\sin\frac{\varphi}{\pi}\theta} d\theta}{\Psi_\varphi(t)} \\ &\le \frac{2\varphi}{\pi^2} \int_0^{\pi/2} \frac{\sin\theta}{\sin\frac{\varphi}{\pi}\theta}\, d\theta \left[\lim_{t\to 0} \frac{\Phi_\varphi(t)}{\Psi_\varphi(t)}\right]. \end{aligned}$$

□

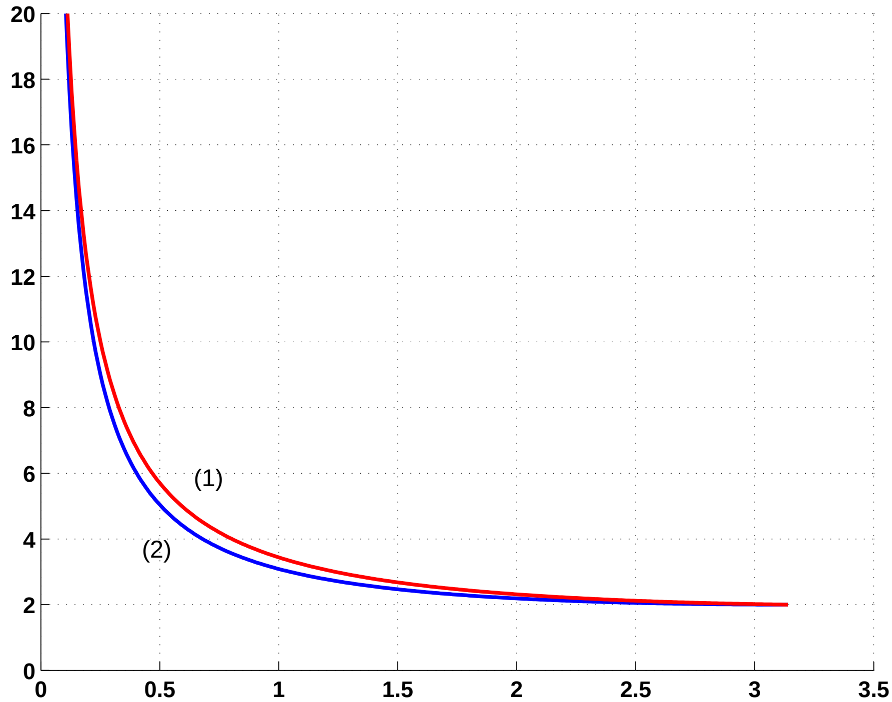

Figure 6. Comparison of the estimate given by Lemma 5.30 (1), and the function $1/\sin\frac{\varphi}{2} + 1$ (2).

We conclude this chapter with a discussion on the large angles, i.e. the cases $\varphi \in (\pi, 2\pi)$. The main difficulties here are that first of all we would need a counterpart for Lemma 5.25, which we do not have. Second, though the geodesics can be determined, the explicit calculation of the distances is hard, quite like Case 5 in Theorem 1.3.

For large angles $\varphi \in (\pi, 2\pi)$ it seems difficult to obtain explicit expressions for the geodesic segments, even if they can be constructed. Because of this we also fail to get sharp results for the uniformity constant. However, the number 2 is of course attained for every angle, and gives a lower bound. One is also tempted to conjecture that the function $\varphi \mapsto A_{S_\varphi}$ is strictly increasing in $[\pi, 2\pi)$. Namely, the limiting domain $\mathbb{R}^n \setminus [0, \infty)$ is not even uniform. Also, it is possible to calculate some explicit lower bounds for the uniformity constants, apart from the obvious bound 2, using specific points with obvious quasihyperbolic geodesics. Natural candidates are provided by the symmetric situations $(r_1, \theta_1) = (1, \theta)$, $(r_2, \theta_2) = (1, \varphi - \theta)$. Letting $\theta \to 0$ we see that this limit does not depend on the angle φ, but is also 2. Numerically one can compute the angle θ that gives the maximal ratio $k_{S_\varphi}/j_{S_\varphi}$ for each angle φ, but a formula for this angle as a function of φ seems to be hard to derive. A good estimate is obtained by $\theta = \frac{2\pi - \varphi}{2}$, which gives a lower bound in terms of the function

$$\varphi \mapsto \frac{2 \log \tan \frac{\varphi}{4} + \varphi - \pi}{\log\left(1 - 2\cos\frac{\varphi}{2}\right)}.$$

Next we employ a result of Gehring and Hag to prove an upper bound.

PROOF OF THEOREM 1.8: The left-hand inequality follows from the discussion above. For the right-hand inequality, denote $S = S_\varphi$ and $S^* = \mathbb{R}^2 \setminus \overline{S_\varphi}$. In [GeHa, Lemma 5.1] it is shown that the mapping $f\colon S^* \to S$

$$f(re^{i\theta}) = -re^{-iL(\varphi)\theta}$$

is a reflection in the boundary ∂S_φ, and also a bilipschitz mapping in the hyperbolic metric. Here $L = L(\varphi) = \max\{(2\pi - \varphi)/\varphi, \varphi/(2\pi - \varphi)\}$ (Note that in [GeHa] the domain S_φ is symmetrically defined about the positive e_1-axis). We first show that $j_{S^*}(x, y) \le L\, j_S(f(x), f(y))$. Let $x = (r_x, \theta_x)$ and $y = (r_y, \theta_y)$ be points in S^*. Since f preserves distances to the origin, we may denote $f(x) = (r_x, \theta'_x)$ and $f(y) = (r_y, \theta'_y)$. Also, we assume that $d(x) \le d(y)$, and then also $d(f(x)) \le d(f(y))$. We want to show that

$$\frac{\sqrt{r_x^2 + r_y^2 - 2r_x r_y \cos|\theta_y - \theta_x|}}{\sqrt{r_x^2 + r_y^2 - 2r_x r_y \cos|\theta'_y - \theta'_x|}} \frac{\sin\theta'_x}{\sin\theta_x} \le L. \tag{5.31}$$

Since angles with vertex at the origin grow in the mapping f, we have that $|\theta'_y - \theta'_x| \ge |\theta_y - \theta_x|$, and thus the ratio of the square roots is ≤ 1. Also, we have $\theta_x \le \theta'_x = L\theta_x$, and so

$$\sin\theta'_x \le \sin(L\theta_x) \le L \sin\theta_x,$$

which proves (5.31), and the inequality

$$j_{S^*}(x, y) \le L\, j_S(f(x), f(y)) \tag{5.32}$$

follows from Bernoulli's inequality (2.14).

Now denote the hyperbolic metric in an arbitrary domain $G \subset \mathbb{R}^2$ by h_G. From the Koebe distortion theorem

$$\frac{1}{2\,\mathrm{dist}(z,\partial G)} \le \rho_G(z) \le \frac{2}{\mathrm{dist}(z,\partial G)},$$

where $\rho_G(z)$ is the metric density of the hyperbolic metric in G, we immediately obtain

$$\frac{1}{2}\,k_G(x,y) \le h_G(x,y) \le 2\,k_G(x,y).$$

From this inequality, together with the fact that f is a hyperbolic bilipschitz mapping, the formula (5.32), and Theorem 1.7, we obtain

$$\begin{aligned}\frac{k_S(x,y)}{j_S(x,y)} &\le 2\,\frac{h_S(x,y)}{j_S(x,y)} \le 2L\,\frac{h_{S^*}(f(x),f(y))}{j_S(x,y)} \\ &\le 2L^2\,\frac{h_{S^*}(f(x),f(y))}{j_{S^*}(f(x),f(y))} \le 4L^2\,\frac{k_{S^*}(f(x),f(y))}{j_{S^*}(f(x),f(y))} \le 4L^2 A_{S^*}.\end{aligned}$$

This completes the theorem. □

We now turn to studying the punctured ball $\mathbf{B}^n_* = \mathbf{B}^n \setminus \{0\}$, the geodesics of which were determined in section 4. Unfortunately the formula for the geodesics given by Lemma 4.12 is rather involved, but the case where the geodesic is tangent to the ball $B(\frac{1}{2})$ is quite easy to compute, and the other cases can be handled using easier estimates.

The natural lower bound is of course given by the constant $\pi/\log 3$, since choosing $x = e_1/a,\ y = -e_1/a$ for $a \ge 2$ gives exactly the maximal constant from the case $\mathbb{R}^n_*$. It is also clear that if both points x and y are located within the ball $B^n(\frac{1}{2})$, the situation is reduced to the case $\mathbb{R}^n_*$. We then need to discuss only the remaining cases, i.e. the case when both points x and y are outside $S^{n-1}(\frac{1}{2})$, and the case when one is outside and the other is not. For this purpose we need the following lemma.

Lemma 5.33. *For each point $x \in \mathbf{B}^2 \setminus B^2(\frac{1}{2})$ there is a unique quasihyperbolic geodesic $\gamma = J_{k_{\mathbf{B}^n}}[x,z]$ and a point $z \in S^1(\frac{1}{2})$ such that $\gamma\ \bigcirc_z\ S^1(\frac{1}{2})$ and the geodesic line γ' containing γ satisfies $\gamma' \perp S^1$. The angle $\varphi = \widehat{x,0,z}$ is given by the expression*

$$\varphi = \arccos\Big(\frac{1-|x|}{|x|}\Big) - \sqrt{2|x|-1}. \qquad (5.34)$$

PROOF: Following [MaOs, p.41] and the formula for the geodesic in Lemma 4.12 we see that solving the integral for $a = 1/2$ gives for the geodesic the equation

$$\theta - \tfrac{\pi}{2} = \arccos\Big(\frac{1-|x|}{|x|}\Big) - \sqrt{2|x|-1},$$

with that the statement concerning the angle is clear. The fact that $\gamma \perp S^1$ is seen by computing the derivative of the geodesic equation

$$\theta'(r) = \frac{1-r}{r\sqrt{2r-1}}.$$

Clearly $\theta'(r) \to 0$ as $r \to 1$. $\square$

Proof of Theorem 1.9: First of all we conclude that it is enough to consider the case $n = 2$ since for every x, y we may restrict to $(\mathbf{B}^n \setminus \{0\}) \cap \Sigma$, where Σ is the 2-dimensional plane determined by $0, x$ and y.

Assume first that $|x| \le \frac{1}{2} \le |y|$. Applying a suitable rotation we may naturally assume that $y = te_1$ for some $t \in [\frac{1}{2}, 1)$. Without loss of generality we may also assume that the point x lies in the upper halfplane. Denote also $s = |x|$ and let $\omega = \widehat{x, 0, y}$. The idea is to estimate the geodesic by the logarithmic spiral connecting x with $\frac{e_1}{2}$ and the geodesic $J_k[\frac{e_1}{2}, y]$. Denoting $z = \frac{e_1}{2}$, and letting H be the halfplane on the left side of the line $L(e_1, e_2)$, we obtain

$$\begin{aligned} k_{B^2_*}(x,y) &\le k_{B^2_*}(x,z) + k_{B^2_*}(z,y) \\ &\le k_{\mathbb{R}^2_*}(x,z) + k_H(z,y) \\ &= k_{\mathbb{R}^2_*}(x,z) + \rho_H(z,y). \end{aligned}$$

Then we may, using Lemma 4.11 and (2.11) define a function $\Phi(\omega, s, s')$ by the formula

$$\begin{aligned} \Phi(\omega, s, s') &= \frac{k_{\mathbb{R}^2_*}(x,z) + \rho_H(z,y)}{j_{B^2_*}(x,y)} \\ &= \frac{\sqrt{\omega^2 + \log^2 2s} + \log \frac{1}{1-2s'}}{\log\left(1 + \frac{\sqrt{s^2 + (t+\frac{1}{2})^2 - 2s(t+\frac{1}{2})\ \cos\omega}}{\min\{s, \frac{1}{2} - s'\}}\right)}, \end{aligned} \tag{5.35}$$

where s and s' take values in $[0, 1/2)$. The extremal situation for this subcase is then obtained by maximizing the function Φ. Differentiating, one can show that $\frac{\partial \Phi}{\partial \omega} > 0$ in all inner points of

$$\{0 \le \omega \le \pi,\ 0 < s \le \tfrac{1}{2},\ 0 \le s' < \tfrac{1}{2}\}.$$

From this follows that Φ is strictly increasing with respect to ω, and that no stationary points exist. Hence we let $\omega = \pi$, and study the boundary of the domain $\{0 < s \le \frac{1}{2},\ 0 \le s' < \frac{1}{2}\}$. Setting $s' = 0$ we see that the maximum of the remaining one-variable function is attained at $s = \frac{1}{2}$, and is $\pi/\log 3$. Similarly, setting $s = \frac{1}{2}$ the maximum is also $\pi/\log 3$, attained at $s' = 0$. Finally, for $s' \to \frac{1}{2}$ it is easy to see that $\Phi(\pi, \cdot\,, s') \to 1$, and also $\Phi(\pi, s, \cdot) \to 0$ as $s \to 0$. Hence we conclude that

$$P = \sup\{\Phi \mid 0 \le \omega \le \pi, 0 < s \le 1/2, 0 \le s' < 1/2\} = \frac{\pi}{\log 3}.$$

Next, assume that $\frac{1}{2} \le |y| \le |x|$. Then it is clear that the k-geodesic $J_k[x,y]$ is contained in the annulus $\mathbf{B}^2 \setminus B^2(\frac{1}{2})$. From Lemma 5.33 we get unique points x' and y' on $S^1(\frac{1}{2})$ and geodesics $J_{k_{\mathbf{B}^n}}[x,x']$ and $J_{k_{\mathbf{B}^n}}[y,y']$ tangential to $S^1(\frac{1}{2})$. Also, the angles $\widehat{x',0,x}$ and $\widehat{y',0,y}$ are given by the expression (5.34). Then we see that $k_{\mathbf{B}^2_*}(x',y') = k_{\mathbb{R}^2_*}(x',y') = \omega - \arccos\left(\frac{1-t}{t}\right) + \sqrt{2t-1} - \arccos\left(\frac{1-s}{s}\right) + \sqrt{2s-1}$. It is also easy to derive formulas for $k_{\mathbf{B}^n}(x,x')$ and $k_{\mathbf{B}^n}(y,y')$, them being

$$\int_{1/2}^{r} \frac{z}{(1-z)\sqrt{2z-1}}\, dz, \tag{5.36}$$

where $r = t$ and $r = s$, respectively.

Hence, we see that defining a function $\Psi(\omega,t,s)$ by

$$\Psi(\omega,t,s) = \frac{k_{\mathbf{B}^2}(x,x') + k_{\mathbb{R}^2_*}(x',y') + k_{\mathbf{B}^2}(y,y')}{j_{\mathbf{B}^2}(x,y)},$$

it is enough to maximize Ψ. However, this function is symmetric with respect to variables t and s, so the stationary points can be found studying

$$\Psi'(\omega,t) = \tfrac{1}{2}\Psi(\omega,t,t) = \frac{\int_{1/2}^{t} \frac{z}{(1-z)\sqrt{2z-1}}\, dz + \frac{\omega}{2} - \arccos\left(\frac{1-t}{t}\right) + \sqrt{2t-1})}{\log\left(1 + \frac{2t\sin\frac{\omega}{2}}{1-t}\right)},$$

Note that studying Ψ' we may restrict to angles

$$\omega \in \Big(2\big(\arccos\left(\tfrac{1-t}{t}\right) + \sqrt{2t-1}\big), \pi\Big],$$

since otherwise the quasihyperbolic geodesic $J_k[x,y]$ is contained in $\mathbf{B}^2 \setminus B^2(\frac{1}{2})$. As for the function Φ, differentiation shows that Ψ' is strictly increasing with respect to ω in the above interval, and hence no stationary points exist.

Setting $\omega = \pi$, and studying the two-variable function $\Psi(\pi,t,s)$, as in the previous case, shows that the maximum is attained at $t = \frac{1}{2}, s = \frac{1}{2}$. Hence,

$$Q = \sup\{\Psi \mid 0 \le \omega \le \pi, 1/2 \le t < 1, 1/2 \le s < 1\} = \frac{\pi}{\log 3},$$

and it is clear that

$$A_{\mathbf{B}^2_*} \le \max\left\{P, Q, \frac{\pi}{\log 3}\right\} = \frac{\pi}{\log 3}.$$

Thus

$$A_{\mathbf{B}^n_*} = \frac{\pi}{\log 3} = A_{\mathbb{R}^n_*}.$$

□

We conclude with a discussion on planar polygonal domains, for which some upper bounds for the uniformity constant can be obtained as corollaries of the results on S_φ. The question of the geodesics is here more delicate, and as they have not been determined our results lack the sharpness of the ones considered in the earlier sections.

Lemma 5.37. *Let $\Delta \subset \mathbb{R}^2$ be a triangle with vertices v_α, v_β and v_γ, where α, β and γ are the corresponding angles. Also assume that $\alpha, \beta, \gamma \in (0, \pi/2]$. Then for any two points x and y on the triangle*

$$|x-y| \geq \sin\theta \ \max\{|x-v_\theta|, |y-v_\theta|\},$$

where $\theta \in \{\alpha, \beta, \gamma\}$ is the angle between the two sides containing x and y.

PROOF: By symmetry, assume that x and y are such that α is the angle between them, so that the situation corresponds to the figure below.

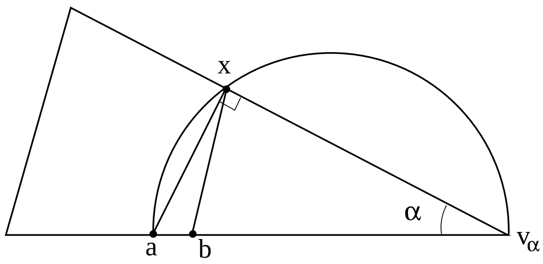

Figure 7.

First assume that $y \in [b, v_\alpha]$, where b is the point such that $|x-v_\alpha| = |b-v_\alpha|$. Then clearly

$$\frac{|x-y|}{\max\{|x-v_\alpha|, |y-v_\alpha|\}} = \frac{|x-y|}{|x-v_\alpha|} \geq \frac{|x-c|}{|x-v_\alpha|} = \sin\alpha.$$

Now assume that y is left of the point b. We get

$$\frac{|x-y|}{\max\{|x-v_\alpha|, |y-v_\alpha|\}} = \frac{|x-y|}{|y-v_\alpha|},$$

and the minimum of this ratio is attained for $y = a$. Thus

$$\frac{|x-y|}{|y-v_\alpha|} \geq \frac{|x-a|}{|a-v_\alpha|} = \sin\alpha.$$

The claim follows. □

Theorem 5.38. *For every triangle Δ with angles $\alpha \leq \beta \leq \gamma$ we have that*

$$k_\Delta(x,y) \leq \frac{A_{S_\alpha} + A_{S_\beta}}{\cos\frac{\gamma}{2}} \, j_\Delta(x,y).$$

for every $x, y \in \Delta$.

PROOF: Let Δ be a triangle, with vertices a, b, c and angles $\alpha \leq \beta \leq \gamma$, respectively, in clockwise direction. Let ℓ_α, ℓ_β and ℓ_γ be the bisectors of the angles, correspondingly, and z be the point of intersection of the bisectors. The bisectors define a partition $\{D_i\}_{i=1,\ldots,6}$ of the triangle Δ. Denote with D_1 the domain touching α on the right of ℓ_α as seen from the vertex a, and the others correspondingly by $D_2, \ldots, D_6$ in clockwise direction.

Let $x, y \in \Delta$ be arbitrary, and assume that $x \in D_1$. Now, if $y \in D_1 \cup D_2$, one easily sees that either A_{S_α} is an upper bound for the uniformity constant, since the S_α-geodesic is within the domain $D_1 \cup D_2$. If $y \in D_6$, there is a unique circular arc $C \perp [a, c]$ such that $C \bigcirc \ell_\alpha$ and $C \bigcirc \ell_\gamma$. First, if either the S_α-geodesic or the S_γ-geodesic is included in $D_1 \cup D_6$, we are done. If not, then $x \in C_1 = \mathrm{comp}(\overline{\Delta(a,c,z)} \setminus C, a)$ and $y \in C_2 = \mathrm{comp}(\overline{\Delta(a,c,z)} \setminus C, c)$, and we see, since the angle $\widehat{a, z, c} > \pi/2$, that the extremal case is represented by the case in the picture.

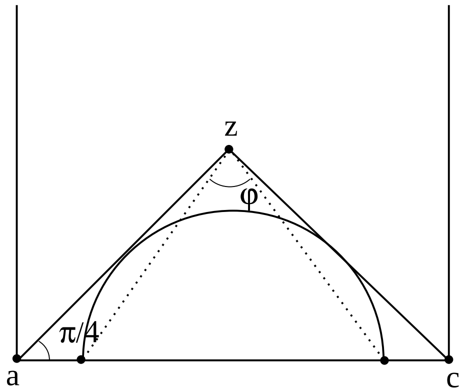

Figure 8.

Then we see that

$$|x - y| \geq \tfrac{2}{\sqrt{3}} \max\{|x - z|, |y - z|\}.$$

Thus

$$\begin{aligned}
\frac{k_\Delta(x,y)}{j_\Delta(x,y)} &\leq \frac{k_\Delta(x,z)}{j_\Delta(x,y)} + \frac{k_\Delta(z,y)}{j_\Delta(x,y)} \leq \frac{\sqrt{3}}{2}\Big(\frac{k_\Delta(x,z)}{j_\Delta(x,z)} + \frac{k_\Delta(z,y)}{j_\Delta(z,y)}\Big) \\
&\leq \frac{\sqrt{3}}{2}\Big(\frac{k_{S_\alpha}(x,z)}{j_{S_\alpha}(x,z)} + \frac{k_{S_\gamma}(z,y)}{j_{S_\gamma}(z,y)}\Big) \leq \frac{2}{\sqrt{3}}\big(A_{S_\alpha} + A_{S_\beta}\big) \\
&\leq \frac{1}{\cos\frac{\gamma}{2}}\big(A_{S_\alpha} + A_{S_\beta}\big),
\end{aligned}$$

where we used the fact that A_{S_θ} is decreasing as a function of θ, and that the largest angle γ must be $\geq \pi/3$.

Finally, if $y \in D_n$ where $n = 3, 4$ or 5, at least one of the triangles in the partition is between the points x and y. It is easy to see that the smallest of the angles at z is $\frac{\pi-\gamma}{2}$, and thus by Lemma 5.37, we see that

$$|x - y| \geq \cos\tfrac{\gamma}{2}\ \max\{|x - z|, |y - z|\}.$$

Then, a similar calculation as in the case of D_6 gives the desired result. □

In the case of a regular n-gon we can use the fact that this figure is close a disk, especially when n is large. We get the following estimate estimate for the uniformity constant.

Theorem 5.39. *For the regular n-gon P_n*
$$\frac{1}{\cos\frac{\pi}{n}} + 1 \leq A_{P_n} \leq \frac{2}{\cos^4\frac{\pi}{n}}.$$

PROOF: Let P_n be a regular n-gon with vertices $\{v_1, v_2, \ldots, v_n\}$, and without loss of generality assume that P_n is origin centered and that $|v_i| = 1$ for all $i = 1, \ldots, n$. For every $z \in \partial P_n$, denote by z' the unique point in $\operatorname{ray}(0,z) \cap S^1$. Let $F\colon P_n \to \mathbf{B}^2$ be the radial mapping which linearly stretches each segment $[0,z]$ onto the segment $[0,z']$. This mapping can easily be extended to a mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, by defining f to be a radial translation outside P_n, i.e. for $x \in \operatorname{ray}(0,z) \cap (\mathbb{R}^2 \setminus \mathbf{B}^2)$ we define $f(x) = x + |z - z'| \cdot z'$. Then the mapping f is obviously bilipschitz, and letting c be the midpoint of say the segment $[v_1, v_2]$, it is clear that the bilipschitz constant is obtained by
$$L = \frac{|f(c)|}{|c|} = \frac{1}{\cos\frac{\pi}{n}}.$$
Then, the upper bound follows from Lemma 3.2 and Remark 2.6 ii). The lower bound follows from Theorem 1.7, since the maximal situation for the domain $S_{\pi(n-2)/n}$ is attained by letting $x \to v_1$ and $y \to c$. □

Remark 5.40. Note that for the equilateral triangle P_3 Theorem 5.39 only gives the upper bound $A_{P_3} \leq 32$, whereas Theorem 5.38 gives $A_{P_3} \leq 4\sqrt{3}$. However, as n grows and P_n approaches to a circle, the estimates in Theorem 5.39 improve, and as expected it is easy to see that $A_{P_n} \to 2$ as $n \to \infty$.

For the rectangle, the approach of pulling back to the halfplane which we used in the case of the angular domain, is also possible. Here we use an elliptic integral, which provides a conformal mapping from the halfplane to a rectangle. In this way the quasihyperbolic distance can be estimated from above. Again the problem is that we do not know which points give the maximal ratio. However, we can develop an estimate for points approaching two corners of the longer side, a situation which probably is maximal in case the ratio of the rectangle sides is small, i.e. the rectangle is close to a square. The hyperbolic metric of a rectangle has recently been studied by A. Beardon in [Be2].

We want to examine the quasihyperbolic metric of the rectangle $[0,a] \times [0,b]$, where $a, b > 0$. Denote for the *complete elliptic integral*
$$\mathcal{K} \;=\; \mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2 t}},$$
and for the *complementary argument* $\sqrt{1-r^2}$, let
$$\mathcal{K}' \;=\; \mathcal{K}'(r) = \mathcal{K}(\sqrt{1-r^2}).$$

Since $\mathcal{K}\colon (0,1) \to (\pi/2, \infty)$ is a strictly increasing homeomorphism, and $\mathcal{K}'\colon (0,1) \to (\pi/2, \infty)$ is a strictly decreasing homeomorphism, there is a uniquely determined number $r \in (0,1)$ such that

$$\frac{2\,\mathcal{K}(r)}{\mathcal{K}'(r)} = \frac{a}{b}.$$

Then it suffices to examine the rectangles $R = [-\mathcal{K}(r), \mathcal{K}(r)] \times [0, \mathcal{K}'(r)]$.

The *incomplete elliptic integral* is defined for any complex number z by the formula

$$\mathcal{F}(z,r) = \int_0^z \frac{dt}{\sqrt{1-t^2}\sqrt{1-r^2t^2}}.$$

It is a conformal mapping $\mathbf{H}^2 \to R$, which has Jacobi's elliptic sine function $\mathrm{sn}\colon R \to \mathbf{H}^2$ as its inverse function. More on the mapping properties of the incomplete elliptic integral can be found in e.g. [KoSt, 4.2].

Theorem 5.41. *Let $f\colon \mathbf{H}^2 \to R$ be the mapping defined by $z \mapsto \mathcal{F}(z,r)$, and let $x, y \in \mathbf{H}^2$ be the points $x = e^{\varphi i}$, $y = e^{(\pi-\varphi)i}$, where $\varphi \in (0, \frac{\pi}{2})$. Then*

$$\begin{aligned} k_R\big(f(x), f(y)\big) &\le \max_{t\in[0,\pi]} \mathcal{S}(t,r)\, \mathcal{T}(t,r)\, \rho_{\mathbf{H}^n}(x,y) \\ &\le \frac{\sqrt{2}}{\sqrt{1-r^2}\,\mathrm{Im}(\mathcal{F}(i,r))}\, \rho_{\mathbf{H}^n}(x,y) \end{aligned}$$

where

$$\mathcal{S}(t,r) = \frac{\sin t}{\sin(\arg \mathcal{F}(e^{it}, r))} \tag{5.42}$$

and

$$\mathcal{T}(t,r) = \frac{1}{\left|\mathcal{F}(e^{it},r)\,\sqrt{1-e^{2it}}\sqrt{1-r^2e^{2it}}\right|}. \tag{5.43}$$

Proof: We follow the technique used in Corollary 5.28. Let $\sigma(z) = \mathrm{dist}(z, \partial R)^{-1}$. By [Väl, 5.6] we see that for all paths $\gamma\colon \Delta \to \mathbf{H}^2$ we have that

$$\int_{f(\gamma)} \sigma(z)|dz| = \int_\gamma \sigma(f(z))|f'(z)||dz|.$$

Let $\alpha\colon [0,\pi] \to \mathbf{H}^2$ be the arc defined by $t \mapsto e^{it}$. Then

$$\frac{|f'(z)||dz|}{\mathrm{dist}(f(z), \partial R)} = \frac{|f'(z)||dz|}{|f(z)|\sin(\arg f(z))} = \frac{|dz|}{z_2}\frac{|f'(z)||z|}{|f(z)|}\frac{\sin(\arg z)}{\sin(\arg f(z))},$$

where $z_2 = \mathrm{dist}(z, \partial\mathbf{H}^2)$. Note that on α we have $|z| = 1$. Also, $|f'(z)|/|f(z)|$ can be expressed as a function of the argument $\theta = \arg z$ by setting

$$\mathcal{T}(\theta,r) = \frac{1}{\left|\mathcal{F}(e^{i\theta},r)\,\sqrt{1-e^{2i\theta}}\sqrt{1-r^2e^{2i\theta}}\right|}.$$

Similarly $\sin(\theta)/\sin(\arg f(z))$ is a function of the argument by

$$\mathcal{S}(\theta, r) = \frac{\sin\theta}{\sin(\arg \mathcal{F}(e^{i\theta}, r))}.$$

Then, letting $\gamma \in \Gamma_{f(x)f(y)}$ we get

$$k_R(f(x), f(y)) = \inf_{\gamma} \int_{\gamma} \sigma(z)|dz| \leq \int_{f(\alpha)} \sigma(z)|dz| = \int_{\alpha} \frac{|dz|}{z_2}\, \mathcal{T}(\theta, r)\, \mathcal{S}(\theta, r)\, d\theta.$$

The product $\mathcal{T}(\theta, r)\ \mathcal{S}(\theta, r)$ is bounded, so using Hölder's $(1, \infty)$-inequality we obtain

$$k_R(f(x), f(y)) \leq \int_0^{\pi} \frac{|dz|}{z_2} \max_{\theta \in [0,\pi]} \mathcal{T}(\theta, r)\ \mathcal{S}(\theta, r) = \rho_{\mathbf{H}^2}(x, y) \max_{\theta \in [0,\pi]} \mathcal{T}(\theta, r)\ \mathcal{S}(\theta, r).$$

Now, one can show that

$$\mathcal{S}(\theta, r) \leq \mathcal{S}(\theta, 0) \leq 2\sqrt{\sin\theta}.$$

Also,

$$|\mathcal{F}(e^{i\theta}, r)| = \operatorname{Im}(\mathcal{F}(i, r)) \quad \text{and} \quad |\sqrt{1 - e^{2i\theta}}| = \sqrt{2\,\sin\theta},$$

and furthermore

$$|\sqrt{1 - r^2 e^{2i\theta}}| \geq \Big|_{\theta=0} |\sqrt{1 - r^2 e^{2i\theta}}| = \sqrt{1 - r^2}.$$

Combining these yields the t-independent estimate in the statement of the theorem. □

From Theorem 5.41 we see that to get an upper bound for the ratio k/j at the corner points, we need to find a constant C such that

$$\frac{\rho_{\mathbf{H}^2}(x, y)}{j_R(f(x), f(y))} \leq C.$$

However, this inequality seems to be nontrivial to prove. By the definition of j_R and (2.9) it holds true with $C = 4$ if

$$\log\cot\tfrac{t}{2} \leq 2\log\Big(1 + \cot\big(\arg \mathcal{F}(e^{it}, r)\big)\Big),$$

that is, if

$$\cot\tfrac{t}{2} \leq \Big(1 + \cot\big(\arg \mathcal{F}(e^{it}, r)\big)\Big)^2.$$

This, in turn, seems to hold for values $r \leq 1/2$, but as r grows the constant C must be chosen larger. In fact it seems that $C \to \infty$ as $r \to 1$.

It should be noted, that the estimate given by Theorem 5.41 is merely an upper estimate for some special points, and does not qualify as a lower bound for the uniformity constant, since we do not know whether the number is attained. However, if one can prove that the situation is maximal, for instance for small ratios a/b, then the theorem gives an upper bound.

For rectangles with large ratio a/b the situation is somewhat different. Let $a \geq 1$, $b = 1$ and let $x, y \in [0,a] \times [0,b]$ be the points $x = (\frac{1}{2}, \frac{1}{2})$, $y = (a - \frac{1}{2}, \frac{1}{2})$. Then it is clear that the k-geodesic is the straight line segment $[x, y]$, and we have

$$\frac{k_R(x,y)}{j_R(x,y)} = \frac{2a-2}{\log(2a-1)}.$$

Clearly this grows without bound as $a \to \infty$. Then we see that the function

$$\frac{4\mathcal{K}(r) - 2\mathcal{K}'(r)}{\mathcal{K}'(r)\ \log\left(\frac{4\ \mathcal{K}(r)}{\mathcal{K}'(r)} - 1\right)}$$

is actually a lower bound for the uniformity constant of a rectangle with parameter $r \in [3 - 2\sqrt{2}, 1)$.

It is actually also possible to obtain an upper bound for the uniformity constant, using a bilipschitz mapping as in Theorem 5.39. The proof will be omitted, as the technique is exactly the same as in the proof for polygons. However, the upper bound in the following theorem gets large quite quickly, and probably is far from optimal.

Theorem 5.44. *Let R be the rectangle $[0,a] \times [0,1]$, where $a \geq 1$. Then, the uniformity constant satisfies*

$$\frac{2a-2}{\log(2a-1)} \leq A_R \leq 2(1+a^2)^2.$$

□

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