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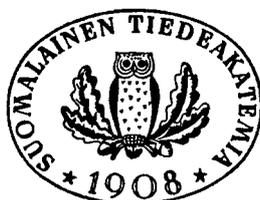
MATHEMATICA

DISSERTATIONES

158

ON NONCOMMUTATIVE BRST-COMPLEX AND
SUPERCONNECTION CHARACTER FORMS

HENRI LIPPONEN



HELSINKI 2013
SUOMALAINEN TIEDEKATEMIA

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HENRI LIPPONEN

University of Helsinki, Department of Mathematics and Statistics

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Helsinki, December 2012

Henri Lipponen

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1. INTRODUCTION

We study Chern-forms associated with the signature operators of Dirac operators on manifolds with boundary, where the Dirac operators are coupled to vector potentials. These Chern-forms live on the space of vector potentials and can also be thought to be regularizations of Chern-forms on the infinite dimensional Grassmannian manifold [Q], [MP], [MR], [St].

In canonical quantization of Fermion fields in background fields, the signature operators as above specify the vacuum in the Fermionic Fock-space [MR], [MP], [M]. Also, the corresponding Chern-forms appear in the study of anomalies. Especially, the Hamiltonian anomaly is associated with the Chern-form of degree two, which represents the curvature of the vacuum line bundle [MP], [CM], [CMM] and is also known as the Schwinger term. In geometry, these forms are related to the determinant line bundles (and their generalizations) [L], [MR], [M2], [St]. In particular, the Schwinger-term is related to the curvature of the determinant line bundle [MP], [St].

Due to the infinite dimensionality we need to regularize the trace. We use, essentially, the zeta function regularization introduced in [MeNi] (see also [MoNi]) to define the *regularized trace*. This regularized trace is used to define the Chern-forms.

However, the above Chern-forms are not closed. The obstruction is measured by the trace anomaly formula [MeNi]. In contrast to the closed manifold case the trace anomaly on manifold with boundary is not local. Namely, there is always the non-local boundary contribution in the trace anomaly formula, in addition to the local term given by the Wodzicki residue [W].

Therefore, at best the Chern-forms are closed modulo the boundary terms from the trace anomaly formula. If this is the case, then the Chern-forms become *transgressive*. Moreover, if the boundary is empty, then the transgressive forms become closed forms.

Similar regularizations of Chern-forms have been studied, in a different context, for example in [CDP], [CM], [P], [PR] in the case of closed manifolds. For the construction of transgression forms see [MeRo3].

Our first objective is to regularize the Chern-forms so that they become transgressive. We discuss two different ways of regularization. In the first approach we use ideas of E. Langmann to construct a noncommutative differential calculus ([La], [La2], [LaMiRy]), which allows us to construct the needed regularizations in a systematic way.

In the second approach we modify the Chern-forms by suitable local counterterms coming from the Wodzicki residues. This approach was introduced in [MP].

The second objective is to find local representatives for the induced forms on the boundary of the regularized Chern-forms that are transgressive, when the regularized Chern-forms are restricted to the gauge directions. Here the induced form on the boundary means the differential of the corresponding regularized Chern-forms restricted to the boundary. If the boundary is empty, then we find local representatives for the restricted regularized Chern-forms.

In the gauge directions (on the space of connections), there is a cohomology theory, the BRST-cohomology [B], [S1], [S2]. This is the relevant cohomology theory in physical applications. Now, the constructing local representatives means, essentially, decomposing the regularized Chern-form in terms of the regularized traces of commutators and BRST-coboundaries. If the manifold is even dimensional, then we have to add a term that depends only on the Maurer-Cartan form [LaMiRy]. If the manifold has a boundary, then we need to take BRST-coboundary of the regularized Chern-forms in order to obtain local representatives (for the BRST-cohomology class) for the induced forms on the boundary.

The existence of such decompositions of the regularized Chern-forms was proved in [LaMiRy]. However, in that paper the explicit construction was left open. In this thesis, the construction of an algorithm that allows us to explicitly represent the regularized Chern-forms in terms of commutators, BRST-coboundaries and terms depending only on the Maurer-Cartan form is given. This algorithm allows us to find a local representative for every regularized Chern-form. We also give a direct construction of the local representations, when the Chern-forms are regularized using the counterterm regularization of Mickelsson-Paycha [MP].

The third objective is to give explicit examples of the local representatives of the regularized Chern-forms. Moreover, we give some examples, where the local representative can be explicitly computed. Particularly, we obtain the standard local formulas for the chiral anomaly (one-form case) and the Schwinger-term (two-form case) in dimensions ≤ 4 . We also give examples on higher dimensions.

This thesis is organized as follows. We first give a review of the cusp calculus of Melrose-Mazzeo [MeMa], [MeNi]. The cusp calculus allows us to discuss the manifold with the boundary case. There are also other options, such as b-calculus [Me3].

After the necessary machinery of the cusp pseudodifferential operators has been introduced, we review the notion of the regularized trace [MeNi], [MoNi] in the cusp calculus (see also [CDP], [MP], [Sc] for the case of a closed manifold). Particularly,

we discuss the trace anomaly formula. The trace anomaly formula is the key tool to obtain the local formulas for the regularized Chern-forms.

Next we give a brief introduction to the BRST-formalism [B]. Here we take the minimalistic approach, giving only those details that are absolutely necessary to follow the computations involving Lie-algebra cocycles. We also give some examples of the standard cocycle formulas relevant in the quantum field theory.

Some basics of the Chern-Weil theory is introduced next. This provides the machinery to construct the standard cocycle formulas. We also use the same philosophy later in the 'noncommutative setup' of Chern-Weil theory [La], [LaMiRy].

After these preliminaries we state our assumptions for the manifolds, gauge transformations and connections. Then, we define the naive Chern-forms and show the need of the regularization in order to obtain transgressive forms. We also discuss the supercommutator, which is used a lot in the computations that follow.

In Section 7, we show some explicit examples how the local representatives are constructed. The technique of 'integration of parts' is also introduced here, which is used to construct the local representatives of the regularized Chern-forms.

In Section 8, we regularize the forms given in Section 7. We prove that these regularizations are transgressive. However, the proof that these regularizations agree with the original forms modulo BRST-coboundaries have to wait until the tools from the later sections become available.

The construction of such tools begins next in Section 8. We develop certain 'noncommutative' BRST-formalism along the lines of [LaMiRy], [La]. We introduce the notions of noncommutative *BRST-complex*, *superconnections* and their Chern-Weil and Chern-Simons forms on even dimensional manifolds. Their basic properties are then studied. Then this theory is applied to the construction of the regularizations that we need.

In this noncommutative BRST-formalism the transgression forms are interpreted as Chern-Simons forms. Moreover, these Chern-Simons forms depend on a choice of the path. It is important that we know this dependence explicitly. To this end, we introduce the higher Chern-Simons forms. These higher Chern-Simons forms give us the path dependence of the Chern-Simons forms by the use of the triangle formula. The triangle formula is also proven in this section.

In Section 9 the integration of these Chern-Simons forms is defined by taking suitable regularized traces of the Chern-Simons forms. The regularized trace of the Chern-Simons form is called an *eta-chain*. We study them briefly, and also introduce a notion of an *eta-cocycle*. This eta-cocycle is precisely the transgressive form that we are after.

In Section 10 we prove the fundamental iteration formulas (see Lemma 3 and Lemma 4) that allow us to represent the above Chern-Simons forms in a *standard form*, that is, in terms of commutators, BRST-coboundaries and terms depending only on the Maurer-Cartan form, explicitly.

In sections 11,12 and 13 we give applications related to the above decompositions of the Chern-Simons forms. Particularly, we construct the local representation formulas for the eta-chains. We also discuss briefly the regularization introduced in [MP]. Some explicit examples of the above local representation formulas are also computed.

In Section 14 we briefly discuss the odd dimensional case. Compared to the even dimensional case the treatment of the odd case consists mostly just introducing notation. Therefore, we just concentrate on a few special cases of the physical interest. The most important case is the Schwinger term. We construct an operator expression for the Schwinger term in any odd dimension. Particularly, we construct a local representation formula for the Schwinger term that agrees with the local representation given in [LaMi], when we restrict to the dimension three.

Next, we briefly discuss the technicalities related to the zero-modes of the Dirac operator. Finally, after a brief summary, we discuss some open problems related to this work.

2. THE CUSP CALCULUS

We give a brief review of the calculus of cusp pseudodifferential operators and some of their basic properties. For more details and additional information see [MeMa] and [MeNi].

Let M be a n -dimensional compact manifold with connected boundary. We assume M is equipped with a boundary defining function $x \in C^\infty(M)$, which is by definition a positive smooth function vanishing at the boundary with $dx \neq 0$ at $x = 0$. Hence it trivializes the conormal bundle of ∂M .

We assume that M is equipped with the exact cusp metric g [MeNi], [MoNi]. This metric is of the following form on a collar neighbourhood $[0, \epsilon)_x \times \partial M$ of the boundary

$$(2.1) \quad g = \frac{dx^2}{x^4} + h.$$

Here $h \in C^\infty([0, \epsilon)_x \times \partial M; T^*(\partial M) \times T^*(\partial M))$ is a smooth Riemannian metric.

Remark 1. The cusp metric above does not describe a manifold with cusps. Rather, it describes a manifold with infinite cylindrical end. Moreover, near the cylindrical end, the cusp metric itself is obtained from the standard metric $dt^2 + h$ on the cylinder $[0, \infty) \times \partial M$ by compactifying the cylinder. This compactification is, essentially, given by defining $t = \frac{1}{x}$.

Definition 1. The elements in the set

$$(2.2) \quad V_c(M) = \{X \in C^\infty(M, TM) \mid Xx \in x^2 C^\infty(M)\},$$

are called *cusp vector fields*.

Proposition 1. The cusp vector fields form a Lie algebra with respect to the Lie-bracket of vector fields.

Proof. Let V, W be cusp vector fields. Then by definition $Vx = x^2g$ and $Wx = x^2g'$ for some smooth real valued functions g, g' on M . Therefore

$$\begin{aligned}
 [V, W]x &= V(Wx) - W(Vx) \\
 &= V(x^2g') - W(x^2g) \\
 (2.3) \quad &= (Vx^2)g' + x^2Vg' - (Wx^2)g - x^2Wg \\
 &= x^2(Vg' - Wg).
 \end{aligned}$$

We see that $(Vg' - Wg)$ is smooth. Thus $[V, W]$ is a cusp vector field. \square

The following local representation of cusp vector fields is usefull. Let $V \in V_c(M)$. We can write near the boundary

$$(2.4) \quad a(x, y)x^2 \frac{\partial}{\partial x} + \sum_{\alpha=1}^{n-1} b_\alpha(x, y) \frac{\partial}{\partial y^\alpha}.$$

Here a, b_α are smooth functions and $\frac{\partial}{\partial y^\alpha}$ are the coordinate tangent vectors to ∂M .

The cusp vector fields can be considered as sections of a vector bundle. This vector bundle is the cusp tangent bundle ${}^c TM$. It is usually constructed as follows [MeMa]. Define I_p to be the ideal of smooth functions on M vanishing at the point p in M . Denote by $I_p V_c(M)$ the finite linear span of products aV , $a \in I_p$ and $V \in V_c(M)$. Then we set

$$(2.5) \quad {}^c T_p M = V_c(M) / I_p V_c(M).$$

Proposition 2. The disjoint union

$$(2.6) \quad {}^c TM = \coprod_{p \in M} ({}^c T_p M),$$

has a unique smooth structure as a vector bundle over M . Furthermore there is a natural linear map $i_p : {}^c T_p M \rightarrow T_p M$, which is an isomorphism on interior. These will give a bundle map $i : {}^c TM \rightarrow TM$ such that for every $V \in C^\infty(M, {}^c TM)$ there is a unique $V' \in V_c(M) \subset C^\infty(M, TM)$ such that

$$(2.7) \quad i_p V'_p = V_p,$$

for all $p \in M \setminus \partial M$.

Proof. See [MeMa, Lemma 2]. \square

The cusp cotangent bundle ${}^c T^*M$ can be defined by duality. Its sections are called cusp covector fields. Any cusp covector field can be represented near the boundary as

$$(2.8) \quad a(x, y) \frac{dx}{x^2} + \sum_{\alpha=1}^{n-1} b_\alpha(x, y) dy^\alpha,$$

for some smooth functions a, b_α on M .

Definition 2. A *cuspidal differential operator* D of order k on M is an operator $D : C^\infty(M) \rightarrow C^\infty(M)$, which is a product of at most k cuspidal vector fields. Locally near the boundary D is of the form

$$(2.9) \quad \sum a_{ij\alpha}(x, y) \left(x^2 \frac{\partial}{\partial x} \right)^i \left(\frac{\partial}{\partial y^\alpha} \right)^j,$$

where $a_{ij\alpha}$ are smooth functions on M , $\frac{\partial}{\partial y^\alpha}$ are the coordinate tangent vectors to ∂M , i, j run from 0 to k and α runs from 1 to $n - 1$. The space of cuspidal differential operators of order k is denoted by

$$(2.10) \quad \text{Diff}_c^k(M).$$

The cuspidal vector fields form a module over smooth functions. Hence, the cuspidal differential operators have this property too. Now, the cuspidal differential operators acting between sections of vector bundles can be defined.

Definition 3. Let E and F be vector bundles over M . Then the space of cuspidal differential operators of order k , acting from sections of E to sections of F is defined as a tensor product

$$(2.11) \quad \text{Diff}_c^k(M; E, F) = \text{Diff}_c^k(M) \otimes_{C^\infty(M)} C^\infty(M; \text{Hom}(E, F)),$$

where $\text{Hom}(E, F)$ denotes the homomorphism bundle over M with fiber $\text{Hom}(E_z, F_z)$ at the point z in M .

Example 1. A basic example of a cuspidal differential operator is the Laplacian

$$(2.12) \quad \Delta_c : C^\infty(M) \rightarrow C^\infty(M),$$

associated to a cuspidal metric.

For example, if the metric h on the boundary ∂M is flat, then the Laplacian takes the form

$$(2.13) \quad \Delta_c = \left(x^2 \frac{\partial}{\partial x} \right)^2 + \sum_{k=1}^{n-1} \left(\frac{\partial}{\partial y^k} \right)^2,$$

near the boundary. More generally, the cuspidal Laplacian is of the form

$$(2.14) \quad \Delta_c = \left(x^2 \frac{\partial}{\partial x} \right)^2 + \Delta_\partial,$$

where Δ_∂ is the Laplacian associated to the Riemannian metric on the boundary.

The definition of the cuspidal pseudodifferential operators is given in terms of their Schwartz kernels. These kernels are defined, in Melrose's framework, using iterative blow-ups. Total of two blow-ups are needed to characterize the kernels of cuspidal pseudodifferential operators. A brief introduction to the ideas involved to define these operators is given. More details can be found in [Me3], [Me4] and [H]. An elementary introduction to blow-ups and b-calculus can be found in [Gr].

Recall how the pseudodifferential operators are defined on a closed manifold. In that case the kernels of pseudodifferential operators are conormal distributions to

the diagonal [H], [Me3]. A similar characterization on the manifold with boundary is discussed.

Now the kernels live on the space $M \times M$, which is a manifold with corner. The corner is $\partial M \times \partial M$. Denote the diagonal in $M \times M$ by Δ . Due to the corner, it no longer makes sense to talk about conormality to the diagonal. There is no way to choose a model fibre to the conormal bundle to Δ at the corner. The problem is that Δ meets both boundary faces $\partial M \times M$ and $M \times \partial M$.

Blowing up the corner $\partial M \times \partial M$ in $M \times M$ solves this problem. The blow-up process separates the boundary faces and creates a new face, the so-called b-front face. There is also a new diagonal, the b-diagonal Δ_b . The b-front face is now the only face that meets the b-diagonal and it does it transversally. Now, the conormal bundle to the b-diagonal can be defined. After this blow-up one could give the definition of the (small) calculus of b-pseudodifferential operators. This is discussed in detail in [Me3].

Blowing up the corner in $M \times M$ produces a new space, the so-called b-stretched double of M . This is denoted by

$$(2.15) \quad M_b^2 = [M \times M, \partial M \times \partial M].$$

This space becomes equipped with a blow-down map

$$(2.16) \quad \beta_b : M_b^2 \rightarrow M \times M.$$

This map can be given as follows. Let x and x' denote the lifts of the boundary defining function x to the left and right factor of $M \times M$ respectively. Put

$$(2.17) \quad \begin{aligned} s &= \frac{x - x'}{x + x'} \\ r &= x + x'. \end{aligned}$$

The above expressions define smooth functions s and r on M_b^2 . The map β_b is given by

$$(2.18) \quad \beta_b(r, s) = \left(\frac{1}{2}r(1 + s), \frac{1}{2}r(1 - s) \right).$$

The map β_b defines polar coordinates to the corner. Here, the variable s is the angular variable that varies between -1 and 1 . These end points correspond to the left and the right boundary faces. Value $s = 0$ means we are on the diagonal. The function r is the radial variable and it varies from 0 to ∞ . It is the defining function for the b-front face. That is

$$(2.19) \quad \text{ff}_b \equiv \{r = 0, s \in [-1, 1]\} = \beta_b^{-1}(\partial M \times \partial M) \approx \partial M \times \partial M \times [-1, 1]_s.$$

The b-diagonal is defined by

$$(2.20) \quad \Delta_b \equiv \text{cl}(\beta_b^{-1}(\text{int}(\Delta))),$$

where cl denotes the taking of the closure. In the above coordinates, the identification $\Delta_b = \{s = 0\}$ holds. Now a transversal part to the diagonal can be chosen.

The transversal part is obtained by moving along the front face. Now the definition of the b-pseudodifferential operators could be given, but we do one more blow-up.

We blow up the fibre diagonal $D = \partial M \times \partial M \times \{0\}$, which is an interior p-submanifold of the b-front face $\partial M \times \partial M \times [-1, 1]_s$ (see [Me4]). This blow-up completely separates the lifted diagonal from the lifts of the old boundary faces $\partial M \times M$ and $M \times \partial M$ to M_b^2 . Again, there is a new front face (the cusp front face) and a new diagonal (the cusp diagonal), and they meet transversally.

In this blow-up M_b^2 is replaced by the following space

$$(2.21) \quad M_c^2 = [M_b^2, D] = [M^2, \partial M \times \partial M, D],$$

which comes with blow-down maps

$$(2.22) \quad \begin{aligned} \beta_{c-b} : M_c^2 &\rightarrow M_b^2 \\ \beta_c : M_c^2 &\rightarrow M^2. \end{aligned}$$

These can be defined using polar coordinates. The function $s = 1 - \frac{x}{x'}$ gives a boundary defining function for the diagonal outside the right and left boundary faces. That is, when $s = 0$ we are on the diagonal. A boundary defining function for the b-front face is still needed. Here $R = x$ is used as such a function. This is to be thought as a radial coordinate. The angular coordinate is $\frac{s}{x}$, which is defined outside the left and right boundary face. Thus, the following coordinates are valid near the points where Δ_b and ff_b meet

$$(2.23) \quad \begin{aligned} S &= \frac{s}{x} = \frac{1}{x} - \frac{1}{x'} \\ R &= x. \end{aligned}$$

The *cusp front face* is by definition

$$(2.24) \quad \text{ff}_c = \beta_{c-b}^{-1} D.$$

In the above coordinates it corresponds setting $R = x = 0$.

The *cusp diagonal* is defined as the closure of the inverse image (under β_c) of the interior of the diagonal of $M \times M$. The cusp diagonal is denoted by $\Delta_c \subset M_c^2$. In the above coordinates the cusp diagonal corresponds to setting $S = 0$.

Definition 4. Densities on M which are near the boundary of the form

$$(2.25) \quad f \frac{dx}{x^2} dy,$$

where f is a smooth function, dy is a density on ∂M are called *cusp densities*. The cusp density bundle is denoted by ${}^c\Omega$.

The definition of the cusp pseudodifferential operators can be now given.

Definition 5. The space of cusp pseudodifferential operators $\Psi_c^m(M)$ acting on functions, is the space of Schwartz kernels on M_c^2 that are conormal to the diagonal

Δ_c and vanishing rapidly with all derivatives in terms of Taylor series on all the boundary faces, except on the cusp front face. More formally

$$(2.26) \quad \Psi_c^m(M) = \{k \in I^{[m]}(M_c^2, \Delta_c; {}^c\Omega_R), k \equiv 0 \text{ on } \partial M_c^2 \setminus \text{ff}_c\},$$

where \equiv means equality in terms of Taylor series and ${}^c\Omega_R$ denotes the cusp density bundle lifted from the right factor in $M \times M$.

Note, that the above definition has an ambiguity. Namely, it does not tell us what kind of symbols we are using. In this thesis, we restrict to classical symbols.

See [LoMoPa, Appendix B] for a description, in the non-compact picture, for the kernels of the cusp pseudodifferential operators.

To define the cusp pseudodifferential operators acting on sections of a vector bundle, we use the fact that $\Psi_c^m(M)$ is a $C^\infty(M^2)$ module [MeNi], [MeMa], which permits the following definition.

Definition 6. The space of cusp pseudodifferential operators $\Psi_c^m(M; E, F)$ acting from the sections of a vector bundle E to sections of a vector bundle F is the space of kernels

$$(2.27) \quad \Psi_c^m(M; E, F) = \Psi_c^m(M) \otimes_{C^\infty(M^2)} C^\infty(M^2, \text{HOM}(E, F)).$$

Here $\text{HOM}(E, F)$ denotes the 'big' homomorphism bundle over $M \times M$ with fibers $\text{Hom}(E_z, F_{z'})$ over $(z, z') \in M \times M$. The space $\Psi_c^m(M; E, E)$ is denoted by $\Psi_c^m(M; E)$.

Example 2. The first example to consider is the identity operator

$$(2.28) \quad \text{Id} : C^\infty(M) \rightarrow C^\infty(M)$$

it acts as

$$(2.29) \quad \text{Id} \cdot f(x, y) = \int_M dx' dy' \delta(x - x') \delta(y - y') f(x', y') \frac{dx}{x^2}.$$

Thus the kernel is

$$(2.30) \quad \delta(x - x') \delta(y - y') dx' dy' \frac{dx}{x^2} dy.$$

We need to lift the kernel to the cusp double space. To do this, we use the coordinates s and x' , where $s = \frac{x-x'}{(x')^2}$ and x' . First, we use the fact that the delta function has homogeneity -1 to get

$$(2.31) \quad \delta(x - x') = \delta\left(\frac{x - x'}{(x')^2} (x')^2\right) = \delta(s(x')^2) = \frac{\delta(s)}{(x')^2}.$$

We still have to compute

$$(2.32) \quad ds = (2s - 1) \frac{dx'}{x'^2}.$$

The use of the fact that the delta function kills the term $2s$ yields

$$(2.33) \quad -\delta(s) \delta(y - y') ds dy' \frac{dx}{x^2} dy.$$

The next example is a cusp differential operator. The kernel of a cusp differential operator of order k is of the form

$$(2.34) \quad -a(x, y) \partial_s^i \delta(s) \partial_y^j \delta(y - y') ds \frac{dx}{x^2},$$

where a is a smooth function on M and $|i| + |j| \leq k$. This representation follows by composing with the identity map above and lifting $x^2 \partial_x$ with the above coordinates.

As in the case without boundary, the cusp pseudodifferential operators of order $-\infty$ are called smoothing operators. In contrast to the closed manifold case these are not the real residual operators. The residual space of cusp pseudodifferential operators consists of smoothing cusp operators whose Schwartz kernels vanish also at the cusp front face in terms of Taylor series. The residual space can be denoted as $x^\infty \Psi_c^{-\infty}(M; E, F)$.

2.1. Basic properties of cusp operators. Following [MeNi] and [MeMa], some basic properties of the cusp pseudodifferential operators are given.

Let E be any vector bundle over M and denote by $C^\infty(M; E)$, $\dot{C}^\infty(M; E)$ the smooth sections and smooth sections that vanish infinite order at the boundary respectively. The distributional sections of E are denoted by $C^{-\infty}(M; E)$ (extendible distributions) and $\dot{C}^{-\infty}(M; E)$ (supported distributions). By the notation $\dot{C}^{-\infty}(M; E)$ we mean the dual of $C^\infty(M; {}^c\Omega \otimes E')$, where E' is a dual bundle of E . Similarly $C^{-\infty}(M; E)$ denotes the dual of $\dot{C}^\infty(M; {}^c\Omega \otimes E')$ (for more details on these spaces see for example [Me4]). The basic mapping properties can be now stated.

Proposition 3. Let $A \in \Psi_c^m(M; E, F)$, for $m \in \mathbb{R}$ and vector bundles E, F over M . Then A defines consistent continuous linear maps

$$(2.35) \quad \begin{aligned} A &: C^\infty(M; E) \rightarrow C^\infty(M; F) \\ A &: \dot{C}^\infty(M; E) \rightarrow \dot{C}^\infty(M; F) \\ A &: \dot{C}^{-\infty}(M; E) \rightarrow \dot{C}^{-\infty}(M; F) \\ A &: C^{-\infty}(M; E) \rightarrow C^{-\infty}(M; F). \end{aligned}$$

The consistency requirement comes from the inclusions of the above spaces.

Proof. For a detailed proof, see [MeMa, Proposition 3] . □

From now on, if A is an element of $\Psi_c^m(M; E, F)$, then we think A as an operator $A : C^\infty(M; E) \rightarrow C^\infty(M; F)$, unless stated otherwise.

It can be proven, see [MeNi], that the boundary defining function acts as multiplier on $\Psi_c^*(M)$. Particularly, any $A \in \Psi_c^m(M; E, F)$ defines continuous linear maps

$$(2.36) \quad A : x^a C^\infty(M; E) \rightarrow x^a C^\infty(M; F),$$

for real numbers m and a .

Proposition 4. Let $A \in x^a \Psi_c^m(M; E, F)$ and $B \in x^b \Psi_c^{m'}(M; F, G)$, where a, b, m and m' are real numbers. Then

$$(2.37) \quad BA \in x^{a+b} \Psi_c^{m+m'}(M; E, G).$$

Proof. See [MeMa, Theorem 2] . □

Proposition 5. The principal symbol map σ extends by continuity from the interior to a well defined map

$$(2.38) \quad \sigma_m : \Psi_c^m(M) \rightarrow S^m({}^cT^*M).$$

It fits in the following exact sequence

$$(2.39) \quad 0 \rightarrow \Psi_c^{m-1}(M) \hookrightarrow \Psi_c^m(M) \rightarrow S^m({}^cT^*M)/S^{m-1}({}^cT^*M) \rightarrow 0.$$

Here the notation $S^m(E)$ means (classical) symbols of order m for any vector bundle E over M .

Extension to the case of vector bundle valued symbols is straightforward.

Example 3. Consider the Laplacian in the previous example. Its principal symbol is, near the boundary,

$$(2.40) \quad \sigma(\Delta_c) = -\lambda^2 - \sum_{k=1}^{n-1} p_k^2.$$

Let $V = a(x, y)x^2\partial_x + \sum_{k=1}^{n-1} b_k(x, y)\partial_{y^k}$ be a cusp vector field then its symbol is

$$(2.41) \quad \sigma(V) = ia(x, y)\lambda + i \sum_{k=1}^{n-1} b_k(x, y)p_k.$$

Let

$$(2.42) \quad D = \sum_{|k|+|l|=m, i} a_{kl}(x, y)(-ix^2\partial_x)^k(-i\partial_{y^i})^l$$

be a cusp differential operator of order m . Then its principal symbol is

$$(2.43) \quad \sigma(D) = \sum_{|k|+|l|=m, i} a_{kl}(x, y)(\lambda)^k(p^i)^l.$$

There exists an 'inverse map' to symbol map. This is the quantization map as in the case of manifold without boundary.

Proposition 6. For any compact manifold with boundary there is a global quantization map

$$(2.44) \quad S^{[m]}({}^cT^*M) \rightarrow \Psi_c^m(M),$$

which is order filtered and induces the (full symbol) isomorphisms

$$(2.45) \quad S^{[m]}({}^cT^*M)/S^{-\infty}({}^cT^*M) \rightarrow \Psi_c^m(M)/\Psi_c^{-\infty}(M),$$

where m is a real number.

Proof. See [MeNi, Proposition 24] . □

The notion of ellipticity is introduced exactly in the same way as in the case without boundary.

Definition 7. Any cusp pseudodifferential operator of order m acting from sections of a vector bundle E to sections of a vector bundle E is called elliptic, if its principal m -symbol is invertible outside the zero section.

Example 4. The identity operator is an elliptic cusp operator of order 0, since its principal symbol is 1. The Laplacian is elliptic of order 2, which is immediate from the formula of its principal symbol.

In contrast to the case of a closed manifold, elliptic cusp operators cannot be inverted modulo residual terms, in general. They can be inverted modulo smoothing cusp pseudodifferential operators. This is not enough, since smoothing operators are not compact (in L^2) in the cusp calculus. Particularly, ellipticity does not imply the Fredholm property. To capture the obstruction to compactness, we need to introduce yet another symbol. This new symbol is called the indicial operator and it is a pseudodifferential operator (actually a family of operators). The model for the operators where the indicial operator takes values is the 1-parameter suspended pseudodifferential operators. They are described in [Me2], [MeMa].

Definition 8. Let $A \in \Psi_c^m(M; E, F)$. Then the restriction of A to the boundary ∂M is the operator

$$(2.46) \quad A_\partial \phi = (A\psi)_{\partial M},$$

where $\phi \in C^\infty(\partial M, E_{\partial M})$ and $\psi \in C^\infty(M, E)$ such that $\psi|_{\partial M} = \phi$.

At the level of Schwartz kernels, this means the restriction of the kernel to the cusp-front face.

Definition 9. Let $A \in \Psi_c^m(M; E, F)$, then the *indicial family* of the operator A is the family of operators

$$(2.47) \quad I(A, \xi) : C^\infty(\partial M; E_{\partial M}) \rightarrow C^\infty(\partial M; F_{\partial M})$$

$$(2.48) \quad I(A, \xi) = (e^{i\frac{\xi}{x}} A e^{-i\frac{\xi}{x}})_{\partial M},$$

where ξ is a real number.

The indicial family of a cusp pseudodifferential operator A is also denoted by A_0 . The parameter ξ in the definition of the indicial family is called the *suspension parameter*.

Example 5. Let us consider a cusp differential operator of order m

$$(2.49) \quad D = \sum_{|k|+|l|=m, i} a_{kl}(x, y) (ix^2 \partial_x)^k (i\partial_{y^i})^l.$$

Its indicial family is

$$(2.50) \quad I(D, \lambda) = \sum_{|k|+|l|=m, i} a_{kl}(0, y)(\lambda)^k (i\partial_{y^i})^l.$$

Example 6. The indicial family of the Laplacian Δ_c in example 1 associated to the exact cusp metric is

$$(2.51) \quad I(\Delta_c, \xi) = -\xi^2 + \Delta_{\partial},$$

where Δ_{∂} denotes the Laplacian on the boundary (associated to the Riemannian metric).

Next, consider the indicial family of the identity map, but this time at the level of Schwartz kernels. The kernel of $e^{i\lambda\frac{1}{x}}\text{Ide}^{-i\lambda\frac{1}{x}}$ is

$$(2.52) \quad e^{i\xi(\frac{1}{x}-\frac{1}{x'})}\delta(x-x')\delta(y-y')dx'dy'\frac{dx}{x^2}dy.$$

At this point restriction to $x = 0$ is not possible. Lift to the cusp double space solves this problem. Coordinates $S = \frac{1}{x} - \frac{1}{x'}$ and x' can be used to lift the kernel to the cusp double space. These coordinates are valid near the diagonal, where the kernel of Id is supported. Lifting the above kernel with the above coordinates gives

$$(2.53) \quad e^{i\xi S}\delta(S)\delta(y-y')dSdy'\frac{dx}{x^2}dy.$$

At the level of kernels, the restriction to the boundary means restriction to the front face. The restriction is done by omitting $\frac{dx}{x^2}$ factor and putting $x = 0$.

We get the following expression for the kernel

$$(2.54) \quad e^{i\xi S}\delta(S)\delta(y-y')dSdy'dy.$$

Proposition 7. The indicial map is multiplicative. That is, for any cusp operators $A \in \Psi_c^*(M; F, G)$ and $B \in \Psi_c^*(M; E, F)$, the indicial map satisfies

$$(2.55) \quad I(AB, \xi) = I(A, \xi)I(B, \xi),$$

where ξ is a real number, and I fits into the following exact sequence

$$(2.56) \quad 0 \rightarrow x\Psi_c^m(M; E, F) \hookrightarrow \Psi_c^m(M; E, F) \rightarrow \Psi_{sus}^m(\partial M; E, F) \rightarrow 0.$$

The space $\Psi_{sus}^m(\partial M; E, F)$ in the sequence needs to be described. This is the suspended algebra mentioned above. Here it is associated to the boundary of the manifold, but the suspended algebra can be defined on any closed manifold X . For detailed treatise, see [Me2], [MeNi] and [LP]. We mostly follow [Me2].

For what follows, let X be a closed manifold. Denote by \mathcal{S} the Schwartz-space and let Ω be a smooth density bundle on X .

Definition 10. By the suspended pseudodifferential operators $\Psi_{sus}^*(X)$, we mean pseudodifferential operators $\Psi^*(X \times \mathbb{R})$ with the following constraints. They are translation invariant in \mathbb{R} and their kernels lie in $C_c^{-\infty}(X^2 \times \mathbb{R}^2; \Omega X) + \mathcal{S}(X^2 \times \mathbb{R}; \Omega X)$.

For $A \in \Psi_{sus}^*(X)$, translation invariance means that the kernels are of the form

$$(2.57) \quad A(x, x', t - t') \in C_c^{-\infty}(X^2 \times \mathbb{R}^2; \Omega X) + \mathcal{S}(Y^2 \times \mathbb{R}^2; \Omega X).$$

Thus, we may write by slight abuse of notation

$$(2.58) \quad A(x, x', t - t') \in C_c^{-\infty}(X^2 \times \mathbb{R}; \Omega X) + \mathcal{S}(Y^2 \times \mathbb{R}; \Omega X).$$

The residual operators $\Psi_{sus}^{-\infty}(X)$ correspond to the kernels in $\mathcal{S}(X^2 \times \mathbb{R}; \Omega X)$.

The action of A on a function $f \in C^\infty(X, \mathbb{R})$ can be written

$$(2.59) \quad Af(x, t) = \int_X \int_{\mathbb{R}} A(x, x', t - s) f(x', s) ds.$$

It follows from the general properties of pseudodifferential operators, that $\Psi_{sus}^*(X)$ is an order filtered algebra of operators

$$(2.60) \quad A : \mathcal{S}(X \times \mathbb{R}) \rightarrow \mathcal{S}(X \times \mathbb{R}).$$

The suspended operators can be defined to act between sections of vector bundles. To this end, let E and F be complex vector bundles over X . To define the vector bundles over $X \times \mathbb{R}$, simply pull back with the canonical projection $X \times \mathbb{R} \rightarrow X$. Now put

$$(2.61) \quad \Psi_{sus}^m(X; E, F) = \Psi_{sus}^m(X) \otimes_{C^\infty(X)} C^\infty(X^2, \text{Hom}(E, F)).$$

Proposition 8. The suspended algebra of operators $\Psi_{sus}^*(X)$ is a naturally complete topological vector space and order filtered and $*$ closed.

Proof. See [Me2, Proposition 1] . □

There is the following relation to the cusp operators, which relates the Taylor expansion at the front face of cusp operators to the suspended operators.

Proposition 9. The choice of a normal fibration near the boundary fixes isomorphism

$$(2.62) \quad \Psi_{sus}^m(\partial M; E, F)[[x]] = x^{-\mathbb{Z}} \Psi_c^m(M; E, F) / x^\infty \Psi_c^m(M; E, F),$$

where the linear variable on the suspension of ∂M is identified with $\frac{1}{x}$ in the product.

Proof. See [MeNi, Proposition 25]. □

For example, let A belong to $\Psi_c^m(M; E, F)$. Its expansion at the front face is denoted by

$$(2.63) \quad A \approx \sum_{k=0}^{\infty} x^k A_{-k},$$

where $A_{-k} \in \Psi_{sus}^m(M; E, F)$ and the sum determines A modulo $x^\infty \Psi_c^m(M; E, F)$. Note that A_0 is the indicial family. The choice of the negative sign is the convention introduced in [MeNi].

Definition 11. Any elliptic cusp pseudodifferential operator with an invertible indicial family is called fully elliptic.

Example 7. Consider the Laplacian Δ_c associated to the cusp metric. Then $\Delta_c + 1$ is fully elliptic. Note that, in general, Δ_c is not fully elliptic. The full ellipticity requires that the associated Laplace operator on the boundary Δ_∂ has no zero eigenvalues.

A fully elliptic cusp pseudodifferential operator can be inverted modulo residual terms. They also define Fredholm operators on appropriate Sobolev spaces.

Proposition 10. Let $A \in \Psi_c^m(M; E, F)$ be a fully elliptic cusp operator of order m . Then there exist a cusp pseudodifferential operator $B \in \Psi_c^{-m}(M; F, E)$ of order $-m$, called a *parametrix*, that inverts A up to a residual term. That is

$$(2.64) \quad AB = 1 + R, BA = 1 + R',$$

where $R' \in x^\infty \Psi_c^{-\infty}(M; E, E)$ and $R \in x^\infty \Psi_c^{-\infty}(M; F, F)$.

Proof. See [MeMa, Proposition 8] . □

It is instructive to see how this can be done. So, suppose $A \in \Psi_c^m(M)$ is fully elliptic. Thus A has a small right parametrix $B \in \Psi_c^{-m}(M)$, that is

$$(2.65) \quad AB = 1 - E, E \in \Psi_c^{-\infty}(M).$$

Since the indicial family of A is invertible, the properties of the indicial family can be used to find a cusp pseudodifferential operator Q , whose indicial family is given by

$$(2.66) \quad I(Q) = I(A)^{-1}I(E).$$

It follows that Q is smoothing, and the choice $P = B + Q$ gives

$$(2.67) \quad AP = AB + AQ = 1 - E + AQ = 1 - R.$$

Now by construction

$$(2.68) \quad I(R) = I(E) - I(AQ) = I(E) - I(A)I(Q) = I(E) - I(A)I(A)^{-1}I(E) = 0.$$

Thus $R \in x \Psi_c^{-\infty}(M)$.

Put $P_k = P(1 + R + R^2 + \cdots + R^{k-1})$ and compute

$$(2.69) \quad AP_k = AP(1 + R + R^2 + \cdots + R^{k-1}) = (1 - R)(1 + R + R^2 + \cdots + R^{k-1}) = 1 - R^k.$$

Now R^k belongs to $x^k \Psi_c^{-\infty}(M)$. Continuing this way, we find a parametrix G , which has the property that $AG = 1 + x^\infty \Psi_c^{-\infty}(M)$.

The residual terms can be arranged as orthogonal projections to the kernel and cokernel of A . Then B is said to be a generalized inverse. It can be proven, that the cusp calculus contains generalized inverses, particularly it is spectrally closed. This result is due to Melrose [LaMonNi].

Next we define the cusp Sobolev spaces, and state the mapping properties with respect to the cusp Sobolev spaces beginning from L^2 -continuity of the cusp operators of order zero. For proofs see [MeMa].

Proposition 11. Any cusp pseudodifferential operator $A \in \Psi_c^0(M; E)$ defines a continuous linear map

$$(2.70) \quad A : L^2(M; E) \rightarrow L^2(M; E),$$

defined with respect to any positive density on M .

Proof. See [MeMa, Theorem 3]. □

Definition 12. Let m be a real number and let n be a positive real number. Then the cusp Sobolev spaces are

$$(2.71) \quad \begin{aligned} x^m H_c^n(M, E) &= \{u \in x^m L^2(M; E); Pu \in L^2(M, E), \text{ for all } P \in \Psi_c^n(M, E)\} \\ x^m H_c^{-n}(M, E) &= \{u \in C^{-\infty}(M; E); u = \sum_{i=1}^N P_i u_i, \\ &\quad u_i \in x^m L^2(M, E), P_i \in \Psi_c^n(M, E)\}. \end{aligned}$$

Proposition 12. Let l, l', m, m' be positive real numbers such that $l' \leq l$ and $m' \leq m$. Then and only then

$$(2.72) \quad x^l H_c^m(M, E) \subset x^{l'} H_c^{m'}(M, E),$$

with the inclusion then continuous. The inclusion is compact if and only if $l' < l$ and $m' < m$.

Now the mapping property of the cusp pseudodifferential operators between these Sobolev spaces can be given.

Proposition 13. Let m be a real number and let $A \in \Psi_c^m(M; E, F)$. Then A defines a continuous mapping between weighted cusp Sobolev spaces

$$(2.73) \quad A : x^l H_c^{m'}(M; E, F) \rightarrow x^l H_c^{m'-m}(M; E, F),$$

for any real numbers l and m' .

Proposition 14. Let $A \in \Psi_c^m(M; E, F)$ be fully elliptic. Then it defines a Fredholm operator with respect to the Sobolev spaces

$$(2.74) \quad A : x^l H_c^{m'}(M; E) \rightarrow x^l H_c^{m'-m}(M; F),$$

for all real numbers m', l .

Proposition 15 (Elliptic regularity). Let $A \in \Psi_c^m(M; E, F)$ be fully elliptic. Then the sections of its kernel lie in $\dot{C}^\infty(M; E)$ and the sections of its cokernel lie in $\dot{C}^\infty(M; F)$.

2.2. The cusp Dirac operator. We assume that the manifold M is spin with a fixed spin structure (see [Me3] and [LM] for more details on spinors). We denote the spin-bundle by S and let E be a Hermitean vector bundle over M . Also, we denote by S_0 and E_0 the induced bundles over the boundary of the spin-bundle S and the vector bundle E , respectively. Furthermore, we assume that all geometric structures are of 'product type' near a fixed collar neighbourhood of the boundary, say $[0, 1)_x \times \partial M$. Particularly, any connection on E satisfies $\nabla_{x^2 \partial_x}^E = 0$ at ∂M .

2.3. The even dimensional case. Assume that M is even dimensional. Then the corresponding spinor bundle is graded $S = S^+ \oplus S^-$. Let

$$(2.75) \quad \check{\mathfrak{D}}_E : C^\infty(M; S \otimes E) \rightarrow C^\infty(M; S \otimes E)$$

be the (twisted) total cusp Dirac operator. It is given by the Levi-Civita connection associated to the exact-cusp metric and a chosen Hermitean connection ∇ on E .

Remark 2. It follows, as in [Me3], that the curvature of the Levi-Civita connection associated to the exact cusp-metric is $C^\infty(M)$ -form on M . The smoothness of the curvature of ∇^E is clear from the assumptions.

The Dirac operator can be written, using a local orthonormal frame on ${}^c TM$ given by e_i and its dual coframe ϕ^i , where $i = 1, \dots, \dim M$ by

$$(2.76) \quad \check{\mathfrak{D}}_E = -i \sum_{i=0}^n \text{cl}(\phi^i) \nabla_{e_i} = -i \sum_{i=0}^n \gamma^i \nabla_{e_i},$$

where $\text{cl}(\phi^i) = \gamma^i$ denotes the Clifford multiplication given by γ -matrices and

$$(2.77) \quad \nabla = \nabla^{LC} \otimes 1 + 1 \otimes \nabla^E,$$

is a connection on $S \otimes E$ given by the Levi-Civita connection ∇^{LC} associated to the exact cusp metric and a Hermitean connection ∇^E on E . Note that the gamma-matrices only act on the spinor part S . Our convention for the Clifford algebra is

$$(2.78) \quad \gamma^a \gamma^b + \gamma^b \gamma^a = 2\delta^{ab},$$

where the gamma matrices are as above. They satisfy $\gamma^2 = 1$ and $\gamma^* = \gamma$.

We denote the corresponding chiral-Dirac operators by

$$(2.79) \quad \check{\mathfrak{D}}_E^\pm : C^\infty(M; S^\pm \otimes E) \rightarrow C^\infty(M; S^\mp \otimes E),$$

as usual.

Remark 3. If we do not fix the Hermitean connection on E , then we obtain a family of Dirac operators on the space of Hermitean connections on E . Later, we adopt this viewpoint.

Recall, from [Me3], that the spinor bundle S splits as two copies of S_0 on the collar neighbourhood of the boundary. This fact is used to write the Dirac operator in terms of the boundary Dirac operator

$$(2.80) \quad \check{\mathfrak{D}}_0 : C^\infty(\partial M; S_0 \otimes E_0) \rightarrow C^\infty(\partial M; S_0 \otimes E_0).$$

This is done via identifications $M_{\pm} : S_{\partial M}^{\pm} \leftrightarrow S_0$ given in [Me3], [MePi]. Here $S_{\partial M}$ denotes the spinor bundle S restricted to the boundary. We denote the total identification by $M : S_{\partial M} \rightarrow S_0 \oplus S_0$. Using this identification, the Dirac operator on the collar neighborhood can be given as (here, we take this as a definition)

$$(2.81) \quad \tilde{\mathfrak{D}}_E = -ix^2\gamma\partial_x + \sigma\tilde{\mathfrak{D}}_0,$$

where σ and γ are 2×2 matrices

$$(2.82) \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Hence, the indicial family of $\tilde{\mathfrak{D}}_E$ is

$$(2.83) \quad I(\tilde{\mathfrak{D}}_E, \xi) = \sigma\tilde{\mathfrak{D}}_0 + \gamma\xi,$$

where ξ is a real number. Particularly, we have for the corresponding chiral Dirac operators

$$(2.84) \quad I(\tilde{\mathfrak{D}}_E^{\pm}, \xi) = \tilde{\mathfrak{D}}_0 \pm i\xi.$$

Using the above representation for the Dirac operator it is not difficult to prove that the corresponding indicial family is invertible (as an operator on L^2) when the boundary Dirac operator $\tilde{\mathfrak{D}}_0$ is invertible. In this case $\tilde{\mathfrak{D}}_E : H_c^m(M; S \otimes E) \rightarrow H_c^{m-1}(M; S \otimes E)$ defines a Fredholm operator, for any real number m .

2.4. The odd dimensional case. Now, we assume M is odd dimensional. Then the boundary spinor bundle $S_{\partial M}$ is graded. As in [MePi2] we define the Clifford action of $T^*\partial M$ on $S_{\partial M}$ by

$$(2.85) \quad \text{cl}_{\partial}(\eta) = i\text{cl}\left(\frac{dx}{x^2}\right)\text{cl}(\eta),$$

where $\eta \in T^*\partial M$.

Put $\sigma = \text{cl}\left(\frac{dx}{x^2}\right)$, then $\sigma^2 = 1$ and $\sigma^* = \sigma$. Therefore σ defines grading on $S_{\partial M}$ to be denoted by

$$(2.86) \quad S_{\partial M} = S_0^+ \oplus S_0^-,$$

where S_0^{\pm} are the ± 1 -eigenspaces of σ .

Now on the collar neighborhood the Dirac operator can be written as

$$(2.87) \quad \tilde{\mathfrak{D}}_E = \frac{1}{i}\sigma x^2\partial_x + \frac{1}{i}\sigma\tilde{\mathfrak{D}}_0.$$

Hence, the corresponding indicial family is

$$(2.88) \quad I(\tilde{\mathfrak{D}}_E, \xi) = \sigma\left(\xi + \frac{1}{i}\tilde{\mathfrak{D}}_0\right).$$

Note that by definition σ anticommutes with $\tilde{\mathfrak{D}}_0$. Therefore

$$(2.89) \quad \tilde{\mathfrak{D}}_0 = \begin{pmatrix} 0 & \tilde{\mathfrak{D}}_0^- \\ \tilde{\mathfrak{D}}_0^+ & 0 \end{pmatrix},$$

where $\tilde{\mathfrak{D}}_0^{\pm} : C^{\infty}(\partial M; S_0^{\pm} \otimes E_0) \rightarrow C^{\infty}(\partial M; S_0^{\mp} \otimes E_0)$.

Again, in order to the above indicial family to be invertible, the boundary Dirac operator $\bar{\partial}_0$ has to be invertible. In this case $\bar{\partial}_E$ defines a Fredholm operator as before.

Example 8 (A grading operator associated to an invertible Dirac operator). Now assume the Dirac operator $\bar{\partial}_E$ is invertible, then we can define the signature operator

$$(2.90) \quad F = \frac{\bar{\partial}_E}{|\bar{\partial}_E|} \in \Psi_c^0(M; S \otimes E),$$

where

$$(2.91) \quad |\bar{\partial}_E| = \sqrt{\bar{\partial}_E^2} \in \Psi_c^1(M; S \otimes E).$$

First, assume M is even dimensional. Then the indicial family of F is

$$(2.92) \quad I(F, \xi) = \frac{\gamma\xi + \sigma\bar{\partial}_0}{\sqrt{\xi^2 + \bar{\partial}_0^2}}.$$

If M is odd dimensional, then

$$(2.93) \quad I(F, \xi) = \sigma \frac{\xi - i\bar{\partial}_0}{\sqrt{\xi^2 + \bar{\partial}_0^2}}.$$

3. TRACE FUNCTIONALS

We review how to regularize the trace in the cusp calculus. Here we follow mostly [MeNi], [MoNi] but see also [Mo], [Mo2] and [MoLa]. We do not discuss the technicalities of defining complex powers or holomorphic families with values in pseudodifferential operators. For the reader who is interested in these technicalities, consult [MeNi] and [ALNV]. For the complex powers in the b-calculus approach, see [Lo] and [Pi2]. The standard reference to complex powers is [Se], but see also [Bu].

Let us recall how the L^2 -trace of a pseudodifferential operator can be defined on a closed n -manifold M . We let A be a scalar valued pseudodifferential operator of order $-\infty$. Then the trace of A can be expressed as the integral over the diagonal of its kernel by Lidskii's theorem

$$(3.1) \quad \mathbf{Tr}A = \int_{\Delta} A = \int_M A,$$

where we have identified the diagonal Δ with M . The trace can be expressed in terms of the symbol of A denoted by a

$$(3.2) \quad \mathbf{Tr}A = \int_M dy \int_{\mathbb{R}^n} dp a(y, p) = \int_{T^*M} a\omega^n,$$

where ω is the canonical symplectic form on T^*M . The above formula holds for operators of order $m < -\dim M$. We want to extend the notion of the trace to arbitrary orders of pseudodifferential operators. This is done using the zeta-function regularization.

Let Q be a pseudodifferential operator of order 1, which is self-adjoint, strictly positive and elliptic. The operator Q is referred as a weight. Regularization of the trace is defined via complex powers of Q [Se], [Sc], [CDP]. For $\tau \in \mathbb{C}$, we consider the expression

$$(3.3) \quad Z(A; \tau) = \mathbf{Tr} A Q^{-\tau},$$

where A is allowed to be a holomorphic family of operators $A : \mathbb{C} \rightarrow \Psi^m(M)$. It is well known that Z is well defined for the real part of τ large enough. Furthermore, Z extends to a meromorphic function to the whole complex plane, with at most simple poles at $\tau \in -\dim M - \mathbb{N}_0$. Near the origin τZ can be expanded as a Taylor series

$$(3.4) \quad \tau Z(A; \tau) = \mathrm{Tr}_R A + \tau \overline{\mathbf{Tr}} A + W \tau^2,$$

where W is holomorphic near the origin. Here, the first term is the residue trace of Wodzicki [W]. The second term is the regularized trace.

The following observation is useful;

$$(3.5) \quad (\mathbf{Tr} \tau A Q^{-\tau})_{\tau=0} = \mathrm{Tr}_R A.$$

Thus

$$(3.6) \quad \begin{aligned} \overline{\mathbf{Tr}} A &= FP_{\tau=0} \mathbf{Tr} A Q^{-\tau} \\ &= (\mathbf{Tr} A Q^{-\tau} - \frac{1}{\tau} \mathrm{res}_{\tau=0} \mathbf{Tr} A Q^{-\tau})_{\tau=0} \\ &= (\mathbf{Tr} A Q^{-\tau} - \frac{1}{\tau} \mathrm{Tr}_R A)_{\tau=0}, \end{aligned}$$

where FP stands for taking the finite part.

The regularized trace $\overline{\mathbf{Tr}}$ is not a trace. To see this, let $A \in \Psi^*(M)$ and $B \in \Psi^*(M)$ be pseudodifferential operators. Then

$$(3.7) \quad \begin{aligned} Z([A, B]; \tau) &= \mathbf{Tr}[A, B] Q^{-\tau} = \mathbf{Tr}[A Q^{-\tau}, B] - \mathbf{Tr} A [Q^{-\tau}, B] \\ &= \mathbf{Tr}[A Q^{-\tau}, B] - \mathbf{Tr} \tau A \left[\frac{Q^{-\tau} - I}{\tau}, B \right]. \end{aligned}$$

This yields

$$(3.8) \quad \mathbf{Tr}[A, B] = \mathrm{Tr}_R A [\log Q, B],$$

where we define $[\log Q, B]$ as in [MeNi] by

$$(3.9) \quad [\log Q, B] = \left(\frac{I - Q^{-\tau} B Q^{\tau}}{\tau} \right)_{\tau=0}.$$

Particularly, the trace anomaly is local, since the Wodzicki residue is. Here the locality means that the expression above depends only a finite number of terms from the asymptotic expansion of the classical pseudodifferential operator $A[\log Q, B]$.

Consider the case of a manifold with boundary, M , with the same assumptions as in the previous section. There is an immediate problem. Namely, the integral (3.2) does not exist, in general. In fact, it does not exist even in the case of the cusp

smoothing operators. Here, the problem arises from the singular density carried by the cusp pseudodifferential operators. To correct this problem; a new regularization has to be introduced. This can be done with the zeta function regularization with respect to the boundary defining function x [MeNi], [MoNi]. For a smoothing cusp pseudodifferential operator A , we use expression

$$(3.10) \quad Z(A; z) = \mathbf{Tr} Ax^z,$$

where z is a complex number. Remember, that A carries a density proportional to x^{-2} . Thus Z defines a holomorphic function on the complex plane when $\operatorname{Re} z > 1$. Furthermore, Z extends as a meromorphic function to the whole complex plane, with at most simple poles at $1 - \mathbb{N}_0$ [MeNi]. We may expand as before

$$(3.11) \quad zZ(A; z) = \overline{\mathbf{Tr}}_{\partial} A + z\overline{\mathbf{Tr}} A + z^2 W',$$

where W' is holomorphic near the origin. The boundary regularized trace is

$$(3.12) \quad \begin{aligned} \overline{\mathbf{Tr}} A &= FP_{z=0} \mathbf{Tr} Ax^z = (\mathbf{Tr} Ax^z - \frac{1}{z} \operatorname{res}_{z=0} \mathbf{Tr} Ax^z)_{z=0} \\ &= (\mathbf{Tr} Ax^z - \frac{1}{z} \overline{\mathbf{Tr}}_{\partial} A)_{z=0}, \end{aligned}$$

where the identification of the residue is seen with similar argument as above.

This definition can be used for the operators of order less than $-\dim M$. The above trace functional does not have the trace property. The corresponding anomaly formula is derived as follows. Let A and B be cusp smoothing operators and let $\operatorname{Re} z \gg 1$. Then

$$(3.13) \quad \begin{aligned} \mathbf{Tr}[A, B]x^z &= \mathbf{Tr}(ABx^z - BAx^z) \\ &= \mathbf{Tr}(A(B - x^z Bx^{-z})x^z + Ax^z B - BAx^z) \\ &= \mathbf{Tr}(AB(z)x^z) + \mathbf{Tr}[Ax^z, B]. \end{aligned}$$

The first term is

$$(3.14) \quad \mathbf{Tr}(AB(z)x^z) = \mathbf{Tr}(Az[\log x, B]x^z) + Wz^2,$$

where W is holomorphic near the origin and (see [MeNi], [MoNi])

$$(3.15) \quad [\log x, B] = \left(\frac{I - x^z Bx^{-z}}{z} \right)_{z=0} \in x\Psi_c^{-\infty}(M).$$

Using unique continuation, it follows that only the first term above contributes to the anomaly. Thus

$$(3.16) \quad \overline{\mathbf{Tr}}[A, B] = \overline{\mathbf{Tr}}_z A[\log x, B] = \overline{\mathbf{Tr}}_{\partial} A[\log x, B].$$

We see later that the expression $\overline{\mathbf{Tr}}_{\partial} A[\log x, B]$ depends on the indicial families of A and B .

A generalization to the case of non-smoothing cusp pseudodifferential operators is needed. This is done as in [MeNi] and [MoNi]. The idea is to combine these two zeta-regularizations by introducing the double zeta function regularization of a holomorphic family of operators $A(\tau, z) : \mathbb{C}^2 \rightarrow \Psi_c^m(M)$

$$(3.17) \quad Z(A; \tau, z) = \mathbf{Tr} A x^z Q^{-\tau}.$$

By [MoNi, Lemma 1] (see also [MeNi, Lemma 4]) $z\tau Z(A; \tau, z)$ is holomorphic near the zero in \mathbb{C}^2 . This is used to define the following four trace-type functionals by using the Taylor series of $z\tau Z(A; \tau, z)$;

$$(3.18) \quad z\tau Z(A; \tau, z) = \mathbf{Tr}_{\partial, \sigma} A + z\overline{\mathbf{Tr}}_{\sigma} A + \tau\overline{\mathbf{Tr}}_{\partial} A + z\tau\overline{\mathbf{Tr}} A + z^2 W + \tau^2 W'.$$

Here W and W' are holomorphic in a neighborhood of the origin. A single operator $A \in \Psi_c^m(M)$ is treated as a constant family. Here $\overline{\mathbf{Tr}} A$ is called the *regularized trace* of A .

Lemma 1. The following identities hold for $A \in \Psi_c^m(M)$

$$(3.19) \quad \begin{aligned} \overline{\mathbf{Tr}} \tau A &= \overline{\mathbf{Tr}}_{\sigma} A \\ \overline{\mathbf{Tr}} z A &= \overline{\mathbf{Tr}}_{\partial} A \\ \overline{\mathbf{Tr}} z \tau A &= \overline{\mathbf{Tr}}_{\partial} \tau A = \overline{\mathbf{Tr}}_{\sigma} z A = \mathbf{Tr}_{\partial, \sigma} A, \end{aligned}$$

where z and τ are complex numbers.

Proof. See [MoNi, Lemma 2] . □

For example, it follows from (3.18) that

$$(3.20) \quad \overline{\mathbf{Tr}} z \tau A = Z(z\tau A; z, 0) = \mathbf{Tr}_{\partial, \sigma} A.$$

Furthermore, it can be seen that

$$(3.21) \quad \overline{\mathbf{Tr}}_{\sigma} z A = Z(zA; 0, 0) = (\partial_{\tau} z\tau Z(A))_{\tau=0, z=0} = \mathbf{Tr}_{\partial, \sigma} A,$$

and

$$(3.22) \quad \overline{\mathbf{Tr}}_{\partial} \tau A = Z(zA; 0, 0) = (\partial_z z\tau Z(A))_{\tau=0, z=0} = \mathbf{Tr}_{\partial, \sigma} A.$$

First we discuss the functional $\overline{\mathbf{Tr}}_{\sigma}$. Observe that

$$(3.23) \quad Z(A; \tau, z) = \frac{1}{z\tau} \mathbf{Tr}_{\partial, \sigma} A + \frac{1}{\tau} \overline{\mathbf{Tr}}_{\sigma} A + \frac{1}{z} \overline{\mathbf{Tr}}_{\partial} A + \overline{\mathbf{Tr}} A + \frac{z}{\tau} W + \frac{\tau}{z} W'.$$

This yields

$$(3.24) \quad \text{res}_{\tau=0} Z(A; \tau, z) = \overline{\mathbf{Tr}}_{\sigma} A + \frac{1}{z} \mathbf{Tr}_{\partial, \sigma} A + zW.$$

Taking the finite part at $z = 0$ yields $\overline{\mathbf{Tr}}_{\sigma} A$. That is

$$(3.25) \quad \text{FP}_{z=0} \text{res}_{\tau=0} \mathbf{Tr} A x^z Q^{-\tau} = \overline{\mathbf{Tr}}_{\sigma} A.$$

The above observation suggests connection with the Wodzicki residue.

Proposition 16. The functional $\overline{\mathbf{Tr}}_\sigma$ is given by a regularized integral of the Wodzicki residue density

$$(3.26) \quad \begin{aligned} \mathbf{wres}A &= \frac{1}{(2\pi)^n} \int_{{}^c S^*M/M} a_{-n} \omega_c^n \\ \overline{\mathbf{Tr}}_\sigma A &= {}^c \int_M \mathbf{wres}A, \end{aligned}$$

where ω_c^n is the symplectic-measure on the unit cusp cotangent bundle ${}^c S^*M$ and ${}^c \int_M$ denotes the Hadamard-regularization (see [MeNi] and [MoNi] for exact formula). Furthermore, the functional $\overline{\mathbf{Tr}}_\sigma$ does not depend on the choice of a weight Q used in the regularization.

Proof. See [MeNi]. □

Later $\mathbf{Wres}A$ is used to denote $\overline{\mathbf{Tr}}_\sigma$. In general, the functional $\overline{\mathbf{Tr}}_\sigma$ is not a trace.

Recall the expansion from (2.63). The value of the $\mathbf{Tr}_{\sigma,\partial}A$ depends only on the operator $A_{-1} \in \Psi_{sus}(\partial M)$. More precisely, only of its symbol of homogeneity $-n$ $a_{-n,-1}|\omega_\partial^{n-1} dt d\xi|$. Here ω_∂ is the symplectic form on $T^*\partial M$ and ξ is the variable dual to t .

Proposition 17. The functional $\mathbf{Tr}_{\sigma,\partial}$ is given by

$$(3.27) \quad \mathbf{Tr}_{\sigma,\partial}A = \frac{1}{(2\pi)^n} \int_{{}^c S^*_{\partial M}M} a_{-n,-1} \nu_c^n,$$

where ν_c^n is a measure obtained by contracting the form $\omega_\partial^{n-1} d\xi$ with the radial vector field on ${}^c T\partial M^* = T^*\partial M \times \mathbb{R}_\xi$.

Proof. See [MeNi, Lemma 8]. □

In order to describe the trace $\overline{\mathbf{Tr}}_\partial$, we must recall some traces defined on the suspended calculus. Background material can be found in [Me2], [MeNi], [MoNi] and [Mo2] but see also [LP]. We follow mostly [MeNi].

Recall, that the suspended operators on ∂M are pseudodifferential operators defined on $\partial M \times \mathbb{R}$, which are translation invariant on the suspension parameter.

Let A be any smoothing suspended pseudodifferential operator acting on smooth compactly supported functions on $\partial M \times \mathbb{R}$. The integral over the diagonal of the kernel of A is of the form

$$(3.28) \quad \int_{\mathbb{R}} \int_{\partial M} \mathbf{tr}A(y, y, t),$$

which usually does not exist.

Thus, a regularization is needed to define this integral.

Proposition 18. Let $A \in \Psi_{sus}^{-\infty}(\partial M; E)$. Then the integral over the diagonal of the Schwartz-kernel of A with the suspension parameter value 0, defines a trace on

$\Psi_{sus}^{-\infty}(\partial M; E)$

$$(3.29) \quad \overline{\mathbf{Tr}}A = \int_{\partial M} \mathbf{tr}A(y, y, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{Tr}\hat{A}(\xi)d\xi,$$

where \hat{A} denotes the indicial family of A

$$(3.30) \quad \hat{A}(\xi) = \int e^{-it\xi} A(y, y', t)dt,$$

and \mathbf{Tr} is the ordinary L^2 trace on ∂M .

Proof. The trace property follows directly from that of \mathbf{Tr} . □

If there is a danger of confusion, the above trace is denoted by $\overline{\mathbf{Tr}}_{sus}$.

The functional $\overline{\mathbf{Tr}}_{\partial}$ can be now described. Let $A \in x^k\Psi_c^m(M)$ be supported on the collar $[0, 1)_x \times \partial M$ (we see later, that this restriction is not essential). Then by definition

$$(3.31) \quad \begin{aligned} \overline{\mathbf{Tr}}_{\partial}A &= \text{FP}_{\tau=0} \text{res}_{z=0} \mathbf{Tr}Ax^zQ^{-\tau} \\ &= \text{FP}_{\tau=0} \text{res}_{z=0} \int_{\Delta_c} \mathbf{tr}\kappa(Ax^zQ^{-\tau}) \\ &= \text{FP}_{\tau=0} \text{res}_{z=0} \int_{\Delta_c} \mathbf{tr}\kappa(Ax^zQ^{-\tau}) \\ &= \text{FP}_{\tau=0} \text{res}_{z=0} \int_0^1 dx \int_{\partial M} \mathbf{tr}\kappa(Ax^zQ^{-\tau}), \end{aligned}$$

where $\kappa(B)$ denotes the kernel of any operator $B \in \Psi_c(M)$. Think x as a defining function to the cusp front face and use it to Taylor expand the kernel in powers of x . The result is the asymptotic series

$$(3.32) \quad \kappa(Ax^{-z}Q^{\tau}) = x^{k+z}\kappa(AQ^{-\tau})_{[-k]} + x^{-k-1+z}\kappa(AQ^{-\tau})_{[-k-1]} \cdots,$$

where the kernel $\kappa(AQ^{\tau})_{[k]}$ is restricted to $x = 0$. Therefore, it lies on the cusp front face.

The residue is non-vanishing only if the following condition is satisfied;

$$(3.33) \quad \text{res}_{z=0} \int_0^1 dx x^{z+l-2} = \text{res}_{z=0} \frac{1}{z+l-1} = 1, \text{ if } l=1$$

where the factor 2 comes from the cusp density. It follows that

$$(3.34) \quad \overline{\mathbf{Tr}}_{\partial}A = \text{FP}_{\tau=0} \int_{\partial M} \mathbf{tr}\kappa(AQ^{-\tau})_{[-1]}.$$

Thus the trace $\overline{\mathbf{Tr}}_{\partial}A$ depends only on the term $(AQ^{-\tau})_{[-1]}$. This corresponds the kernel that is the coefficient of x of the Taylor expansion above (3.32). If there is no such a term, then the trace is identically zero. When $A' \in x\Psi_c^m(M)$, then the kernel of the indicial operator of $x^{-1}A'Q^{-\tau}$ is precisely $\kappa(A'Q^{-\tau})_{[-1]}$ term in the expansion. Put for convenience $A = x^{-1}A'$. We can write using the multiplicativity of the indicial operator and the definition of the trace in the suspended algebra that

$$\begin{aligned}
(3.35) \quad \overline{\mathbf{Tr}}_{\partial} A' &= \frac{1}{2\pi} \text{FP}_{\tau=0} \int_{-\infty}^{\infty} d\xi \mathbf{Tr} I(AQ^{-\tau}) \\
&= \frac{1}{2\pi} \text{FP}_{\tau=0} \int_{-\infty}^{\infty} d\xi \mathbf{Tr} I(A) I(Q^{-\tau}).
\end{aligned}$$

Observe that the last expression is the regular value of the zeta-function regularization of $\overline{\mathbf{Tr}}A$ in the suspended algebra. We thus have (by abuse of notation)

$$(3.36) \quad \overline{\mathbf{Tr}}_{\partial} A' = \text{FP}_{\tau=0} \overline{\mathbf{Tr}}_{sus}(AQ^{-\tau}).$$

3.1. The trace anomaly formula. The trace anomalies associated with the trace functionals $\overline{\mathbf{Tr}}$, $\overline{\mathbf{Tr}}_{\partial}$, $\overline{\mathbf{Tr}}_{\sigma}$ and $\mathbf{Tr}_{\partial,\sigma}$ are now discussed. It turns out that only the functional $\mathbf{Tr}_{\sigma,\partial}$ is a trace.

The trace anomaly formula for the regularized trace $\overline{\mathbf{Tr}}$ defined on $\Psi_c^*(M)$ is treated first. To this end, we let A and B be cusp pseudodifferential operators in $\Psi_c^*(M)$, acting on smooth functions, and let $n = \dim M$.

The following computation is standard (see [MeNi] and [MoNi])

$$\begin{aligned}
(3.37) \quad \mathbf{Tr}[A, B]x^z Q^{-\tau} &= \mathbf{Tr} A B x^z Q^{-\tau} - \mathbf{Tr} B A x^z Q^{-\tau} \\
&= \mathbf{Tr} A (B - x^z B x^{-z}) x^z Q^{-\tau} + \mathbf{Tr} A x^z B Q^{-\tau} \\
&\quad + \mathbf{Tr} (Q^{\tau} B Q^{-\tau} - B) A x^z Q^{-\tau} - \mathbf{Tr} Q^{\tau} B Q^{-\tau} A x^z Q^{-\tau} \\
&= \mathbf{Tr} A B(z) + \mathbf{Tr} B(\tau) A - \mathbf{Tr} [Q^{\tau} B Q^{-\tau}, A x^z Q^{-\tau}].
\end{aligned}$$

Here the following notation has been used

$$\begin{aligned}
(3.38) \quad B(z) &= (B - x^z B x^{-z}) \\
B(\tau) &= (Q^{\tau} B Q^{-\tau} - B).
\end{aligned}$$

Note also that $\mathbf{Tr}[Q^{\tau} B Q^{-\tau}, A x^z Q^{-\tau}]$ is vanishing, when the real parts of τ and z are large enough. Thus, by analytic continuation its regularized value is 0.

Take Taylor expansions in the corresponding variables to get

$$\begin{aligned}
(3.39) \quad \mathbf{Tr} A B(z) &= -\mathbf{Tr} A [\log x, B] z x^z Q^{-\tau} + z^2 R(z) \\
\mathbf{Tr} B(\tau) A &= \mathbf{Tr} [\log Q, B] A \tau x^z Q^{-\tau} + \tau^2 R'(\tau),
\end{aligned}$$

where R and R' are holomorphic near the origin. Thus, for $\text{Re } z$ and $\text{Re } \tau$ large enough

$$\begin{aligned}
(3.40) \quad \mathbf{Tr}[A, B]x^z Q^{-\tau} &= -\mathbf{Tr} A [\log x, B] z x^z Q^{-\tau} + \mathbf{Tr} [\log Q, B] A \tau x^z Q^{-\tau} \\
&\quad - \tau^2 R'(\tau) + z^2 R(z).
\end{aligned}$$

We obtain after regularization

$$\begin{aligned}
(3.41) \quad \overline{\mathbf{Tr}}[A, B] &= -\overline{\mathbf{Tr}} A [\log x, B] z + \overline{\mathbf{Tr}} [\log Q, B] A \tau \\
&= -\overline{\mathbf{Tr}}_{\partial} A [\log x, B] + \overline{\mathbf{Tr}}_{\sigma} [\log Q, B] A,
\end{aligned}$$

where Lemma 1 has been used.

We need the fact that the boundary term $\overline{\mathbf{Tr}}_{\partial} A[\log x, B]$ depends only on the indicial families of the operators A and B . This follows from (3.36) and the lemma below.

Lemma 2. Let $A \in \Psi_c^m(M; E)$, then the cusp pseudodifferential operator $[\log x, A]$ is in $x\Psi_c^m(M; E)$. Furthermore, we have

$$(3.42) \quad I(x^{-1}[\log x, A])(\xi) = i \frac{d}{d\xi} I(A)(\xi).$$

Proof. See [MoLa], for example. \square

Particularly, if the above indicial family of A is independent of the suspension variable ξ , then the indicial family of $x^{-1}[\log x, A]$ vanishes. Therefore, the regularized trace $\overline{\mathbf{Tr}}_{\partial} A[\log x, B]$ is zero if the indicial family of B does not depend on the suspension parameter.

Proposition 19 (The trace anomaly formula). Given two cusp operators A and B in $\Psi_c^*(M; E)$, then the trace anomaly takes the following form

$$(3.43) \quad \overline{\mathbf{Tr}}[A, B] = \overline{\mathbf{Tr}}_{\sigma}[\log Q, B]A - \overline{\mathbf{Tr}}_{\partial} A[\log x, B].$$

Particularly, if the indicial family of A or B does not depend on the suspension variable, then $\overline{\mathbf{Tr}}_{\partial} A[\log x, B] = 0$.

Proof. \square

Remark 4. We need a slight generalization of the above formula. Assume the bundle E is \mathbb{Z}_2 -graded (say the spinor bundle of an even dimensional spin-manifold), with Γ the operator giving the grading. Then we define the *supertrace* by $\overline{\mathbf{Tr}}_s A = \overline{\mathbf{Tr}} \Gamma A$, where A is a cusp pseudodifferential operator acting on sections of E and the weight Q is always taken as even with respect to the \mathbb{Z}_2 -grading. When A, B are cusp pseudodifferential operators of pure type, then the \mathbb{Z}_2 -graded version of the trace anomaly formula holds

$$(3.44) \quad \overline{\mathbf{Tr}}_s[A, B] = \overline{\mathbf{Tr}}_{\sigma, s}[\log Q, B]A - \overline{\mathbf{Tr}}_{\partial, s} A[\log x, B],$$

where $\overline{\mathbf{Tr}}_{\partial, s} A[\log x, B] = \overline{\mathbf{Tr}}_{\partial} \Gamma A[\log x, B]$. Naturally, the commutator used in the \mathbb{Z}_2 -graded case is the supercommutator.

The anomaly for $\overline{\mathbf{Tr}}_{\sigma}$ is treated next. Let A and B be cusp operators in $\Psi_c^*(M)$. Then by definition

$$(3.45) \quad \begin{aligned} \overline{\mathbf{Tr}}_{\sigma}[A, B] &= \text{res}_{z=0} \text{FP}_{\tau=0} \mathbf{Tr}[A, B] x^z Q^{-\tau} \\ &= -\text{res}_{z=0} \text{FP}_{\tau=0} (\mathbf{Tr} A[\log x, B] z x^z Q^{-\tau} + \mathbf{Tr}[\log Q, B] A \tau x^z Q^{-\tau}) \\ &\quad + \text{res}_{z=0} \text{FP}_{\tau=0} (W z^2 + W' \tau^2). \end{aligned}$$

Taking the residue kills all the terms except $\mathbf{Tr}[\log Q, B]A\tau x^z Q^{-\tau}$, since the meromorphic function $(z, \tau) \rightarrow \mathbf{Tr}[\log Q, B]Ax^z Q^{-\tau}$ has only simple poles. Thus

$$\begin{aligned}
(3.46) \quad \overline{\mathbf{Tr}}_\sigma[A, B] &= \text{res}_{z=0} \text{FP}_{\tau=0}(\mathbf{Tr}[\log Q, B]A\tau x^z Q^{-\tau}) \\
&= \text{FP}_{\tau=0} \overline{\mathbf{Tr}}_\partial[\log Q, B] \tau A Q^{-\tau} \\
&= \mathbf{Tr}_{\partial, \sigma}[\log Q, B]A,
\end{aligned}$$

where the definition of $\overline{\mathbf{Tr}}_\partial$ and the relations in Lemma 1 has been used. It follows that $\overline{\mathbf{Tr}}_\sigma$ is not a trace.

Proposition 20. Given two cusp operators A and B in $\Psi_c^*(M; E)$, then the trace anomaly of $\overline{\mathbf{Tr}}_\sigma$ is

$$(3.47) \quad \overline{\mathbf{Tr}}_\sigma[A, B] = \mathbf{Tr}_{\partial, \sigma}[\log Q, B]A.$$

Proof. □

Similarly one can derive the anomaly formula for $\overline{\mathbf{Tr}}_\partial$.

Proposition 21. Given two cusp operators A and B in $\Psi_c^*(M; E)$, then the trace anomaly of $\overline{\mathbf{Tr}}_\partial$ is

$$(3.48) \quad \overline{\mathbf{Tr}}_\partial[A, B] = -\mathbf{Tr}_{\partial, \sigma}[\log x, B]A.$$

Proof. See [MeNi] and [MoNi]. □

The functional $\mathbf{Tr}_{\partial, \sigma}$ is a trace (see also [MeNi, Lemma 6]).

Proposition 22. Given two cusp operators A and B in $\Psi_c^*(M; E)$, then the trace anomaly of $\mathbf{Tr}_{\partial, \sigma}$ vanishes.

Proof. Indeed, by definition and using earlier computations

$$\begin{aligned}
(3.49) \quad \mathbf{Tr}_{\partial, \sigma}[A, B] &= \text{res}_{z=0} \text{res}_{\tau=0} \mathbf{Tr}[A, B]x^{-z} Q^{-\tau} \\
&= \text{res}_{z=0} \text{res}_{\tau=0} (\mathbf{Tr}A[\log x, B]zx^z Q^{-\tau} - \mathbf{Tr}[\log Q, B]A\tau x^z Q^{-\tau} \\
&\quad + Wz^2 + W'\tau^2).
\end{aligned}$$

Again, taking residues kills the last two terms. This gives

$$\begin{aligned}
(3.50) \quad \overline{\mathbf{Tr}}_{\partial, \sigma}[A, B] &= \text{res}_{\tau=0} (\text{res}_{z=0} \mathbf{Tr}A[\log x, B]zx^z Q^{-\tau}) \\
&\quad - \text{res}_{z=0} (\text{res}_{\tau=0} \mathbf{Tr}[\log Q, B]A\tau x^z Q^{-\tau}).
\end{aligned}$$

Note that

$$(3.51) \quad (z, \tau) \rightarrow \mathbf{Tr}A[\log x, B]zx^z Q^{-\tau}$$

is regular in z and

$$(3.52) \quad (z, \tau) \rightarrow \mathbf{Tr}[\log Q, B]A\tau x^z Q^{-\tau}$$

is regular in τ . Thus, taking the residues kills the both terms. This means

$$(3.53) \quad \mathbf{Tr}_{\partial, \sigma}[A, B] = 0.$$

□

4. A BRIEF INTRODUCTION TO THE BRST-FORMALISM

The necessary machinery to handle the gauge-group cocycle computations is now introduced. Here, we assume that the manifold M is \mathbb{R}^n . If we need to specify the dimension, we denote M by M^n . We further assume that all the fields to be smooth and with compact support. Furthermore, we let E be a product complex vector bundle $M \times \mathbb{C}^N$ over M , where N is rank of E .

We denote the set of based gauge transformations of E as \mathbf{G} and the set of Hermitean connections on E as B . We think B as a (smooth) Frechet manifold and the connections as matrix-valued differential one-forms. The gauge transformations are thought as smooth functions $g : M \rightarrow \text{End}(E)$ that obtain a value identity at some point p of M (hence the word based), because we consider a product bundle. The space $\text{End}(E)$ is identified with complex $N \times N$ -matrices.

The gauge transformations act on the space of connections B via conjugation $A \rightarrow g^{-1}dg + g^{-1}Ag$, where $A \in B$. Now, fix any connection A in B and consider an 1-parameter group of gauge transformations g_t such that at $t = 0$ they are identity maps. Differentiation of the action of g on A , denoted by $g \cdot A$, gives rise to vector fields that are called infinitesimal gauge transformations. We denote by $\xi = \dot{A}_t|_{t=0}$, where $A_t = g_t \cdot A$. Then $\xi \cdot A = d_A \xi$, where $\xi \in T_A B$. Here we have used the standard exterior covariant derivate of $\text{End}(E)$ -valued forms. These vector fields are called *vertical vector fields* in TB or simply *vertical*.

Next, we define an 1-form θ on B , that is only defined in the vertical directions. This form is identified with the Maurer-Cartan form on the gauge group. Define the 1-form θ on B as a tautological form to the vertical directions, that is, $\theta_b(\xi) = \xi$ if ξ is vertical at $T_b B$. Usually we just write $\theta_b = \theta$, where the dependence of the parameter $b \in B$ is understood.

We define the vertical exterior differential δ on B by the usual Palais formula. We do not use this formula explicitly, so we will not state it (see for example [B]). We note that δ is the restriction of the exterior derivative on B into the gauge orbits. This is the 'gauge variation'. We need to introduce a sign convention $\hat{\delta} = (-1)^p \delta$, when acting to the ordinary p-forms. This assures that $\mathbf{d} = d + \hat{\delta}$ satisfies $\mathbf{d}^2 = 0$.

Proposition 23 (BRST-algebra). Let $A \in B$ and let d_A be the exterior covariant differential of $\text{End}(E)$ -valued forms. Let F denote the curvature of A . Then the

following identities are satisfied

$$\begin{aligned}
d_A\theta &= d\theta + [A, \theta] \\
\hat{\delta}A &= -\delta A = -d_A\theta \\
\hat{\delta}d_A\theta &= -\delta d_A\theta = 0 \\
\hat{\delta}\theta &= \delta\theta = -\theta^2 = -\frac{1}{2}[\theta, \theta] \\
\hat{\delta}F &= \delta F = [F, \theta].
\end{aligned}
\tag{4.1}$$

Particularly, we have

$$\begin{aligned}
d_AX &= dX + [A, X] \\
\delta_X A &= d_AX \\
\delta_Y d_A\theta &= 0 \\
\delta\theta(X, Y) &= -[X, Y] \\
\delta_X F &= [F, X],
\end{aligned}
\tag{4.2}$$

when X and Y are vertical vector fields on B . Here we have denoted $\delta_X = i_X\delta$.

Proof. We only check the identity $\hat{\delta}F = [F, \theta]$. This is an easy computation

$$\begin{aligned}
\hat{\delta}F &= \hat{\delta}(dA + A \wedge A) = -d\hat{\delta}A + \hat{\delta}A \wedge A - A \wedge \hat{\delta}A \\
&= dd_A\theta - d_A\theta \wedge A + A \wedge d_A\theta \\
&= d[A, \theta] - d\theta \wedge A + [A, \theta] \wedge A + A \wedge d\theta + A \wedge [A, \theta] \\
&= [dA, \theta] - [A, d\theta] + [A, d\theta] + [A \wedge A, \theta] \\
&= [dA + A \wedge A, \theta] = [F, \theta].
\end{aligned}
\tag{4.3}$$

□

We give few examples of standard δ -cocycles relevant in quantum field theory (see [B], for example). Below, the \mathbf{tr} denotes the trace acting on elements of $\text{End}(E)$.

Proposition 24. Let $\omega_2 = \mathbf{tr}\theta d\theta$. Then $\hat{\delta}\omega_2$ is d-exact.

Proof. Using the above relations and the cyclicity of the trace, we get

$$\begin{aligned}
\hat{\delta}\omega_2 &= \mathbf{tr}\hat{\delta}\theta d\theta - \mathbf{tr}\theta \hat{\delta}d\theta \\
&= -\mathbf{tr}\theta^2 d\theta + \mathbf{tr}\theta d\hat{\delta}\theta \\
&= -\mathbf{tr}\theta^2 d\theta - \mathbf{tr}\theta d\theta^2 \\
&= -\mathbf{tr}d\theta\theta^2 - \mathbf{tr}\theta d\theta^2 \\
&= -d\mathbf{tr}\theta^3.
\end{aligned}
\tag{4.4}$$

□

Thus the integral of ω_2 over M^1 defines a cocycle. This is one of the standard expressions for the central term in the affine Kac-Moody algebra up to a normalization. When we evaluate ω_2 with respect to two vertical vector fields X and Y , we obtain

$$(4.5) \quad \int_{M^1} \omega_2(X, Y) = \int_{M^1} \mathbf{tr}(XdY - YdX).$$

Proposition 25. Let $\omega_1 = \mathbf{tr}Ad\theta$ ¹. Then $\hat{\delta}\omega_1$ is d-exact.

Proof. By a direct computation

$$(4.6) \quad \begin{aligned} \hat{\delta}\omega_1 &= \mathbf{tr}\hat{\delta}Ad\theta - \mathbf{tr}A\hat{\delta}d\theta \\ &= -\mathbf{tr}d\theta d\theta - \mathbf{tr}[A, \theta]d\theta + \mathbf{tr}Ad\hat{\delta}\theta \\ &= -d\mathbf{tr}\theta d\theta - \mathbf{tr}[A, \theta]d\theta + \mathbf{tr}Ad(\theta^2) \\ &= -d\mathbf{tr}\theta d\theta - \mathbf{tr}[A, \theta]d\theta + \mathbf{tr}A[d\theta, \theta] \\ &= -d\mathbf{tr}\theta d\theta + \mathbf{tr}[Ad\theta, \theta] \\ &= -d\mathbf{tr}\theta d\theta. \end{aligned}$$

□

Therefore, the expression

$$(4.7) \quad \int_{M^2} \omega_1(X) = \int_{M^2} \mathbf{tr}AdX,$$

defines a δ -cocycle, where X is a vertical vector field on B . This cocycle is related to the chiral anomaly.

Proposition 26. Let $\omega_2 = \mathbf{tr}Ad\theta d\theta$. Then the form $\hat{\delta}\omega_2$ is d-exact.

Proof. As before, we compute

$$(4.8) \quad \begin{aligned} \hat{\delta}\omega_2 &= \mathbf{tr}\hat{\delta}Ad\theta d\theta - \mathbf{tr}A\hat{\delta}d\theta d\theta - \mathbf{tr}Ad\theta\hat{\delta}d\theta \\ &= \mathbf{tr}\hat{\delta}Ad\theta d\theta + \mathbf{tr}Ad\hat{\delta}\theta d\theta + \mathbf{tr}Ad\theta\hat{\delta}\theta \\ &= -\mathbf{tr}d_A\theta d\theta d\theta - d\mathbf{tr}A\hat{\delta}\theta d\theta + \mathbf{tr}dA\hat{\delta}\theta d\theta - d\mathbf{tr}Ad\theta\hat{\delta}\theta + \mathbf{tr}dAd\theta\hat{\delta}\theta. \end{aligned}$$

The first term gives

$$(4.9) \quad \begin{aligned} -\mathbf{tr}d_A\theta d\theta d\theta &= -\mathbf{tr}d\theta d\theta d\theta - \mathbf{tr}A\theta d\theta d\theta - \mathbf{tr}\theta Ad\theta d\theta \\ &= -\mathbf{tr}d\theta d\theta d\theta - \mathbf{tr}A(\theta d\theta d\theta - \mathbf{tr}d\theta d\theta\theta) \\ &= -\mathbf{tr}d\theta d\theta d\theta - \mathbf{tr}A(-d\theta^2 d\theta + d\theta\theta d\theta - d\theta d\theta^2 - d\theta\theta d\theta) \\ &= -\mathbf{tr}d\theta d\theta d\theta + \mathbf{tr}A(d\theta^2 d\theta + d\theta d\theta^2) \\ &= -d\mathbf{tr}\theta d\theta d\theta + \mathbf{tr}Ad(\theta^2 d\theta + d\theta\theta^2). \end{aligned}$$

¹Here $\mathbf{tr}Ad\theta \equiv \mathbf{tr}Ad\theta_A$.

Thus

$$(4.10) \quad \begin{aligned} \hat{\delta}\omega_2 &= -d\mathbf{tr}\theta d\theta d\theta - d\mathbf{tr}A(\theta^2 d\theta + d\theta\theta^2) - d\mathbf{tr}A(\hat{\delta}\theta d\theta + d\theta\hat{\delta}\theta) \\ &= -d\mathbf{tr}\theta d\theta d\theta. \end{aligned}$$

□

Therefore, we obtain a cocycle in a 3-dimensional manifold. This reads

$$(4.11) \quad \int_{M^3} \omega_2(X, Y) = \int_{M^3} \mathbf{tr}A(dXdY - dYdX),$$

when evaluated with respect to vertical vector fields X and Y . The above expression is the Mickelsson-Faddeew cocycle up to a normalization.

5. DESCENT EQUATIONS IN THE FINITE DIMENSIONAL CASE

In the previous section we gave some examples of the most important cocycle formulas (in the case of closed manifolds). How are such formulas obtained? There is an efficient method for constructing such expressions. It was first discovered by physicists Zumino (see chapter Chiral Anomalies And Differential Geometry in [TJZW] for example) and Stora [S1], [S2]. The idea of their construction was to represent the cocycles as secondary characteristic classes. Then the cocycle property follows directly from the so-called descent equations [B], [Hou]. Later in the seminal paper of Atiyah and Singer [AS2] the relation to the families index theorem was explained.

We now review the construction of cocycles from the descent equations. The first step is the derivation of a transgression formula. The descent equations follow from the transgression formula. From the descent equations we can read the cocycle property directly in the case of closed manifolds. For later purposes, we formulate this in terms of superconnection formalism (though this would not be completely necessary). However, our approach to superconnections differs from standard treatises such as [BGV].

In this section, the same assumptions as in the previous section are used.

We define an algebra of $\text{End}(E)$ -valued differential forms Ω on $M \times B$ generated by elements $1, A, dA$ and θ , where A is an element of B and θ is the Maurer-Cartan form interpreted as $\text{End}(E)$ -valued form via the use of evaluation map. That is, elements of Ω are polynomials in the variables A, dA and θ . Note that the curvature $F = dA + A^2$ is in Ω .

We equip Ω with differentials $d, \hat{\delta}$ and the total differential \mathbf{d} defined earlier. Now Ω is graded in terms of form degrees of M and form degrees of B (ghost degree). Also, Ω is graded by the total form degree. The total form degree is just a sum of the ghost degree and the form degree on M .

Define a graded commutator on Ω by

$$(5.1) \quad [\omega, \eta] = \omega\eta - (-1)^s\eta\omega,$$

where ω and η are pure elements of Ω and the sign s is computed from

$$(5.2) \quad s = p(\omega) \cdot p(\eta) + (\partial\omega) \cdot (\partial\eta),$$

where $p(\omega)$ is the form degree on M and $\partial\omega$ is the ghost degree of any pure element ω of Ω , respectively.

Next, we define an odd covariant derivative $d_{\mathbb{A}}$ acting on Ω by

$$(5.3) \quad d_{\mathbb{A}}\omega = \mathbf{d}\omega + [\mathbb{A}, \omega] \in \Omega,$$

where ω is any element of Ω , \mathbb{A} is an odd (in total degree) element of Ω or zero, and where the commutator is the above graded commutator.

Usually, we take \mathbb{A} to be

$$(5.4) \quad \mathbb{A} = tA + \theta$$

or

$$(5.5) \quad \mathbb{A} = t\theta,$$

where $t \in [0, 1]$.

We equip Ω with \mathbb{Z}_2 -grading coming from the parity grading of the total form degree. Denote by $\Omega = \Omega_+ \oplus \Omega_-$ this grading.

Now define Γ as the parity operator acting on forms Ω by

$$(5.6) \quad \Gamma\omega_p = (-1)^p\omega_p,$$

where ω has form degree p on M . Note that $\Gamma^2 = 1$, $\hat{\delta} = \Gamma\delta$, and that Γ anticommutes with d . Also, Γ anticommutes with any odd form, in the form degree of M , on Ω .

Now, replace θ by $\Gamma\theta$. Then $d_{\mathbb{A}}$, defined above, also satisfies graded Leibniz rule with respect to grading $\Omega_+ \oplus \Omega_-$

$$(5.7) \quad d_{\mathbb{A}}(\omega\eta) = (d_{\mathbb{A}}\omega)\eta + (-1)^{\partial\omega}\omega d_{\mathbb{A}}(\eta),$$

where $\omega, \eta \in \Omega$ and $\partial\omega$ denotes the total parity of ω .

Furthermore, for pure elements ω and η of Ω

$$(5.8) \quad [\omega, \eta] = \omega\eta - (-1)^{\partial\omega \cdot \partial\eta}\eta\omega,$$

and $\partial\omega, \partial\eta$ denotes the total parities of ω and η respectively.

We think $d_{\mathbb{A}}$ as a superconnection on Ω . Also, we call the form \mathbb{A} in the definition of a superconnection as superconnection.

From now on, we interpret θ as $\Gamma\theta$. This convention simplifies many computations involving the sign rule. Later, we shall use a similar interpretation, in the infinite dimensional situation. Essentially, the substitution $\theta \rightarrow \Gamma\theta$ allows us to think θ as an 1-form on M , from the computational point of view.

The curvature for $d_{\mathbb{A}}$ as above, is defined as usual (formally $d_{\mathbb{A}}^2 = [\mathbb{F}, \cdot]$). Let us denote the corresponding curvatures by \mathbb{F} . The curvatures corresponding to the

above superconnections are easily calculated

$$(5.9) \quad \mathbb{F} = \mathbf{d}\mathbb{A} + \mathbb{A} \wedge \mathbb{A} = tF + (t^2 - t)A^2 + (1 - t)d\theta,$$

where F is the curvature corresponding to A . Similarly we compute

$$(5.10) \quad \mathbb{F} = \mathbf{d}\mathbb{A} + \mathbb{A} \wedge \mathbb{A} = td\theta + (t^2 - t)\theta^2.$$

It is not difficult to prove that the Bianchi-identity $d_{\mathbb{A}}\mathbb{F} = 0$ is satisfied.

We introduce the Chern-Weil type form

$$(5.11) \quad \text{ch}_k(\mathbb{A}) = \mathbf{tr}_s \mathbb{F}^k,$$

where \mathbf{tr}_s is the ordinary trace and k is a positive integer. The subscript s is to remind us from the \mathbb{Z}_2 -grading coming from Ω .

It is useful to introduce the so-called (graded) symmetric trace \mathbf{Str}_s of k -objects, where k is a non-negative integer [B], [Hou]. The symmetric trace means taking the graded symmetrization of k objects and then taking of the trace. In our case the grading comes from the \mathbb{Z}_2 -grading of Ω . Particularly, $\mathbf{Str}_s(\omega_1, \omega_2, \dots, \omega_k)$ means symmetrization when each ω_i is even, and antisymmetrization when each ω_i is of odd element of Ω respectively. Usually, we do not show the number k , when we use the symmetrized trace (later we have to keep track of k). By definition, the symmetric trace is completely (graded) symmetric; we can change the places of any two arguments by using the sign rule. This is best understood by example. Consider the case $k = 4$ and suppose ω_1, ω_2 are even and ω_3, ω_4 odd then

$$(5.12) \quad \mathbf{Str}_s(\omega_1, \omega_2, \omega_3, \omega_4) = \mathbf{Str}_s(\omega_1, \omega_3, \omega_2, \omega_4) = -\mathbf{Str}_s(\omega_1, \omega_2, \omega_4, \omega_3),$$

where the sign change comes when passing ω_3 over ω_4 .

When some of the ω_i 's are the same, we collect them. For example, if $\omega_1 = \omega_2 = \omega$, then we denote

$$(5.13) \quad \mathbf{Str}_s(\omega, \omega, \omega_3, \omega_4) = \mathbf{Str}_s(\omega^2, \omega_3, \omega_4).$$

Sometimes there is expressions like a^2 in the symmetrization, then we can denote $[a^2]^m$ to emphasize that we mean m -times a^2 . We cannot collect odd objects. Namely, if any two odd objects are the same (say ω_3, ω_4 above), the symmetrization process kills the whole expression.

The most important property of the symmetrized trace is the integration by parts formula. For example, in the first case above

$$(5.14) \quad \begin{aligned} \mathbf{dStr}_s(\omega_1, \omega_2, \omega_3, \omega_4) &= \mathbf{Str}_s(\mathbf{d}\omega_1, \omega_2, \omega_3, \omega_4) + \mathbf{Str}_s(\omega_1, \mathbf{d}\omega_2, \omega_3, \omega_4) \\ &\quad + \mathbf{Str}_s(\omega_1, \omega_2, \mathbf{d}\omega_3, \omega_4) - \mathbf{Str}_s(\omega_1, \omega_2, \omega_3, \mathbf{d}\omega_4), \end{aligned}$$

where the sign comes from passing \hat{d} over the odd object ω_3 .

The integration by parts formula becomes useful when several arguments are the same. For example, consider the following expression that has k times a supercurvature \mathbb{F}

$$(5.15) \quad \mathbf{dStr}_s(\mathbb{F}^k) = k\mathbf{Str}_s(\mathbf{d}\mathbb{F}, \mathbb{F}^{k-1}) = k\mathbf{Str}_s([\mathbb{A}, \mathbb{F}], \mathbb{F}^{k-1}),$$

where we have used the Bianchi identity. This expression is ofcourse zero. This fact depends on the trace property. It is easier to use the trace property at the beginning (switch from \mathbf{d} to $d_{\mathbb{A}}$). We also read from the above, that we are just taking commutator $[\mathbb{A}, \mathbb{F}^k]$. Later, we need to keep track of the commutators. Then, we need to separate the symmetrization process. In the above example this is done by writing $\mathbf{Str}_s(\mathbb{F}^k) = \mathbf{tr}_s S_k(\mathbb{F}^k)$. In this notation

$$(5.16) \quad \mathbf{dStr}_s(\mathbb{F}^k) = k\mathbf{tr}_s S_k([\mathbb{A}, \mathbb{F}], \mathbb{F}^{k-1}) = \mathbf{tr}_s[\mathbb{A}, S_k(\mathbb{F}, \mathbb{F}^{k-1})] = 0.$$

Particularly, the integration by parts formula above holds for the graded symmetrizer $S_k(\omega_1, \dots, \omega_k)$, where $\omega_1, \dots, \omega_k$ are pure elements of Ω . It is important to note that the partial integration formula holds for the derivations of the form $[\omega, \cdot]$ acting on Ω , where ω can be even or odd element of Ω .

Next step is the derivation of the transgression formula. This is obtained as follows. For $t \in [0, 1]$, let $\mathbb{A}_t = \mathbb{A}$ be an one parameter family of superconnections on Ω . First observe that

$$(5.17) \quad \begin{aligned} \partial_t \mathbb{F} &= d\dot{\mathbb{A}} + \dot{\mathbb{A}} \wedge \mathbb{A} + \mathbb{A} \wedge \dot{\mathbb{A}} \\ &= d_{\mathbb{A}} \dot{\mathbb{A}}, \end{aligned}$$

since \mathbb{A} is odd.

Differentiation with respect to a parameter gives

$$(5.18) \quad \begin{aligned} \partial_t \mathbf{ch}_k(\mathbb{A}) &= k\mathbf{Str}_s(\dot{\mathbb{F}}, \mathbb{F}^{k-1}) = k\mathbf{Str}_s(d_{\mathbb{A}} \dot{\mathbb{A}}, \mathbb{F}^{k-1}) \\ &= \mathbf{d}k\mathbf{Str}_s(\dot{\mathbb{A}}, \mathbb{F}^{k-1}). \end{aligned}$$

Here we have used the integration by parts, the Bianchi identity and the cyclicity of the trace.

Thus, we get the *transgression formula*

$$(5.19) \quad \mathbf{ch}_k(\mathbb{A}_1) - \mathbf{ch}_k(\mathbb{A}_0) = \int_0^1 \partial_t \mathbf{ch}_k(\mathbb{A}_t) = \mathbf{d} \int_0^1 k\mathbf{Str}_s(\dot{\mathbb{A}}, \mathbb{F}^{k-1}).$$

Put

$$(5.20) \quad \mathbf{ch}_k^1(\mathbb{A}) = \int_0^1 k\mathbf{Str}_s(\dot{\mathbb{A}}, \mathbb{F}^{k-1}).$$

The differential form $\mathbf{ch}_k^1(\mathbb{A})$ is called the *Chern-Simons form* of the superconnection \mathbb{A}_t . Now

$$(5.21) \quad \mathbf{dch}_k^1(\mathbb{A}) = \mathbf{ch}_k(\mathbb{A}_1) - \mathbf{ch}_k(\mathbb{A}_0) \equiv \partial \mathbf{ch}_k(\mathbb{A}).$$

The descent equations follow by projecting to the ghost degree $m + 1$ in the transgression formula

$$(5.22) \quad \hat{\delta} \mathbf{ch}_{k,[m]}^1(\mathbb{A}) = -d\mathbf{ch}_{k,[m+1]}^1(\mathbb{A}) + \partial \mathbf{ch}_{k,[m+1]}(\mathbb{A}).$$

If $\partial \mathbf{ch}_{k,[m+1]}(\mathbb{A}) = 0$, then

$$(5.23) \quad \hat{\delta} \mathbf{ch}_{k,[m]}^1(\mathbb{A}) = -d\mathbf{ch}_{k,[m+1]}^1(\mathbb{A}).$$

Thus $\text{ch}_k^1(\mathbb{A})$ defines δ -cocycles modulo d .

Therefore, we can construct δ -cocycles by integrating the Chern-Simons forms over the manifold. In the case of a boundary we do not get cocycles. Then the Chern-Simons forms become transgressive in the sense that δ -coboundaries of the Chern-Simons forms defined above lie on the boundary, when we perform the integral. On the boundary the resulting form may represent a non-trivial cocycle.

In the case m odd, we integrate $\text{ch}_{k,[m]}^1(\mathbb{A})$ over the even dimensional manifolds. In the case m even, we integrate $\text{ch}_{k,[m]}^1(\mathbb{A})$ over the odd dimensional manifolds. The dimension of the manifold have to match with the de Rham form degree of $\text{ch}_{k,[m]}^1(\mathbb{A})$ in order to get (possible) nonvanishing result.

We obtain from the Stokes theorem (when dimensions agree)

$$(5.24) \quad \hat{\delta} \int_M \text{ch}_{k,[m]}^1(\mathbb{A}) = - \int_{\partial M} \text{ch}_{k,[m+1]}^1(\mathbb{A}).$$

Now choose $\mathbb{A} = tA + \theta$. Then we have

$$(5.25) \quad \mathbf{dch}_k^1(\mathbb{A}) = \mathbf{tr}_s F^k - \mathbf{tr}_s (d\theta)^k.$$

Projecting $\text{ch}_k^1(\mathbb{A})$ to the ghost degree m we get for $0 < m < k$.

$$(5.26) \quad \hat{\delta} \text{ch}_{k,[m]}^1(\mathbb{A}) = -d\text{ch}_{k,[m+1]}^1(\mathbb{A}).$$

Particularly, for the ghost degree zero

$$(5.27) \quad d\text{ch}_{k,[0]}^1(\mathbb{A}) = \mathbf{tr}_s F^k.$$

For the choice $\mathbb{A} = t\theta$ we get

$$(5.28) \quad d\text{ch}_k^1(\mathbb{A}) = \mathbf{tr}_s (d\theta)^k.$$

Thus, for $k \leq m \leq 2k - 1$

$$(5.29) \quad \hat{\delta} \text{ch}_{k,[m]}^1(\mathbb{A}) = -d\text{ch}_{k,[m+1]}^1(\mathbb{A}).$$

Combining the above results we get a $\hat{\delta}$ -cocycle modulo d for the each ghost degree m , where $0 < m \leq 2k - 1$.

For example, if $\mathbb{A} = tA + \theta$, then we split the curvature in terms of the ghost degree

$$(5.30) \quad \mathbb{F} = \mathbb{F}_{[0]} + \mathbb{F}_{[1]},$$

where

$$(5.31) \quad \begin{aligned} \mathbb{F}_{[0]} &= tF + (t^2 - t)A^2 \\ \mathbb{F}_{[1]} &= (1 - t)d\theta. \end{aligned}$$

Now, we have to expand

$$(5.32) \quad \text{ch}_{k,[m]}^1(\mathbb{A}) = k \int_0^1 \mathbf{Str}_s(A, \mathbb{F}^{k-1})_{[m]} = k \binom{k-1}{m} \int_0^1 \mathbf{Str}_s(A, \mathbb{F}_{[0]}^{k-1-m}, \mathbb{F}_{[1]}^m).$$

Expand by introducing the necessary normalization constants $\phi_{k,m,l}$ coming from the integration and the expansion of the powers of supercurvature \mathbb{F}

$$(5.33) \quad \text{ch}_{k,[m]}^1(\mathbb{A}) = \sum_{l=0}^{k-1-m} \phi_{k,m,l} \mathbf{Str}_s(A, F^l, [A^2]^{k-1-m-l}, [d\theta]^m).$$

The constants are

$$(5.34) \quad \begin{aligned} \phi_{k,m,l} &= k \binom{k-1}{m} \binom{k-1-m}{l} \int_0^1 t^l (t^2 - t)^{k-1-m-l} (1-t)^m \\ &= k \binom{k-1}{m} \binom{k-1-m}{l} (-1)^{k-m-l-1} B(k-m, k-l). \end{aligned}$$

Here B is the standard beta-function.

Example 9. Choose $k = 3$ and $m = 1$ to obtain

$$(5.35) \quad \text{ch}_{3,[1]}^1(\mathbb{A}) = \phi_{3,1,1} \mathbf{Str}_s(A, F, d\theta) + \phi_{3,1,0} \mathbf{Str}_s(A, A^2, d\theta).$$

Example 10. Choose $k = 4$ and $m = 1$ to obtain

$$(5.36) \quad \text{ch}_{4,[1]}^1(\mathbb{A}) = \phi_{3,1,2} \mathbf{Str}_s(A, F^2, d\theta) + \phi_{3,1,1} \mathbf{Str}_s(A, F, A^2, d\theta) + \phi_{3,1,0} \mathbf{Str}_s(A, (A^2)^2, d\theta).$$

Example 11. If $k = 3$ and $m = 2$ we obtain

$$(5.37) \quad \text{ch}_{3,[2]}^1(\mathbb{A}) = \phi_{3,2,0} \mathbf{Str}_s(A, (d\theta)^2).$$

Example 12. Similarly, if $k = 4$ and $m = 2$ we obtain

$$(5.38) \quad \text{ch}_{4,[2]}^1(\mathbb{A}) = \phi_{4,2,1} \mathbf{Str}_s(A, F, (d\theta)^2) + \phi_{k,m,0} \mathbf{Str}_s(A, A^2, (d\theta)^2).$$

The δ -cocycle property of the above forms can be read at once from the descent equations. It is not so easy to verify the cocycle property by hand, as we saw earlier. However, the use of the symmetrized trace makes these computations easier. For example, consider the case $\text{ch}_{3,[2]}^1(\mathbb{A})$ above. We compute using the 'partial integration' technique (normalization is 1)

$$(5.39) \quad \begin{aligned} \hat{\delta} \text{ch}_{3,[2]}^1(\mathbb{A}) &= \mathbf{Str}_s(\hat{\delta}A, (d\theta)^2) - 2\mathbf{Str}_s(\hat{\delta}A, \hat{\delta}d\theta, d\theta) \\ &= \mathbf{Str}_s(-d_A\theta, (d\theta)^2) - 2\mathbf{Str}_s(\hat{\delta}A, [d\theta, \theta], d\theta) \\ &= -d\mathbf{Str}_s(\theta, (d\theta)^2) - \mathbf{Str}_s([A, \theta], (d\theta)^2) - 2\mathbf{Str}_s(\hat{\delta}A, [d\theta, \theta], d\theta) \\ &= -d\mathbf{Str}_s(\theta, (d\theta)^2). \end{aligned}$$

We have used identity $\hat{\delta}d\theta = [d\theta, \theta]$, the cyclicity of the trace and the fact

$$(5.40) \quad [S_3(A, (d\theta)^2), \theta] = S_3([A, \theta], (d\theta)^2) - 2S_3(\hat{\delta}A, [d\theta, \theta], d\theta).$$

The lesson in this section is that we can use the descent equations and Chern-Simons forms to construct δ -cocycles over manifold M , when M is closed. When M has a boundary, we can construct δ -transgression forms. That is, forms whose

δ -coboundaries live on the boundary. These boundary terms are automatically cocycles.

The above BRST-computations use heavily the cyclicity of the trace. This causes trouble in the infinite dimensional situations. Then, use of the symmetrization operator gives us a book keeping device to keep track of commutators. It also makes the BRST-computations more tractable.

6. FORMS ASSOCIATED WITH FAMILIES OF DIRAC OPERATORS

Let M be a connected compact spin-manifold with connected boundary. Assume M is equipped with a fixed boundary defining function and an exact cusp metric. Furthermore, let E denote a Hermitean vector bundle $M \times \mathbb{C}^N$ over M of rank N that is equipped with a fixed Hermitean metric. Let S denote the total spinor bundle and let \mathcal{H} denote the (complex) Hilbert space $L_c^2(M; S \otimes E)$, defined using the cusp metric and Hermitean metric on E . We assume that all geometric structures are of 'product type'. Particularly, the Hermitean connections B on E and the gauge transformations are independent of x near the fixed collar neighborhood $[0, 1)_x \times \partial M$ of the boundary.

We consider a family of cusp-pseudodifferential operators $F \in \Psi_c^0(\mathbb{M}/B; S \otimes E)$ acting on (as bounded operators) \mathcal{H} , where $\mathbb{M} = M \times B$. Each F_b is assumed to be self-adjoint, fully elliptic and $F_b^2 = 1$, where b is any element in B . If the manifold M is even dimensional, then there exists a Hermitean operator Γ acting on \mathcal{H} such that $\Gamma^2 = 1$ and it anticommutes with each F_b . The operator Γ comes from the \mathbb{Z}_2 -grading of spinors. If the manifold M is odd dimensional, then we do not have Γ . The operators Γ and F are usually referred to as grading operators. One can also think grading operators F as points of suitable infinite dimensional Grassmannians [Q2], [MR], [MeNi] and [St].

The grading operator family F comes usually from a family of elliptic cusp-pseudo-differential operators defined over the above fibration. The correspondence is going from the original family to the signature family of the corresponding operator. More precisely, if $D \in \Psi_c^m(\mathbb{M}/B; S \otimes E)$, then $F = \frac{D}{|D|}$.

This family comes, in our case, from a family of Dirac operators $\mathfrak{D}_E \in \Psi_c^1(\mathbb{M}/B; S \otimes E)$. We assume the full ellipticity unless stated otherwise. Thus $\mathfrak{D}_{E,b}$ defines a Fredholm operator $\mathfrak{D}_{E,b} : H_c^m(M; S \otimes E) \rightarrow H_c^{m-1}(M; S \otimes E)$, for $b \in B$ and $m \in \mathbb{R}$. We can also think $\mathfrak{D}_{E,b}$ as an unbounded differential operator acting on \mathcal{H} , where the domain is specified essentially by the standard APS-boundary conditions [MePi], [MePi2] and [MeRo2].

As is customary, the chiral Dirac family is denoted (in the even dimensional case) by $\mathfrak{D}_E^\pm \in \Psi_c^1(\mathbb{M}/B; S^\pm \otimes E, S^\mp \otimes E)$.

The construction of F assumes that \mathfrak{D}_E has no zero-eigenvalues. Later, we show how this condition can be relaxed. For now, we assume that \mathfrak{D}_E does not have zero-eigenvalues.

Let us denote by d the exterior derivative of B . If necessary, we use d_B to distinguish it from the exterior derivative of M . The gauge group acts on the

Dirac operators and grading operators by conjugation. The differentiation of the action of the gauge group on the grading operators gives commutators. Particularly, when we restrict to the vertical directions, they are of the form $[F, X]$, where $X \in \Psi_c^0(\mathbb{M}/B; S \otimes E)$ is a vertical vector field. Here X acts as a multiplication operator on \mathcal{H} . We see that $[F, X]$ is a cusp pseudodifferential operator of order -1 and X is a cusp pseudodifferential operator of order 0 .

From the product geometry assumption above, it follows that the indicial family of X can be identified with its restriction to the boundary. Particularly, the indicial family of X is independent of the suspension parameter. This fact is used to simplify trace anomaly formulas.

Usually, we write $[F, \theta]$ when we restrict dF to the vertical directions, where θ is the Maurer-Cartan form defined earlier. The Maurer-Cartan form also acts as a multiplication operator on \mathcal{H} .

Let k be a strictly positive integer. We study the following forms on B

$$(6.1) \quad \omega_k = F(dF)^{k-1},$$

where $k - 1$ is to be taken odd if the manifold M is even dimensional, otherwise $k - 1$ is taken to be even.

The above form ω_k is a cusp pseudodifferential operator of order $-k + 1$. Thus, even in the case of a closed manifold, ω_k is not generally in the trace class.

Let \mathfrak{D}_0 denote the Dirac operator coupled to the canonical flat connection in B . To regularize the trace, we use the weight $Q = \sqrt{\mathfrak{D}_0^2}$ in the zeta function regularization described earlier. In the even dimensional case we have to use the supertrace $\overline{\mathbf{Tr}}\Gamma$. In the odd case we use $\overline{\mathbf{Tr}}$. We denote these 'traces' by $\overline{\mathbf{Tr}}_s$.

The weight Q does not depend on the parameters B . It follows that d and the regularized trace commute (assuming the form under the trace is of constant order).

Put

$$(6.2) \quad \eta_k = \overline{\mathbf{Tr}}_s \omega_k = \overline{\mathbf{Tr}}_s F(dF)^{k-1}.$$

Thus we obtain $(k - 1)$ -form on the base. In general, the forms η_k are not closed. However, they might be transgressive in the following sense.

Definition 13. For a strictly positive integer k , the form $\eta_k = \overline{\mathbf{Tr}}_s F(dF)^{k-1}$, defined above, is called *transgressive* if $d\eta_k$ vanishes, when evaluated with respect to vector fields X_1, X_2, \dots, X_k on B , where each vector field X_i vanishes over the boundary.

Remark 5. Often, we call an expression as a *boundary term*, if it vanishes when the boundary is empty. Also an expression, whose regularized trace defines a boundary term, is called a boundary term.

In other words, $d\eta_k$ lies on the boundary. If the boundary is empty, then η_k defines a closed form on the base B . Observe that $d\eta_k$ defines a closed form on a boundary, if η_k is transgressive.

To find out when η_k is transgressive, we express $d\eta_k$ as a supercommutator. To this end, we use the following supercommutator relations

$$(6.3) \quad \begin{aligned} 1 &= F^2 = \frac{1}{2}[F, F] \\ 0 &= dFF + FdF = [dF, F], \end{aligned}$$

where the sign in the supercommutator $[A, B] = AB - (-)^s BA$ is fixed by s . It is computed from the parities and the form degrees of A and B by the formula :

$$(6.4) \quad s = p(A)p(B) + \partial A \partial B.$$

Here $p(\cdot)$ denotes the parity, and $\partial(\cdot)$ denotes the d-form degree. The parities are computed from the number of F 's. That is, F is odd, FdF is even and so on. From now on, we use this supercommutator.

Thus

$$(6.5) \quad \begin{aligned} d\omega_k &= (dF)^k = F^2(dF)^k = \frac{1}{2}[F, F](dF)^k \\ &= (-1)^k \frac{1}{2}[F(dF)^k, F]. \end{aligned}$$

The use of the trace anomaly formula shows

$$(6.6) \quad \begin{aligned} \Xi &= d\overline{\mathbf{Tr}}_s \omega_k = (-1)^k \frac{1}{2} \mathbf{Wres}_s[l, F]F(dF)^k - (-1)^k \frac{1}{2} \overline{\mathbf{Tr}}_{\partial, s} F(dF)^k [\log x, F] \\ &= \Xi_{k, \sigma} + \Xi_{k, \partial}. \end{aligned}$$

Here we have denoted

$$(6.7) \quad \begin{aligned} l &= \log Q \\ \Xi_{k, \sigma} &= (-1)^k \mathbf{Wres}_s[l, F]F(dF)^k \\ \Xi_{k, \partial} &= -(-1)^k \overline{\mathbf{Tr}}_{\partial, s} F(dF)^k [\log x, F]. \end{aligned}$$

Proposition 27. For a positive integer $k \geq 1$, the form $\eta_k = \overline{\mathbf{Tr}}_s F(dF)^{k-1}$ has a differential

$$(6.8) \quad d\eta_k = \Xi_{k, \sigma} + \Xi_{k, \partial},$$

where $\Xi_{k, \sigma}$ is a local interior term

$$(6.9) \quad \Xi_{k, \sigma} = (-1)^k \frac{1}{2} \mathbf{Wres}_s[l, F]F(dF)^k,$$

and $\Xi_{k, \partial}$ is a (non-local) boundary contribution

$$(6.10) \quad \Xi_{k, \partial} = -(-1)^k \frac{1}{2} \overline{\mathbf{Tr}}_{\partial, s} F(dF)^k [\log x, F].$$

Proof. □

Corollary 1. Suppose $k > n-2$, where $n > 0$ denotes the dimension of the manifold M and k is a positive integer. Then the forms $\eta_k = \overline{\mathbf{Tr}}_s F(dF)^{k-1}$ are transgressive.

Proof. This follows from the computation of order of the operator $[l, F]F(dF)^k$, which is $-k - 2$. \square

7. ETA-FORMS AND LOCALIZATION : A FIRST LOOK

The locality of the *eta-forms* $\eta_k = \overline{\mathbf{Tr}}_s F(dF)^{k-1}$ is now studied, when $k = 2, 3, 4$. This is only interesting if we restrict η_k forms to the vertical directions. Here, we abuse notation and denote this restriction with the same notation. Then, we try to express the restricted forms in terms of 'traces' of commutators (expressed in terms of F and θ) and δ -coboundaries. If the form degree is odd (M even dimensional and k even), there is a correction term, that depends only on the Maurer-Cartan form θ .

Remark 6. If we choose $k = 1$ above, then we obtain just the regularized trace of the grading operator. The form η_1 is interpreted as (at least formally) the difference of the number of the positive and negative eigenvalues of the Dirac operator. However, if the boundary is not empty then the Dirac operator has also continuous spectrum. Therefore, the above interpretation really is only formal.

7.1. Eta 1-form. Recall, when dealing with the odd forms (k even), the manifold M is assumed to be even dimensional. We begin from the form

$$(7.1) \quad \omega_2 = FdF.$$

Restricting this form to the vertical directions gives (using, for example, partial integration)

$$(7.2) \quad \begin{aligned} \omega_2 &= F[F, \theta] \\ &= 2\theta - [F\theta, F]. \end{aligned}$$

Thus

$$(7.3) \quad \eta_2 = \overline{\mathbf{Tr}}_s \omega_2 = 2\overline{\mathbf{Tr}}_s \theta - \overline{\mathbf{Tr}}_s [F\theta, F].$$

Modulo the boundary term coming from the trace anomaly formula, we have

$$(7.4) \quad \eta_2 = 2\overline{\mathbf{Tr}}_s \theta - \mathbf{Wres}_s[l, F]F\theta.$$

Compute

$$(7.5) \quad \delta\overline{\mathbf{Tr}}_s \theta = \overline{\mathbf{Tr}}_s \delta\theta = \frac{1}{2}\overline{\mathbf{Tr}}_s [\theta, \theta].$$

Now, the boundary term coming from the trace anomaly $\overline{\mathbf{Tr}}_s [\theta, \theta]$ vanishes, since the indicial family of θ is independent of the suspension parameter. Thus, we obtain, modulo δ -coboundaries at the boundary, a local representation for the form $\overline{\mathbf{Tr}}_s dFdF$, when restricted to vertical directions. The local representative is

$$(7.6) \quad -\delta\mathbf{Wres}_s[l, F]F\theta + \frac{1}{2}\mathbf{Wres}_s[l, \theta]\theta,$$

where $l = \log Q = \sqrt{\overline{\partial}_0^2}$ as before.

The form $\delta\eta_2$ is only interesting when η_2 is transgressive. Thus we have to restrict to the dimension two by Corollary 1. In this case $\delta\eta_2$ may represent a non-trivial δ -cocycle on the boundary of M . This is easily verified formally, when we consider a flat metric. Then the residue density $\mathbf{wres}_s[l, F]F\theta$ gives the standard expression for the chiral anomaly in dimension two. Then δ -coboundary of the residue density is exact by the descent equations (alternatively by a direct computation). The term $\overline{\mathbf{Tr}}_s\theta$ can be completely ignored in the flat case, since the Wodzicki residue density coming from $\overline{\mathbf{Tr}}_s[\theta, \theta]$ is identically zero by the Clifford algebra.

7.2. Eta 2-form. When we consider even eta-forms, there is no grading operator Γ . Particularly, all supertraces become ordinary traces. We consider the form (see also [MP])

$$(7.7) \quad \omega_3 = FdFdF.$$

Restriction to the vertical directions gives

$$(7.8) \quad \omega_3 = F[F, \theta][F, \theta].$$

Thus

$$(7.9) \quad \begin{aligned} \omega_3 &= -2[F, \theta]\theta - [F[F, \theta]\theta, F] \\ &= -2F\theta\theta + 2\theta F\theta - [F[F, \theta]\theta, F] \\ &= 4\theta F\theta - 2[F\theta, \theta] - [F[F, \theta]\theta, F]. \end{aligned}$$

Put $\Psi = -4F\theta$, then

$$(7.10) \quad \delta\Psi = 4\theta F\theta.$$

Thus

$$(7.11) \quad \omega_3 = \delta\Psi - 2[F\theta, \theta] - [F[F, \theta]\theta, F].$$

Therefore

$$(7.12) \quad \eta_3 = \delta\overline{\mathbf{Tr}}\Psi - 2\overline{\mathbf{Tr}}[F\theta, \theta] - \overline{\mathbf{Tr}}[F[F, \theta]\theta, F].$$

A more systematic way is provided by the method of integration by parts. First integrate by parts with respect to $[F, \cdot]$, to get

$$(7.13) \quad \begin{aligned} \omega_3 &= F[F, \theta][F, \theta] = -[F, F\theta[F, \theta]] + [F, F]\theta[F, \theta] \\ &= -[F, F\theta[F, \theta]] + 2\theta[F, \theta], \end{aligned}$$

where we used $[F, F] = 2$.

Next, integrate by parts with respect to $[\cdot, \theta]$, to obtain

$$(7.14) \quad \begin{aligned} \theta[F, \theta] &= [\theta F, \theta] - [\theta, \theta]F \\ &= [\theta F, \theta] - 2\theta^2 F \\ &= [\theta F, \theta] + 2\delta\theta F, \end{aligned}$$

where we used $\delta\theta = -\theta^2$. Using $\delta F = [F, \theta]$, we get

$$(7.15) \quad \begin{aligned} 2\delta\theta F &= \delta(2\theta F) + 2\theta\delta F \\ &= \delta(2\theta F) + 2\theta[F, \theta]. \end{aligned}$$

Thus

$$(7.16) \quad \theta[F, \theta] = [\theta F, \theta] + 2\delta(\theta F) + 2\theta[F, \theta].$$

This yields

$$(7.17) \quad \theta[F, \theta] = -[\theta F, \theta] - 2\delta(\theta F).$$

Combine everything to get

$$(7.18) \quad \omega_3 = F[F, \theta][F, \theta] = -[F, F\theta[F, \theta]] - 2[\theta F, \theta] - 4\delta(\theta F).$$

We now have a representation in terms of 'traces' of commutators and coboundaries. There are no terms depending only on θ . Thus η_3 is local modulo boundary terms and δ -coboundaries by the trace anomaly formula. Particularly, $\delta\eta_3$ is local modulo δ -coboundaries. The form η_3 is transgressive, when dimension of M is three by Corollary 1. In dimension three, the form η_3 essentially represents the so-called Schwinger term. This is discussed later. We eventually show that η_3 is equivalent to the standard Schwinger term in [MR] and [LaMi], when there is no boundary.

7.3. Eta 3-form. Consider

$$(7.19) \quad \omega_4 = F(dF)^3.$$

Restrict to the vertical directions to get

$$(7.20) \quad \omega_4 = F[F, \theta]^3.$$

As before, we begin with partial integration with respect to $[F, \cdot]$ to get

$$(7.21) \quad \begin{aligned} \omega_4 &= F[F, \theta][F, \theta][F, \theta] \\ &= -[F, F\theta[F, \theta]][F, \theta] + [F, F]\theta[F, \theta][F, \theta] \\ &= -[F, F\theta[F, \theta]][F, \theta] + 2\theta[F, \theta][F, \theta], \end{aligned}$$

where we used $[F, F] = 2$. In the last term, we integrate by parts with respect to $[\cdot, \theta]$, $[F, \cdot]$ and δ

$$(7.22) \quad \begin{aligned} \theta[F, \theta][F, \theta] &= [\theta[F, \theta]F, \theta] - \theta[[F, \theta], \theta]F + [\theta, \theta][F, \theta]F \\ &= [\theta[F, \theta]F, \theta] - \theta[F, [\theta, \theta]]F + 2\theta^2[F, \theta]F \\ &= [\theta[F, \theta]F, \theta] - [F, \theta[\theta, \theta]F] - \theta[\theta, \theta][F, F] - 2\delta\theta[F, \theta]F \\ &= [\theta[F, \theta]F, \theta] - [F, \theta[\theta, \theta]F] - 2\theta[\theta, \theta] - 2\delta(\theta[F, \theta]F) + 2\theta[F, \theta]\delta F \\ &= [\theta[F, \theta]F, \theta] - [F, \theta[\theta, \theta]F] - 2\theta[\theta, \theta] - 2\delta(\theta[F, \theta]F) + 2\theta[F, \theta][F, \theta], \end{aligned}$$

where we have used $[F, F] = 2$, $[[F, \theta], \theta] = [F, [\theta, \theta]]$ and $\delta F = [F, \theta]$. That is

$$(7.23) \quad \theta[F, \theta][F, \theta] = -[\theta[F, \theta]F, \theta] + [F, \theta[\theta, \theta]F] + 2\theta[\theta, \theta] + 2\delta(\theta[F, \theta]F).$$

Putting everything together yields

$$(7.24) \quad \omega_4 = -[F, F\theta[F, \theta][F, \theta]] - 2[\theta[F, \theta]F, \theta] + 2[F, \theta[\theta, \theta]F] + 4\theta[\theta, \theta] + 4\delta(\theta[F, \theta]F).$$

The form η_4 is transgressive in dimensions two and four by Corollary 1. Then $\delta\eta_4$, when restricted to the boundary, can be represented with a local expression modulo coboundaries.

The above integration by parts technique ables us to decompose the forms ω_k in terms of commutators, δ -coboundaries and terms depending only on θ .

The general case of decomposing ω_k can be found in Appendix.

The problem with these forms is that they are not transgressive, in general. Therefore, a regularization of the forms ω_k is needed in order to quarantee the transgressive property in higher dimensions. Unfortunately, these regularized forms are more complicated than the above forms ω_k . An another problem is that the residues coming from the above type commutators are difficult to compute, even in simple cases. Later, we replace the forms ω_k with expressions that yield more managable formulas for the residues.

The regularization of the forms ω_k and their decompositions in terms of commutators, δ -coboundaries and ' θ -terms' go in hand in hand. These problems are solved later in a similar framework introduced in [La] and [LaMiRy]. See also appendix, for an alternative way.

8. REGULARIZATION OF THE FORMS $F(dF)^m$

We introduce one way to regularize the forms $F(dF)^m$. Later, we also show an another way, introduced in [MP]. To this end, we must introduce (formal) graded symmetrization operators S_n , for each integer $n > 0$.

First, define S_n formally by (acting on elements x_1, x_2, \dots, x_n of some \mathbb{Z}_2 -graded algebra A)

$$(8.1) \quad S_n(x_1, x_2, \dots, x_n) = \frac{1}{n!} \sum_P (-1)^{N_f(P)} (x_{P_1^{-1}} x_{P_2^{-1}} \cdots x_{P_n^{-1}}),$$

where P runs over all the permutations of integers from one to n and $P_i = P(i)$, for $1 \leq i \leq n$. Here, the sign of the permutation is computed from the number $N_f(P)$ of fermionic pairs $(x_i, x_j), i < j$ such that $P_i^{-1} > P_j^{-1}$. The pair is fermionic (we also use the word odd) if the corresponding elements are odd with respect to the given \mathbb{Z}_2 -grading of A .

We apply this construction on the cusp algebra (actually, a certain subalgebra of it). For now, we take the sign rule (6.4) to determine if a pair is odd. A pair is odd if the sign rule gives minus sign when the corresponding elements switch places (this is a slight generalization of the above construction).

Put $\epsilon = F_0$, where $0 \in B$ denotes the canonical flat connection. Note that ϵ has odd parity and form degree zero. Now, consider the expressions

$$(8.2) \quad \hat{\omega}_{p,q} = S_k((F - \epsilon), [(F - \epsilon)^2]^p, (dF)^q),$$

where $k = p + q + 1$ and p, q are positive integers. Note that the transposition of the elements $(F - \epsilon)$ and dF is odd. Therefore, there are several sign changes involved, if we expand $\hat{\omega}_{p,q}$. Later, we use similar trick as in Section 5 to simplify the signs. We also see the similarity with the Chern-Simons forms, if we think $F - \epsilon$ as a 'flat connection' and dF as a piece of 'supercurvature' of a suitable 'superconnection'. We make this precise later.

If we choose $p = 0$, then

$$(8.3) \quad \begin{aligned} \hat{\omega}_{0,q} &= S_k((F - \epsilon), (dF)^q) = S_k(F, (dF)^q) - d(S_k(\epsilon, F, (dF)^{q-1})) \\ &= F(dF)^q - d(S_k(\epsilon, F, (dF)^{q-1})). \end{aligned}$$

Thus modulo coboundary we get the forms $F(dF)^q$. So, we can always improve the regularity by one degree, since $F - \epsilon$ has order -1 .

In the general case $\hat{\omega}_{p,q}$ are, as pseudodifferential operators, of order $-1 - 2p - q$. Greater the choice of p , more regular is $\hat{\omega}_{p,q}$. Are the forms $\hat{\omega}_{p,q}$ equivalent to the forms of the type $\hat{\omega}_{0,q}$? This question turns out to be far more difficult. We are able to settle it much later, when the tools from Section 11 are available. For now, let us check, that the forms $\hat{\omega}_{p,q}$ can be used to construct cocycles. We have to prove that $d\hat{\omega}_{p,q}$ are either vanishing or are given in terms of (super)commutators.

We compute

$$(8.4) \quad d\hat{\omega}_{p,q} = S_k(dF, [(F - \epsilon)^2]^p, (dF)^q) + pS_k(F - \epsilon, [dF, F - \epsilon], [(F - \epsilon)^2]^{p-1}, (dF)^q),$$

where we have used the definition of the supercommutator in

$$(8.5) \quad d(F - \epsilon)^2 = dF(F - \epsilon) + (F - \epsilon)dF = [dF, F - \epsilon].$$

We integrate by parts to get

$$(8.6) \quad \begin{aligned} &pS_k(F - \epsilon, [dF, F - \epsilon], [(F - \epsilon)^2]^{p-1}, (dF)^q) \\ &= \frac{p}{q+1} [S_k(F - \epsilon, [(F - \epsilon)^2]^{p-1}, (dF)^{q+1}), F - \epsilon] \\ &- \frac{p}{q+1} S_k([F - \epsilon, F - \epsilon], [(F - \epsilon)^2]^{p-1}, (dF)^{q+1}) \\ &= \frac{p}{q+1} [S_k(F - \epsilon, [(F - \epsilon)^2]^{p-1}, (dF)^{q+1}), F - \epsilon] \\ &- \frac{2p}{q+1} S_k([(F - \epsilon)^2]^p, (dF)^{q+1}). \end{aligned}$$

This yields

$$(8.7) \quad \begin{aligned} d\hat{\omega}_{p,q} &= \frac{q+1-2p}{q+1} S_k([(F-\epsilon)^2]^p, (dF)^{q+1}) \\ &\quad + \frac{p}{q+1} [S_k(F-\epsilon, [(F-\epsilon)^2]^{p-1}, (dF)^{q+1}), F-\epsilon]. \end{aligned}$$

Using the $F^2 = 1$ trick, we get

$$(8.8) \quad \begin{aligned} S_k([(F-\epsilon)^2]^p, (dF)^{q+1}) &= \frac{1}{2} [F, F] S_k([(F-\epsilon)^2]^p, (dF)^{q+1}) \\ &= (-1)^{q+1} \frac{1}{2} [F S_k([(F-\epsilon)^2]^p, (dF)^{q+1}), F]. \end{aligned}$$

Combining the above results, we have

$$(8.9) \quad \begin{aligned} d\hat{\omega}_{p,q} &= (-1)^{q+1} \frac{q+1-2p}{q+1} \frac{1}{2} [F S_k([(F-\epsilon)^2]^p, (dF)^{q+1}), F] \\ &\quad + \frac{p}{q+1} [S_k(F-\epsilon, [(F-\epsilon)^2]^{p-1}, (dF)^{q+1}), F-\epsilon]. \end{aligned}$$

The above supercommutators are ordinary commutators when q is even (M odd dimensional). When q is odd (M even dimensional), then the supercommutators are anticommutators.

When we choose p high enough the cocycle property modulo possible boundary terms is satisfied. A choice $-2p - q - 1 \leq -\dim M$ works (use the trace anomaly formula). Thus we have constructed the regularizations for all of the forms $F(dF)^m$ ($m > 0$).

We still need to find out the relation between the forms $F(dF)^m$ and their regularizations. Furthermore, we would like to construct representations in terms of supercommutators, coboundaries and forms depending only on θ for the forms $\hat{\omega}_{p,q}$. It turns out that these questions are more or less the same. The explicit solution however, requires a lot of work and preparation. This preparation keeps us busy for the few next sections. The case when F is not defined is postponed till the end.

9. NONCOMMUTATIVE BRST-COMPLEX

We begin a systematic way to construct cocycles via transgression type arguments. This approach was introduced in [La], [LaMiRy]. The idea is to use a similar Chern-Weil type calculus as in the finite dimensional case. Particularly, we need an analogue of Chern-Simons transgression forms. Recall, that in the finite dimensional case the cocycles (transgression forms when there is a boundary) were precisely Chern-Simons forms.

First, we have to develop a differential graded algebra (DGA for short), where all this machinery of characteristic classes is then applied. Luckily for us, this has been done in [La] and [LaMiRy]. This DGA contains two differentials, \hat{d} and $\hat{\delta}$. Here \hat{d} is an analogue of the de Rham differential and $\hat{\delta}$ is the gauge variation with a certain sign convention to be discussed below. This leads us to the notion of

the non-commutative BRST-complex introduced in [La], [LaMiRy]. This complex comes with several natural gradings. For us, the most important grading is the total \mathbb{Z}_2 -grading. The differentials become odd with respect to this grading.

This complex of 'differential'-forms contains expressions, what we could call 'connections'. These 'connections' are odd with respect to the total \mathbb{Z}_2 -grading. This leads us to a notion of superconnection. The Chern-Weil theory then follows as in the case of the superconnection formalism of Quillen [Q]. Particularly, we define certain 'characteristic classes' for these superconnections, called Chern-forms and Chern-Simons forms. Then, we need to establish some of their basic properties. Particular importance are the descent equations, the triangle formula and certain homotopy invariance formulas.

In this formalism the notion of a trace is part of the integration of 'forms'. Therefore, there is some difference compared to the standard Chern-Weil theory.

After the Chern-Simons forms are defined, the next step is to pair them with regularized traces. This object formed by taking a regularized trace of the Chern-Simons form is called an eta-chain. These eta-chains give us the cocycles and transgression forms in the end.

9.1. BRST-complex. Assume that the manifold M is even dimensional. Following essentially [LaMiRy], [La] we define a subalgebra Ω of $\Psi_c^*(\mathbb{M}/B; S \otimes E)$ (with unit) generated by the elements $a = F - \epsilon, \theta, \epsilon$. The elements of Ω are constructed from the generators by a finite number of additions and compositions. We let the operators act on the Hilbert space \mathcal{H} . For example, the expressions $\epsilon F \theta \epsilon \theta, \epsilon + F \theta$ and $\theta^{32} F \epsilon$ are elements of Ω .

We give the generators degrees 1, 1, 1 (their parities). We define a differential \hat{d} on the generators as follows. Let ω_{\pm} be any generator of parity \pm . Then we put

$$(9.1) \quad \begin{aligned} \hat{d}\omega_+ &= \epsilon\omega_+ - \omega_+\epsilon \\ \hat{d}\omega_- &= \epsilon\omega_- + \omega_-\epsilon. \end{aligned}$$

That is, we define \hat{d} as the graded commutator $\hat{d}\omega_{\pm} = [\epsilon, \omega_{\pm}]$. We denote by Ω_{\pm} the forms that have parity \pm and put $\Omega = \Omega_+ \oplus \Omega_-$. Then it is clear that \hat{d} defines a linear map $\hat{d} : \Omega_{\pm} \rightarrow \Omega_{\mp}$ and satisfies the graded Leibniz rule

$$(9.2) \quad \hat{d}(\omega\eta) = \hat{d}\omega\eta \pm \omega\hat{d}\eta,$$

for any ω in Ω_{\pm} and η in Ω . The operator \hat{d} extends to the whole Ω in an obvious way and also $\hat{d}^2 = 0$, since

$$(9.3) \quad \begin{aligned} \hat{d}\hat{d}\omega_+ &= \hat{d}(\epsilon\omega_+ - \omega_+\epsilon) = \epsilon^2\omega_+ + \epsilon\omega_+\epsilon - \epsilon\omega_+\epsilon - \omega_+\epsilon^2 \\ &= 1\omega_+ - \omega_+1 = 0 \\ \hat{d}\hat{d}\omega_- &= \hat{d}(\epsilon\omega_- + \omega_-\epsilon) = \epsilon^2\omega_- - \epsilon\omega_-\epsilon + \epsilon\omega_-\epsilon - \omega_-\epsilon^2 \\ &= 1\omega_- - \omega_-1 = 0. \end{aligned}$$

Remark 7. In order to make the abstract BRST-algebra compatible with the previous notation, we have to think θ as $\Gamma\theta$. This interpretation is used in [LaMiRy]. This has an effect when we define the supertrace on forms Ω .

Remark 8. Note that $[F, \theta] = F\theta + \theta F$ using the above grading. Using our previous supercommutator $[F, \Gamma\theta] = -\Gamma(F\theta - \theta F)$. Particularly, the Γ -factor needed in the supertrace is automatically included in odd degree forms in the ghost.

We say that a form ω is closed if $\hat{d}\omega = 0$. If $\omega = \hat{d}\eta$, then ω is called exact or \hat{d} -coboundary. Let us look some examples. First take $\omega = \hat{d}a\hat{d}\theta$. Then ω has positive parity and is exact, since $\omega = \hat{d}(a\hat{d}\theta)$. Thus ω is closed. Next, consider ϵ , then $\hat{d}\epsilon = \epsilon^2 + \epsilon^2 = 2$, since ϵ is odd. Finally, if $\omega = \epsilon a\hat{d}\theta$, then ω has positive parity. Thus

$$(9.4) \quad \hat{d}\omega = \hat{d}\epsilon a\hat{d}\theta - \epsilon \hat{d}a\hat{d}\theta - \epsilon a \hat{d}^2\theta = 2a\hat{d}\theta - \epsilon \hat{d}a\hat{d}\theta.$$

Next we introduce the BRST-differential $\hat{\delta}$ (which we have already discussed). We define the differential $\hat{\delta}$ by equations

$$(9.5) \quad \begin{aligned} \hat{\delta}F &= -[F, \theta] = -(F\theta + \theta F) \\ \hat{\delta}\epsilon &= 0. \end{aligned}$$

To make this definition consistent with previous notation; we think

$$(9.6) \quad \hat{\delta} = \Gamma\delta, \theta \rightarrow \Gamma\theta,$$

where δ is the BRST-differential defined earlier. So the above means

$$(9.7) \quad \Gamma\delta F = \Gamma[F, \theta]_- = -[F, \Gamma\theta]_+ = \hat{\delta}F.$$

We need the following formulas.

Proposition 28. For $a = F - \epsilon$ and θ in Ω , the following identities are valid

$$(9.8) \quad \begin{aligned} \hat{\delta}a &= -\hat{d}\theta - [a, \theta] \\ \hat{\delta}\theta &= -\theta^2 \\ f &= \hat{d}a + a^2 = 0. \end{aligned}$$

Here, the commutator is the graded commutator with respect to the \mathbb{Z}_2 -grading of Ω .

Proof. We only prove the first and the third line. By definition

$$(9.9) \quad \hat{\delta}a = \hat{\delta}(F - \epsilon) = \hat{\delta}F = -[F, \theta] = -[F - \epsilon, \theta] - [\epsilon, \theta].$$

To check the vanishing of 'curvature' f of a , we need only to observe that $a^2 = (F - \epsilon)^2 = 2 - [F, \epsilon]$ and

$$(9.10) \quad \hat{d}a = [\epsilon, F - \epsilon] = [\epsilon, F] - [\epsilon, \epsilon] = [\epsilon, F] - 2.$$

□

Remark 9. The above vanishing of curvature is used constantly in the form $\hat{d}a = -a^2$.

Thus Ω becomes a bicomplex, with differentials \hat{d} and $\hat{\delta}$. With respect to $\hat{\delta}$, the bicomplex is \mathbb{N} -graded. This grading is essentially the number of θ s, and is referred as the ghost degree. We denote the projection to the ghost degree m by $\Omega_{[m]}$ and generally in any expression the subscript $[m]$ means the component of degree m in the ghost.

It follows from the above conventions that differentials $\hat{\delta}$ and \hat{d} anticommute. Thus $d = \hat{d} + \hat{\delta}$ defines a total differential on Ω .

We define two sets in Ω . The first is $[\Omega, \Omega]$ consisting of all the supercommutators in Ω . The second is $d\Omega$ consisting of all $\hat{\delta}$ and \hat{d} coboundaries in Ω .

Now, given elements ω and η in Ω of parities m and n respectively, we define the graded commutator using the following sign rule

$$(9.11) \quad [\omega, \eta] = \omega\eta - (-1)^{mn}\eta\omega.$$

This defines the graded commutator in the whole complex by extending linearly.

It is very important to become comfortable with this notion. So we give several examples and computations. First we observe, that for any form ω in Ω

$$(9.12) \quad [\epsilon, \omega] = \hat{d}\omega.$$

For example, consider $\omega = \epsilon a \hat{d}\theta$, then ω is even and ϵ is odd. Thus

$$(9.13) \quad [\epsilon, \omega] = \epsilon\omega - (-1)^{1 \cdot 0}\omega\epsilon = \epsilon\omega - \omega\epsilon = \hat{d}\omega.$$

Consider the following operator in Ω given by $[\theta, \cdot]$. It maps $\omega \rightarrow [\theta, \omega]$, for ω in Ω . It is clear that $[\theta, \cdot]$ defines an odd map with respect to \mathbb{Z}_2 -grading on Ω . Suppose ω has a parity \pm , then we have

$$(9.14) \quad [\theta, \omega\eta] = [\theta, \omega]\eta \pm \omega[\theta, \eta].$$

Thus the operator $[\theta, \cdot]$ defines a graded derivation with respect to the \mathbb{Z}_2 -grading of Ω . Similarly, we have the operators $[\cdot, \theta]$, $[a, \cdot]$ and $[\cdot, a]$. We can also consider general graded derivations $[\omega, \cdot]$, for some fixed ω in Ω .

Example 13 (Switching derivations). Often we end up in a situation where we must switch derivations. For example, from \hat{d} derivations to $[\theta, \cdot]$ derivations or vice versa. By 'switching', we mean the following type computations. Suppose ω is any element of Ω . Then we consider an expression

$$(9.15) \quad \omega \hat{d}\theta = \omega[\epsilon, \theta] = [\omega\epsilon, \theta] + [\omega, \theta]\epsilon.$$

Here we have switched from \hat{d} to $[\cdot, \theta]$. Now, suppose ω is an odd element of Ω , and η is any element of Ω . Then

$$(9.16) \quad \epsilon[\omega, \eta] = -[\omega, \epsilon\eta] + [\omega, \epsilon]\eta = -[\omega, \epsilon\eta] + [\epsilon, \omega]\eta = -[\omega, \epsilon\eta] + \hat{d}\omega\eta.$$

Example 14. Consider the expression θ^2 . Then

$$(9.17) \quad \hat{d}\theta^2 = \hat{d}\theta\theta - \theta\hat{d}\theta = [\hat{d}\theta, \theta].$$

On the other hand

$$(9.18) \quad \hat{d}\theta^2 = -\hat{d}\hat{\delta}\theta = \hat{\delta}\hat{d}\theta.$$

Thus

$$(9.19) \quad \hat{\delta}\hat{d}\theta = [\hat{d}\theta, \theta].$$

Another way is to compute

$$(9.20) \quad \hat{d}\theta^2 = \frac{1}{2}\hat{d}[\theta, \theta] = \frac{1}{2}[\hat{d}\theta, \theta] - \frac{1}{2}[\theta, \hat{d}\theta] = \frac{1}{2}[\hat{d}\theta, \theta] + \frac{1}{2}[\hat{d}\theta, \theta] = [\hat{d}\theta, \theta].$$

The computation above and the following computation are fundamental in Section 11.

Example 15. Consider the expression a^2 . We compute its $\hat{\delta}$ coboundary.

$$(9.21) \quad \begin{aligned} \hat{\delta}a^2 &= -\hat{\delta}\hat{d}a = \hat{d}\hat{\delta}a = -\hat{d}\hat{d}\theta - \hat{d}[a, \theta] = -\hat{d}[a, \theta] = -[\hat{d}a, \theta] + [a, \hat{d}\theta] \\ &= -[\hat{d}a, \theta] - [\hat{d}\theta, a] = +[a^2, \theta] - [\hat{d}\theta, a]. \end{aligned}$$

Particularly, we get

$$(9.22) \quad [a^2, \theta] = \hat{\delta}a^2 + [\hat{d}\theta, a].$$

This is a relation that we need later.

9.2. Superconnections. If we think \hat{d} as the de Rham differential and the operator $\hat{d}_a = \hat{d} + [a, \cdot]$ (acting on Ω) as a covariant derivative, then \hat{d}_a^2 should be interpreted as a curvature. This is, however, zero identically, since by definition

$$(9.23) \quad \begin{aligned} \hat{d}_a^2\omega &= \hat{d}_a\hat{d}\omega + \hat{d}_a[a, \omega] \\ &= \hat{d}^2\omega + [a, \hat{d}\omega] + \hat{d}[a, \omega] + [a, [a, \omega]] \\ &= [a, \hat{d}\omega] + \hat{d}[a, \omega] + [a, [a, \omega]] \\ &= [\hat{d}a, \omega] + \frac{1}{2}[[a, a], \omega] \\ &= [\hat{d}a + \frac{1}{2}[a, a], \omega] \\ &= [0, \omega] = 0. \end{aligned}$$

So, in this sense the expression $F - \epsilon$ is a flat connection. Now, consider the expression $\mathbb{A} = ta$, where t is a real parameter. Define a covariant derivative by

$$(9.24) \quad \hat{d}_{\mathbb{A}} = \hat{d} + [ta, \cdot].$$

We define its curvature by

$$(9.25) \quad \mathbb{F} = \hat{d}\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}] = \hat{d}\mathbb{A} + \mathbb{A}^2.$$

An easy computation gives

$$(9.26) \quad \mathbb{F} = t\hat{d}a + t^2a^2 = -ta^2 + t^2a^2 = (t^2 - t)a^2.$$

These are the simplest examples of superconnections on the complex $(\Omega_{[0]}, \hat{d})$.

Definition 14. A *superconnection* on Ω is a linear map $d_{\mathbb{A}} : \Omega \rightarrow \Omega$ defined by

$$(9.27) \quad d_{\mathbb{A}}\omega = d\omega + [\mathbb{A}, \omega] = \hat{d}\omega + \hat{\delta}\omega + [\mathbb{A}, \omega],$$

where \mathbb{A} is any odd form in Ω and ω is any form on Ω . We consider 0 as an odd and even form on Ω . The even form on Ω

$$(9.28) \quad \mathbb{F} = d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}] = (\hat{d} + \hat{\delta})\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}],$$

is called the *curvature* of the superconnection \mathbb{A} in Ω .

Remark 10. It is convenient to call the odd form \mathbb{A} in the definition of superconnection also a superconnection.

Similarly, for $\mathbb{A} \in \Omega_-$, we also define the *partial superconnections* $\hat{d}_{\mathbb{A}}$ and $\hat{\delta}_{\mathbb{A}}$ by

$$(9.29) \quad \begin{aligned} \hat{d}_{\mathbb{A}}\omega &= \hat{d} + [\mathbb{A}, \omega] \\ \hat{\delta}_{\mathbb{A}}\omega &= \hat{\delta} + [\mathbb{A}, \omega]. \end{aligned}$$

Proposition 29. For ω, η in Ω , superconnections $\mathbb{A} \in \Omega_-$ satisfy the following identities

$$(9.30) \quad \begin{aligned} [\mathbb{A}, \omega\eta] &= [\mathbb{A}, \omega]\eta + (-1)^{\partial\omega}\omega[\mathbb{A}, \eta] \\ d_{\mathbb{A}}(\omega\eta) &= d_{\mathbb{A}}\omega\eta + (-1)^{\partial\omega}\omega d_{\mathbb{A}}\eta \\ d_{\mathbb{A}}^2\omega &= [\mathbb{F}, \omega] \\ d_{\mathbb{A}}\mathbb{F} &= d\mathbb{F} + [\mathbb{A}, \mathbb{F}] = 0, \end{aligned}$$

where $\partial\omega$ means the parity of ω . Here, the last identity is the Bianchi identity.

Proof. This is a standard computation. \square

Observe that we can write the Bianchi identity also in terms of the partial superconnections

$$(9.31) \quad \begin{aligned} \hat{d}\mathbb{F} &= -\hat{\delta}_{\mathbb{A}}\mathbb{F} \\ \hat{\delta}\mathbb{F} &= -\hat{d}_{\mathbb{A}}\mathbb{F}. \end{aligned}$$

Proposition 30 (A variation of a supercurvature). Suppose we have an one parameter family of superconnections $t \rightarrow \mathbb{A}_t$ on Ω , for $t \in [0, 1]$. Then, the following identity holds for the supercurvature \mathbb{F}_t of \mathbb{A}_t

$$(9.32) \quad \dot{\mathbb{F}}_t = d_{\mathbb{A}_t}\dot{\mathbb{A}}_t,$$

where $\partial_t\mathbb{A}_t = \dot{\mathbb{A}}_t$.

Proof. The proof is a simple computation

$$\begin{aligned}
(9.33) \quad \dot{\mathbb{F}}_t &= \partial_t(d\mathbb{A}_t + \frac{1}{2}[\mathbb{A}_t, \mathbb{A}_t]) \\
&= d\partial_t\mathbb{A}_t + \frac{1}{2}[\partial_t\mathbb{A}_t, \mathbb{A}_t] + \frac{1}{2}[\mathbb{A}_t, \partial_t\mathbb{A}_t] \\
&= d\partial_t\mathbb{A}_t + [\mathbb{A}_t, \partial_t\mathbb{A}_t] \\
&= d_{\mathbb{A}_t}\dot{\mathbb{A}}_t.
\end{aligned}$$

□

Example 16. Consider $\mathbb{A} = ta$ on Ω , for $t \in [0, 1]$. Then the curvature is

$$\begin{aligned}
(9.34) \quad \mathbb{F} &= d\mathbb{A} + \mathbb{A}^2 = \hat{d}\mathbb{A} + \hat{\delta}\mathbb{A} + \mathbb{A}^2 \\
&= t\hat{d}a + t\hat{\delta}a + t^2a^2 \\
&= (t^2 - t)a^2 + t\hat{\delta}a.
\end{aligned}$$

Example 17. Next consider $\mathbb{A} = ta + \theta$ on Ω , for $t \in [0, 1]$. Then

$$\begin{aligned}
(9.35) \quad \mathbb{F} &= d\mathbb{A} + \mathbb{A}^2 = \hat{d}\mathbb{A} + \hat{\delta}\mathbb{A} + \mathbb{A}^2 \\
&= t\hat{d}a + \hat{d}\theta + t\hat{\delta}a + \hat{\delta}\theta + t^2a^2 + t[a, \theta] + \theta^2 \\
&= t\hat{d}a + t^2a^2 + (\hat{\delta}\theta + \theta^2) + t(\hat{\delta}a + [a, \theta]) + \hat{d}\theta \\
&= (t^2 - t)a^2 + (-\theta^2 + \theta^2) + t(-\hat{d}\theta - [a, \theta] + [a, \theta]) + \hat{d}\theta \\
&= (t^2 - t)a^2 + (1 - t)\hat{d}\theta.
\end{aligned}$$

Example 18. For $t \in [0, 1]$, the superconnection $\mathbb{A} = t\theta$ on Ω , the corresponding curvature is

$$\begin{aligned}
(9.36) \quad \mathbb{F} &= d\mathbb{A} + \mathbb{A}^2 = \hat{d}\mathbb{A} + \hat{\delta}\mathbb{A} + \mathbb{A}^2 \\
&= t\hat{d}\theta + t\hat{\delta}\theta + t^2\theta \\
&= t\hat{d}\theta - t\theta^2 + t^2\theta \\
&= t\hat{d}\theta + (t^2 - t)\theta^2.
\end{aligned}$$

Example 19. For $t \in [0, 1]$, the superconnection $\mathbb{A} = t(a + \theta)$ on Ω , the corresponding curvature is

$$\begin{aligned}
(9.37) \quad \mathbb{F} &= d\mathbb{A} + \mathbb{A}^2 = \hat{d}\mathbb{A} + \hat{\delta}\mathbb{A} + \mathbb{A}^2 \\
&= (t^2 - t)a^2 + t(\hat{\delta}a + \hat{d}\theta + [a, \theta]) + (t^2 - t)\theta^2, \\
&= (t^2 - t)a^2 + (t^2 - t)\theta^2.
\end{aligned}$$

Example 20. Finally, consider a two-parameter superconnection $\mathbb{A} = t_1 a + t_2 \theta$ on Ω , where $t_1, t_2 \in [0, 1]$. Now, the curvature reads

$$\begin{aligned}
(9.38) \quad \mathbb{F} &= d\mathbb{A} + \mathbb{A}^2 = \hat{d}\mathbb{A} + \hat{\delta}\mathbb{A} + \mathbb{A}^2 \\
&= t_1 \hat{d}a + t_2 \hat{d}\theta + t_1 \hat{\delta}a + t_2 \hat{\delta}\theta + t_1^2 a^2 + t_1 t_2 [a, \theta] + t_2^2 \theta^2 \\
&= (t_1^2 - t_1) a^2 + (t_2^2 - t_2) \theta^2 + t_1 \hat{\delta}a + t_1 t_2 [a, \theta] + t_2 \hat{d}\theta.
\end{aligned}$$

From this superconnection we get all the above superconnections and more as special cases.

The above superconnections are the only ones that we consider now on.

9.3. The total superconnection. The total superconnection can be thought as a super connection on $\Omega^2 \equiv \Omega \otimes \Omega(I_{t_1}) \otimes \Omega(I_{t_2})$, where $\Omega(I)$ denotes forms on the interval $[0, 1]$ of \mathbb{R} . We equip Ω^2 with natural \mathbb{Z}_2 grading coming from \mathbb{Z}_2 -grading of Ω and grading of differential forms of intervals.

First, we define the total differential on Ω^2 by

$$(9.39) \quad \mathbf{d} = d_t + \hat{d} + \hat{\delta} = dt_1 \otimes \partial_{t_1} + dt_2 \otimes \partial_{t_2} + \hat{d} + \hat{\delta},$$

acting on forms on Ω^2 . With respect to the total \mathbb{Z}_2 -grading of Ω^2 , \mathbf{d} is odd.

The *total superconnection* is by definition a covariant derivative (acting on Ω^2)

$$(9.40) \quad \mathbf{d}_{\mathbf{A}} = \mathbf{d} + [\mathbf{A}, \cdot],$$

where the commutator is the same supercommutator on Ω as above, and

$$(9.41) \quad \mathbf{A} = t_1 a + t_2 \theta = \mathbb{A}(t_1, t_2).$$

If there is no danger of confusion, we simply denote \mathbf{A} by \mathbb{A} . We also call the superconnection \mathbf{A} as the total superconnection.

The curvature is defined as usual

$$\begin{aligned}
(9.42) \quad \mathbf{F} &= \mathbf{d}\mathbf{A} + \mathbf{A}^2 \\
&= dt_1 a + dt_2 \theta + (t_1^2 - t_1) a^2 + t_1 \hat{\delta}a + t_1 t_2 [a, \theta] + t_2 \hat{d}\theta + (t_2^2 - t_2) \theta^2 \\
&= d_t \mathbb{A} + \mathbb{F}.
\end{aligned}$$

Note that the curvature contains components that are odd with respect to the \mathbb{Z}_2 -grading of Ω .

Observe that

$$(9.43) \quad \mathbf{F}_{[0]} = dt_1 \partial_1 \mathbb{A} + \mathbb{F}_{[0]} = dt_1 a + (t_1^2 - t_1) a^2,$$

where $\partial_1 = \partial_{t_1}$ and $\partial_2 = \partial_{t_2}$, is a pseudodifferential operator valued form whose components have orders -1 and -2 . The component of ghost degree one is

$$(9.44) \quad \mathbf{F}_{[1]} = dt_2 \partial_2 \mathbb{A} + \mathbb{F}_{[1]} = dt_2 \theta + t_1 \hat{\delta}a + t_1 t_2 [a, \theta] + t_2 \hat{d}\theta,$$

that has components of order 0 and -1 . The degree two part in the ghost is

$$(9.45) \quad \mathbf{F}_{[2]} = \mathbb{F}_{[2]} = (t_2^2 - t_2) \theta^2,$$

that has order 0.

The total superconnection satisfies the usual identities as before. Particularly, we have the Bianchi identity $\mathbf{d}_\mathbb{A}\mathbf{F} = 0$.

9.4. Superconnection character forms. The Chern-Weil type forms on the complex Ω are now introduced. These forms are, as in the standard case, polynomial expressions on the supercurvature \mathbb{F} of some superconnection \mathbb{A} on Ω . The difference to the ordinary case is that we do not take the trace. We first define the basic Chern-forms.

Definition 15. Let \mathbb{A} be any superconnection on Ω and let k be a positive integer. We define the basic Chern-form of degree k by

$$(9.46) \quad c_k(\mathbb{A}) = \mathbb{F}^k = S_k(\mathbb{F}, \dots, \mathbb{F}).$$

Here we use the familiar symmetrization operator but with different grading rule (in this expression every form is even so there is no sign changes). The sign rule comes from the parity grading of Ω . The substitutions $\theta \rightarrow \Gamma\theta$ and $\delta \rightarrow \Gamma\delta$ can be used to return to the original sign rule (6.4). However, in our case the parity grading on Ω yields simpler calculations.

When a superconnection depends on a parameter t running from 0 to 1, it is useful to define the following boundary operator acting on the Chern form as follows

$$(9.47) \quad \partial c_k(\mathbb{A}_t) = c_k(\mathbb{A}_1) - c_k(\mathbb{A}_0).$$

Example 21. Consider the superconnection $\mathbb{A}_t = ta + \theta$ on Ω , where $t \in [0, 1]$. Then $\mathbb{A}_1 = a + \theta$ and $\mathbb{A}_0 = \theta$. It is easy to see that \mathbb{A}_1 is flat and that $\mathbb{F}_0 = \hat{d}\theta$. This gives us

$$(9.48) \quad \partial c_k(ta + \theta) = c_k(a + \theta) - c_k(\theta) = -(\hat{d}\theta)^k.$$

Example 22. Now, consider the superconnection $\mathbb{A}_t = t\theta$ on Ω . Then $\mathbb{A}_1 = \theta$ and $\mathbb{A}_0 = 0$. Thus $\mathbb{F}_0 = \hat{d}\theta$ and $\mathbb{F}_1 = 0$. Therefore, we get

$$(9.49) \quad \partial c_k(t\theta) = c_k(\theta) - c_k(0) = (\hat{d}\theta)^k.$$

If we combine the paths above, that is, first go from $a + \theta$ to θ and then to 0, we get the trivial Chern-form. When we combine paths as above it is useful to use a notation

$$(9.50) \quad c_k(\mathbb{A}_0, \mathbb{A}_1, \dots, \mathbb{A}_m)$$

to denote the endpoints of paths. Then the boundary operator is

$$(9.51) \quad \partial c_k(\mathbb{A}_0, \mathbb{A}_1, \dots, \mathbb{A}_m) = c_k(\mathbb{A}_m) - c_k(\mathbb{A}_0).$$

Similarly, we define Chern-forms for the total superconnection.

A basic property of Chern forms is that they are d -closed modulo commutators. This follows from the Bianchi identity

$$\begin{aligned}
d\mathbb{F}^k &= d_{\mathbb{A}}(\mathbb{F}^k) - [\mathbb{A}, \mathbb{F}^k] \\
(9.52) \quad &= \sum_{m=0}^{k-1} (\mathbb{F}^m)(d_{\mathbb{A}}\mathbb{F})(\mathbb{F}^{k-m-1}) - [\mathbb{A}, \mathbb{F}^k] \\
&= -[\mathbb{A}, \mathbb{F}^k].
\end{aligned}$$

Proposition 31. Let \mathbb{A} be any superconnection on Ω and let k be a positive integer. Then the corresponding Chern-forms satisfy

$$(9.53) \quad dc_k(\mathbb{A}) = -[\mathbb{A}, c_k(\mathbb{A})].$$

This can be, also, read as

$$(9.54) \quad \hat{\delta}c_k(\mathbb{A}) = -\hat{d}_{\mathbb{A}}c_k(\mathbb{A}) = -\hat{d}c_k(\mathbb{A}) - [\mathbb{A}, c_k(\mathbb{A})].$$

Proof.

□

Proposition 32. Let \mathbb{A} be a superconnection on Ω depending on a parameter $t \in \mathbb{R}$. Then the corresponding Chern-forms satisfy the following transgression formula

$$(9.55) \quad \partial_t c_k(\mathbb{A}) = d_{\mathbb{A}}\omega_k^1(\mathbb{A}),$$

where $\omega_k \in \Omega$ is called a transgression form or a Chern-Simons form.

Proof. This is a standard computation using the Bianchi identity and the formula (9.32)

$$\begin{aligned}
(9.56) \quad \partial_t c_k(\mathbb{A}) &= kS_k(\dot{\mathbb{F}}, \mathbb{F}^{k-1}) \\
&= kS_k(d_{\mathbb{A}}\dot{\mathbb{A}}, \mathbb{F}^{k-1}) \\
&= kd_{\mathbb{A}}S_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1}).
\end{aligned}$$

Thus

$$(9.57) \quad \omega_k^1(\mathbb{A}) = kS_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1}).$$

□

We also use the integrated version of the transgression formula

$$(9.58) \quad c_k(\mathbb{A}_1) - c_k(\mathbb{A}_0) = \int_0^1 d_{\mathbb{A}}\omega_k^1(\mathbb{A}),$$

or

$$(9.59) \quad \partial c_k(\mathbb{A}) = \int_0^1 d_{\mathbb{A}}\omega_k^1(\mathbb{A}).$$

Definition 16. Let \mathbb{A} be any superconnection on Ω depending on parameter $t \in [0, 1]$ and let $k \geq 1$ be an integer. We call the expression

$$(9.60) \quad c_k^1(\mathbb{A}) = \int_0^1 \omega_k^1(\mathbb{A}) = k \int_0^1 S_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1}),$$

as the (integrated) *Chern-Simons form* associated with the superconnection \mathbb{A} .

Remark 11. When dealing with Chern-Simons forms, we usually assume that the corresponding superconnection \mathbb{A} depends on parameter $t \in [0, 1]$. If the situation demands it, we use the notation \mathbb{A}_t .

We always assume that the variable k in the definition of Chern-Simons forms is an integer greater than one.

The transgression formula can also be written as

$$(9.61) \quad dc_k^1(\mathbb{A}) = \partial c_k(\mathbb{A}) - \int_0^1 [\mathbb{A}, \omega_k^1(\mathbb{A})].$$

That is, we have

$$(9.62) \quad dc_k^1(\mathbb{A}) = \partial c_k(\mathbb{A}) + [\Omega, \Omega].$$

Example 23. For $t \in [0, 1]$, an integer $k \geq 1$ and an integer m , $0 \leq m \leq k - 1$, consider the superconnection $\mathbb{A} = ta + \theta$ on Ω , then

$$(9.63) \quad \omega_k^1(\mathbb{A}) = k S_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1}) = k S_k(a, ((t^2 - t)a^2 + (1 - t)\hat{d}\theta)^{k-1}).$$

Projection to the ghost degree m reads

$$(9.64) \quad \begin{aligned} \omega_{k,[m]}^1(\mathbb{A}) &= S_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1})_{[m]} \\ &= k \binom{k-1}{m} (t^2 - t)^{k-m-1} (1-t)^m S_k(a, (a^2)^{k-m-1}, (\hat{d}\theta)^m) \\ &= (-1)^{k-m-1} \binom{k-1}{m} t^{k-m-1} (1-t)^{k-1} S_k(a, (a^2)^{k-m-1}, (\hat{d}\theta)^m). \end{aligned}$$

This gives

$$(9.65) \quad c_{k,[m]}^1(\mathbb{A}) = \phi_{k,m} S_k(a, (a^2)^{k-m-1}, (\hat{d}\theta)^m),$$

where

$$(9.66) \quad \phi_{k,m} = \binom{k-1}{m} (-1)^{n-m-1} B(n-m, n).$$

Here $B(n-m, n)$ denotes the standard beta function

$$(9.67) \quad B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1}.$$

Put $m = k - 1$ to get

$$(9.68) \quad \omega_{k,[k-1]}^1(\mathbb{A}) = k(1-t)^{k-1} S_k(a, (\hat{d}\theta)^{k-1}).$$

Integration gives

$$(9.69) \quad c_{k,[k-1]}^1(\mathbb{A}) = S_k(a, (\hat{d}\theta)^{k-1}).$$

Example 24. For $t \in [0, 1]$ and k, m as above, consider the superconnection $\mathbb{A} = ta$ on Ω , then

$$(9.70) \quad \omega_k^1(\mathbb{A}) = kS_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1}) = kS_k(a, ((t^2 - t)a^2 + t\hat{d}a)^{k-1}).$$

This yields

$$(9.71) \quad \begin{aligned} \omega_{k,[m]}^1(\mathbb{A}) &= kS_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1})_{[m]} \\ &= k \binom{k-1}{m} (t^2 - t)^{k-m-1} t^m S_k(a, (a^2)^{k-m-1}, (\hat{d}a)^m) \\ &= (-1)^{k-m-1} k \binom{k-1}{m} t^{k-1} (1-t)^{k-m-1} S_k(a, (a^2)^{k-m-1}, (\hat{d}a)^m). \end{aligned}$$

Integrated form is

$$(9.72) \quad c_{k,[m]}^1(\mathbb{A}) = \phi_{k,m} S_k(a, (a^2)^{k-m-1}, (\hat{d}a)^m),$$

where

$$(9.73) \quad \phi_{k,m} = (-1)^{n-m-1} k \binom{k-1}{m} B(n-m, n).$$

Particularly, when $m = k - 1$, we have

$$(9.74) \quad \omega_{k,[k-1]}^1(\mathbb{A}) = kt^{k-1} S_k(a, (\hat{d}a)^{k-1})$$

and

$$(9.75) \quad c_{k,[k-1]}^1(\mathbb{A}) = S_k(a, (\hat{d}a)^{k-1}).$$

Note that

$$(9.76) \quad \begin{aligned} c_{k,[k-1]}^1(\mathbb{A}) &= S_k(F - \epsilon, ([F, \theta])^{k-1}) \\ &= S_k(F, ([F, \theta])^{k-1}) - \hat{\delta} S_k(\epsilon, F, ([F, \theta])^{k-2}). \end{aligned}$$

Thus, we end up studying already familiar forms of the type² $F[F, \theta]^{2k+1}$.

Example 25. For $t \in [0, 1]$, k as before and an integer m , $k \leq m \leq 2k - 1$, consider the superconnection $\mathbb{A} = t\theta$ on Ω , then

$$(9.77) \quad \omega_k^1(\mathbb{A}) = S_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1}) = S_k(\theta, ((t^2 - t)\theta^2 + t\hat{d}\theta)^{k-1}).$$

²Remember the convention $\theta \rightarrow \Gamma\theta!$

Especially, we have

$$\begin{aligned}
(9.78) \quad \omega_{k,[m]}^1(\mathbb{A}) &= kS_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1})_{[m]} \\
&= k \binom{k-1}{m-k} (t^2 - t)^{m-k} t^{2k-m-1} S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^{2k-m-1}) \\
&= (-1)^{m-k} (1-t)^{m-k} k \binom{k-1}{m-k} t^{k-1} S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^{2k-m-1}).
\end{aligned}$$

Integrated form is

$$(9.79) \quad c_{k,[m]}^1(\mathbb{A}) = \phi_{k,m} S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^{2k-m-1}),$$

where

$$(9.80) \quad \phi_{k,m} = (-1)^{m-k} B(m-k+1, k) k \binom{k-1}{m-k}.$$

When $m = k$, we get

$$(9.81) \quad \omega_{m,[m]}^1(\mathbb{A}) = mt^{m-1} S_m(\theta, (\hat{\delta}a)^{m-1})$$

and

$$(9.82) \quad c_{m,[m]}^1(\mathbb{A}) = S_m(\theta, (\hat{d}\theta)^{m-1}).$$

The importance of the Chern-Simons forms is that they give us a source of possible cocycles. This follows directly from the transgression formula, since it reads (when the contribution from the Chern-form can be ignored)

$$(9.83) \quad \hat{\delta}c_k^1(\mathbb{A}) = \hat{d}\Omega + [\Omega, \Omega].$$

The message here is that modulo $\hat{d}\Omega + [\Omega, \Omega]$ the Chern-Simons form satisfies cocycle property with respect to $\hat{\delta}$. This is the key observation in the construction of $\hat{\delta}$ -cocycles.

As an immediate consequence, we get the noncommutative descent equations [LaMiRy], [La] .

Corollary 2 (Descent equations). Let \mathbb{A} be a superconnection on Ω depending on a parameter $t \in [0, 1]$. The Chern-Simons forms $c_k^1(\mathbb{A})$ satisfy a system of equations, called the (noncommutative) *descent equations*,

$$(9.84) \quad \hat{d}c_{k,[m]}^1(\mathbb{A}) + \hat{\delta}c_{k,[m-1]}^1(\mathbb{A}) = \partial c_k(\mathbb{A})_{[m]} - \int_0^1 [\mathbb{A}, \omega_k^1(\mathbb{A})]_{[m]},$$

where m is a positive integer.

Proof. This is just the projection of the equation (9.61) to the ghost degree m . \square

9.5. The triangle formula. There is a slightly different way to approach Chern-Simons forms. First, consider any superconnection \mathbb{A} on Ω parametrized by an unit interval, which we denote by Δ^1 . We think \mathbb{A} as a superconnection pulled back from the total superconnection \mathbf{A} via this parametrization. Now, when we form the pulled back curvature, we have to pull back the differential d_t in the definition of supercurvature \mathbf{F} .

We can now define the Chern-Simons form associated with the superconnection \mathbb{A} (with slight abuse of notation) by

$$(9.85) \quad c_k^1(\mathbb{A}) = \int_{\Delta^1} \mathbf{F}^k = \int_{\Delta^1} S_k(\mathbf{F}, \dots, \mathbf{F}).$$

We also use a notation $c_k^1(\Delta^1)$. When we consider affine parametrizations between superconnections \mathbb{A}_0 and \mathbb{A}_1 we denote the corresponding Chern-Simons forms by $c_k^1(\mathbb{A}_0, \mathbb{A}_1)$. In this case we can write

$$(9.86) \quad c_k^1(\mathbb{A}_0, \mathbb{A}_1) = \int_0^1 ds \omega_k^1(\mathbb{A}_0, \mathbb{A}_1),$$

where

$$(9.87) \quad \omega_k^1(\mathbb{A}_0, \mathbb{A}_1) = k S_k(\partial_s \mathbb{A}, \mathbb{F}^{k-1}),$$

and \mathbb{A} is the s-dependent superconnection

$$(9.88) \quad \mathbb{A} = \mathbb{A}_0 + (\mathbb{A}_1 - \mathbb{A}_0)s,$$

whose curvature is \mathbb{F} .

Note that $c_k^1(\mathbb{A}_0, \mathbb{A}_1) = -c_k^1(\mathbb{A}_1, \mathbb{A}_0)$.

The transgression formula can be written, using Stokes theorem,

$$(9.89) \quad \begin{aligned} dc_k^1(\mathbb{A}_0, \mathbb{A}_1) &= - \int_{\Delta^1} d_t \mathbf{F}^k - \int_{\Delta^1} [\mathbf{A}, \mathbf{F}^k] \\ &= - \int_{\partial \Delta^1} \mathbf{F}^k - \int_{\Delta^1} [\mathbf{A}, \mathbf{F}^k] \\ &= c_k(\partial \Delta^1) - \int_{\Delta^1} ds [\mathbb{A}, \omega_k^1(\mathbb{A})]. \end{aligned}$$

It is also useful to denote

$$(9.90) \quad \begin{aligned} c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2) &= c_k^1(\mathbb{A}_0, \mathbb{A}_1) + c_k^1(\mathbb{A}_1, \mathbb{A}_2) \\ c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3) &= c_k^1(\mathbb{A}_0, \mathbb{A}_1) + c_k^1(\mathbb{A}_1, \mathbb{A}_2) + c_k^1(\mathbb{A}_2, \mathbb{A}_3) \end{aligned}$$

and so on.

Next, we need to define higher Chern-Simons forms. These are defined by pulling back the superconnection \mathbf{A} , via suitable parametrization, to the two dimensional standard simplex and integrating over it. The parametrizations are given by the vertices $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2 \in \Omega_-$ as follows.

Let $\Delta^2 = \{(s_1, s_2) | 0 \leq s_1 + s_2 \leq 1\}$ denote the standard 2-simplex. We identify this 2-simplex with a triangle in Ω_- given by the vertices $(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ (order is important) and parametrized by $(s_1, s_2) \in \Delta^2$ using

$$(9.91) \quad \mathbf{A}(\Delta^2) = \mathbb{A}_0 + (\mathbb{A}_1 - \mathbb{A}_0)s_1 + (\mathbb{A}_2 - \mathbb{A}_1)s_2.$$

The curvature is

$$(9.92) \quad \begin{aligned} \mathbf{F}(\Delta^2) &= \mathbf{d}\mathbf{A}(\Delta^2) + \frac{1}{2}[\mathbf{A}(\Delta^2), \mathbf{A}(\Delta^2)] \\ &= d_s \mathbf{A}(\Delta^2) + d\mathbf{A}(\Delta^2) + \frac{1}{2}[\mathbf{A}(\Delta^2), \mathbf{A}(\Delta^2)] \\ &= (\mathbb{A}_1 - \mathbb{A}_0)ds_1 + (\mathbb{A}_2 - \mathbb{A}_1)ds_2 + d\mathbf{A}(\Delta^2) + \frac{1}{2}[\mathbf{A}(\Delta^2), \mathbf{A}(\Delta^2)] \\ &= (\mathbb{A}_1 - \mathbb{A}_0)ds_1 + (\mathbb{A}_2 - \mathbb{A}_1)ds_2 + \mathbb{F}(\Delta^2), \end{aligned}$$

where

$$(9.93) \quad \mathbb{F}(\Delta^2) = d\mathbf{A}(\Delta^2) + \frac{1}{2}[\mathbf{A}(\Delta^2), \mathbf{A}(\Delta^2)].$$

We usually drop Δ^2 from the curvature and connection above.

We define the boundary operator ∂ acting on triangles $(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ as above by

$$(9.94) \quad \partial(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2) = -(\hat{\mathbb{A}}_0, \mathbb{A}_1, \mathbb{A}_2) + (\mathbb{A}_0, \hat{\mathbb{A}}_1, \mathbb{A}_2) - (\mathbb{A}_0, \mathbb{A}_1, \hat{\mathbb{A}}_2),$$

where $\hat{\mathbb{A}}_i$ means that the corresponding vertex is omitted. We identify $\partial(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ with $\partial\Delta^2$. Moreover, each element on the right-hand side can be identified with Δ . Particularly, we identify

$$(9.95) \quad c_k^1(\partial\Delta^2) = c_k^1(\mathbb{A}_0, \mathbb{A}_1) + c_k^1(\mathbb{A}_1, \mathbb{A}_2) + c_k^1(\mathbb{A}_2, \mathbb{A}_0) = c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_0).$$

Definition 17. Let Δ^2 be given by the ordered triple of superconnections $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2$ as explained above. We define the *second order Chern-Simons form* $c_k^2(\Delta^2)$ on Ω , for any integer $k \geq 2$ with the formula

$$(9.96) \quad c_k^2(\Delta^2) = \int_{\Delta^2} \mathbf{F}^k,$$

where $\mathbf{F} = \mathbf{F}(\Delta^2)$ as above.

We can write

$$(9.97) \quad c_k^2(\Delta^2) = \int_{\Delta^2} ds_1 ds_2 \omega_k^2(\mathbf{A}(\Delta^2)),$$

where ω_k^2 takes the following form

$$(9.98) \quad \omega_k^2(\mathbf{A}(\Delta^2)) = k(k-1)S_k(\partial_1 \mathbf{A}(\Delta^2), \partial_2 \mathbf{A}(\Delta^2), \mathbb{F}^{k-2}(\Delta^2)),$$

where $\partial_1 = \partial_{s_1}$ and $\partial_2 = \partial_{s_2}$.

It is not usually necessary to use the pulled back version of the total curvature. It is easier to work with the total superconnection and do the pull back in the end.

Usually, it is more convenient to denote the forms $c_k^2(\Delta^2)$ by the expression (by using our identifications)

$$(9.99) \quad c_k^2(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2),$$

where the integral is over the triangle defined by the vertices $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2$. This form is of importance in the homotopy formula of Chern-Simons-forms, especially in the triangle formula.

Example 26. Consider the triangle $\Delta = (0, \theta, a + \theta)$ on Ω , then

$$(9.100) \quad \mathbf{A}(\Delta^2) = s_1\theta + s_2a,$$

where s_1 and s_2 are as above.

We have

$$(9.101) \quad \begin{aligned} c_k^2(\Delta^2) &= \int_{\Delta^2} \mathbf{F}^k \\ &= k(k-1) \int_{\Delta^2} S_k(ds_1\partial_1\mathbf{A}, ds_2\partial_2\mathbf{A}, \mathbb{F}^{k-2}) \\ &= k(k-1) \int_0^1 ds_1 \int_0^{1-s_1} ds_2 S_k(\theta, a, \mathbb{F}^{k-2}), \end{aligned}$$

where

$$(9.102) \quad \mathbb{F} = (s_2^2 - s_2)a^2 + (s_1^2 - s_1)\theta^2 + s_2\hat{\delta}a + s_1s_2[a, \theta] + s_1\hat{d}\theta.$$

We now prove a relation of the form $dc_k^2(\Delta^2) = -c_k^1(\partial\Delta^2)$ modulo commutators. This follows from the Stokes theorem and from the Bianchi identity. Explicitly, we have

$$(9.103) \quad \begin{aligned} dc_k^2(\Delta^2) &= (\hat{d} + \hat{\delta}) \int_{\Delta^2} \mathbf{F}^k = \int_{\Delta^2} (d_{\mathbf{A}}\mathbf{F}^k - d_t\mathbf{F}^k - [\mathbf{A}, \mathbf{F}^k]) \\ &= \int_{\Delta^2} (-d_t\mathbf{F}^k - [\mathbf{A}, \mathbf{F}^k]) \\ &= - \int_{\partial\Delta^2} \mathbf{F}^k - \int_{\Delta^2} [\mathbf{A}, \mathbf{F}^k]. \end{aligned}$$

Thus $dc_k^2(\Delta^2) = -c_k^1(\partial\Delta^2)$ modulo commutators. Rearranging terms in the above computation yields

$$(9.104) \quad \int_{\Delta^2} ((\hat{d} + \hat{\delta})\mathbf{F}^k + [\mathbf{A}, \mathbf{F}^k]) = - \int_{\partial\Delta^2} \mathbf{F}^k.$$

That is

$$(9.105) \quad \int_{\Delta^2} d_{\mathbf{A}}\mathbf{F}^k = - \int_{\partial\Delta^2} \mathbf{F}^k.$$

The above formula gives us the triangle formula.

Proposition 33 (Triangle formula). Suppose that a simplex Δ^2 is given by the ordered triple $(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ of superconnections on Ω . Then

$$(9.106) \quad \int_{\partial\Delta^2} \mathbf{F}^k = c_k^1(\mathbb{A}_0, \mathbb{A}_1) + c_k^1(\mathbb{A}_1, \mathbb{A}_2) + c_k^1(\mathbb{A}_2, \mathbb{A}_0) = c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_0).$$

That is

$$(9.107) \quad \int_{\Delta^2} d_{\mathbb{A}} \mathbf{F}^k = - \int_{\partial\Delta^2} \mathbf{F}^k = c_k(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_0).$$

Proof.

□

Remark 12. Note the resemblance with the Chern-Simons forms $c_k^1(\mathbb{A})$ on Ω . Particularly, the definition is a direct higher dimensional analogue

$$(9.108) \quad c_k^n(\Delta^n) = \int_{\Delta^n} \mathbf{F}^k,$$

$n = 1$ is the case of the standard Chern-Simons forms and $n = 2$ is of its higher dimensional analogue. Corresponding transgression formulas are also similar

$$(9.109) \quad \int_{\Delta^n} d_{\mathbb{A}} \mathbf{F}^k = - \int_{\partial\Delta^n} \mathbf{F}^k.$$

The triangle formula becomes important when we compare different Chern-Simons forms. The triangle formula itself is a particular case of homotopy invariance property of the Chern-Simons forms. To understand this we use the following set up. We consider two fixed superconnections $\mathbb{A}_0, \mathbb{A}_1$ on Ω . These will be the endpoints of two different paths of superconnections. We take these paths to be of the special form.

Consider Chern-Simons forms of the following type

$$(9.110) \quad \begin{aligned} \eta_1 &= c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2) = c_k^1(\mathbb{A}_0, \mathbb{A}_1) + c_k^1(\mathbb{A}_1, \mathbb{A}_2), \\ \eta_2 &= c_k^1(\mathbb{A}_0, \mathbb{A}_3, \mathbb{A}_2) = c_k^1(\mathbb{A}_0, \mathbb{A}_3) + c_k^1(\mathbb{A}_3, \mathbb{A}_2). \end{aligned}$$

We want a manageable expression for the difference $\eta_1 - \eta_2$. This expression can be found as follows.

We add and subtract the term $c_k^1(\mathbb{A}_2, \mathbb{A}_0)$ and apply the triangle formula to get

$$\begin{aligned}
\eta_1 - \eta_2 &= c_k^1(\mathbb{A}_0, \mathbb{A}_1) + c_k^1(\mathbb{A}_1, \mathbb{A}_2) + c_k^1(\mathbb{A}_2, \mathbb{A}_0) \\
&\quad - c_k^1(\mathbb{A}_2, \mathbb{A}_0) - c_k^1(\mathbb{A}_0, \mathbb{A}_3) - c_k^1(\mathbb{A}_3, \mathbb{A}_2) \\
&= c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_0) - c_k^1(\mathbb{A}_0, \mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_0) \\
&= \int_{\partial\Delta_{012}^2} \mathbf{F}^k - \int_{\partial\Delta_{032}^2} \mathbf{F}^k \\
(9.111) \quad &= \int_{\Delta_{012}^2} d_{\mathbb{A}} \mathbf{F}^k - \int_{\Delta_{032}^2} d_{\mathbb{A}} \mathbf{F}^k \\
&= \int_{I^2} d_{\mathbb{A}} \mathbf{F}^k \\
&= \int_{I^2} d_{\mathbb{A}} \omega_k^2(\mathbf{A}).
\end{aligned}$$

In the above calculation $\partial\Delta_{012}$ means the boundary of the triangle corresponding to the term $c^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ and similarly for $\partial\Delta_{032}$. The integral over I^2 means integral over the union of the above triangles.

The message here is that

$$(9.112) \quad \eta_1 - \eta_2 = d\Omega + [\Omega, \Omega].$$

This means that $\eta_1 - \eta_2$ is $\hat{\delta}$ -coboundary modulo $\hat{d}\Omega + [\Omega, \Omega]$. Particularly we have

$$(9.113) \quad \eta_1 - \eta_2 = \hat{\delta} \int_{I^2} \omega_k^2(\mathbf{A}) + \int_{I^2} [\mathbf{A}, \omega_k^2(\mathbf{A})] + \hat{d}\Omega.$$

The above formula gives us a tool that allows us to compare two Chern-Simons forms, when we are in the situation as above.

Proposition 34 (Homotopy invariance of Chern-Simons forms). For superconnections $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2$ and \mathbb{A}_3 on Ω , the Chern-Simons form $c_k^1(\mathbb{A}_0, \mathbb{A}_2)$ is independent of the chosen path between \mathbb{A}_0 and \mathbb{A}_2 in the following sense. The Chern-Simons form $c_k^1(\mathbb{A}_0, \mathbb{A}_2)$ satisfies

$$\begin{aligned}
(9.114) \quad c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2) &= c_k^1(\mathbb{A}_0, \mathbb{A}_2) + d\Omega + [\Omega, \Omega] \\
c_k^1(\mathbb{A}_0, \mathbb{A}_3, \mathbb{A}_2) &= c_k^1(\mathbb{A}_0, \mathbb{A}_2) + d\Omega + [\Omega, \Omega],
\end{aligned}$$

where I^2 has the same meaning as above.

We have an explicit relation between these Chern-Simons forms. Particularly, we have

$$(9.115) \quad c_k^1(\mathbb{A}_0, \mathbb{A}_3, \mathbb{A}_2) - c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2) = d \int_{I^2} \omega_k^2(\mathbf{A}) + \int_{I^2} [\mathbf{A}, \omega_k^2(\mathbf{A})].$$

Proof. Note that the relation between $c_k^1(\mathbb{A}_0, \mathbb{A}_2)$ and $c_k^1(\mathbb{A}_0, \mathbb{A}_4, \mathbb{A}_2)$ is just the triangle formula. The relation between $c_k^1(\mathbb{A}_0, \mathbb{A}_3, \mathbb{A}_2)$ and $c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ was shown above. \square

Now, let us give some examples. Note that the higher 'Chern-Simons density' $\omega_k^2(\mathbf{A})$ is always the same, when it is not pulled back. Let us compute this. By definition the Chern-Simons density $\omega_k^2(\mathbf{A})$ is the component of supercurvature \mathbf{F}^k proportional to $dt_1 dt_2$. Thus

$$(9.116) \quad \begin{aligned} \omega_k^2(\mathbf{A}) &= dt_1 dt_2 k(k-1) S_k(\partial_1 \mathbf{A}, \partial_2 \mathbf{A}, \mathbb{F}^{k-2}) \\ &= dt_1 dt_2 k(k-1) S_k(a, \theta, \mathbb{F}^{k-2}). \end{aligned}$$

We usually drop the measure $dt_1 dt_2$ to simplify notation. Now, we can pull $\omega_k^2(\mathbf{A})$ back to Δ^2 , if needed. It is this way, usually, how we think forms $c_k^2(\Delta^2)$.

We can further project this into a fixed ghost degree m to get

$$(9.117) \quad \omega_{k,[m]}^2(\mathbf{A}) = \sum_{m_1+2m_2=m-1} \frac{k(k-1)(k-2)}{m_1! m_2! (k-m_1-m_2)!} S_k(a, \theta, \mathbb{F}_{[0]}^{k-2-m_1-m_2}, \mathbb{F}_{[1]}^{m_1}, \mathbb{F}_{[2]}^{m_2}).$$

Here, the sum means all the non negative integers m_1, m_2 that satisfies $m_1 + 2m_2 = m - 1$ and $k - 2 - m_1 - m_2 \geq 0$ (or take convention that a negative power of a supercurvature is zero). Then the above expression contains all the terms of degree m in the ghost. Let us consider few special cases.

Example 27. First, consider the case $m = 1$. Then we have

$$(9.118) \quad \begin{aligned} \omega_{k,[1]}^2(\mathbf{A}) &= k(k-1) S_k(a, \theta, \mathbb{F}_{[0]}^{k-3}) \\ &= k(k-1) (t_1^2 - t_1)^{k-3} S_k(a, \theta, (a^2)^{k-3}). \end{aligned}$$

We see later that this term relates two different expressions for the Schwinger term, relevant in odd dimensional manifolds.

Example 28. Next consider the case $m = k - 2$. Then we have

$$(9.119) \quad \omega_{k,[k-2]}^2(\mathbf{A}) = \sum_{m_1+2m_2=k-3} \frac{k(k-1)(k-2)}{m_1! m_2! (k-2-m_1-m_2)!} S_k(a, \theta, \mathbb{F}_{[0]}^{k-2-m_1-m_2}, \mathbb{F}_{[1]}^{m_1}, \mathbb{F}_{[2]}^{m_2}).$$

This expression appears when we compare Chern-Simons forms of type $S_k(a, (\hat{d}\theta)^{k-1})$ and $S_k(a, (\hat{\delta}a)^{k-1})$.

Corollary 3. Consider the superconnections $\mathbb{A} = ta + \theta$ and $\mathbb{A}' = ta$ on Ω , where $t \in [0, 1]$, then

$$(9.120) \quad c_{k,[m]}^1(\mathbb{A}') - c_{k,[m]}^1(\mathbb{A}) = (d \int_{I^2} \omega_k^2(\mathbf{A}) + \int_{I^2} [\mathbf{A}, \omega_k^2(\mathbf{A})])_{[m]},$$

where $0 \leq m < k$ and the integration is over the unit square in the coordinates (t_1, t_2) and the total superconnection is given in the form as in its definition.

Proof. This is a direct application of the homotopy invariance. Choose $\mathbb{A}_0 = 0, \mathbb{A}_1 = \theta, \mathbb{A}_3 = a, \mathbb{A}_2 = a + \theta$ and apply Proposition 34 and project to the ghost degree m . \square

Looking back to the formulas of the Chern-Simons forms gives us immediately the equivalence of the forms of the type $F(dF)^m$ (in the old notation) and $S_{m+1}(a, (\hat{d}\theta)^m)$ and their 'regularizations' modulo $d\Omega + [\Omega, \Omega]$, when restricted to the vertical directions.

10. ETA-CHAINS AND ETA-COCYCLES

Integration of the forms Ω is now introduced. This is done using the regularized trace, defined earlier. The notion of 'integration' of forms is important, since it is needed in order to construct δ -cocycles.

10.1. Basic definitions. Let \mathbb{A} be a superconnection on Ω depending on a parameter $t \in [0, 1]$. Consider the basic Chern-forms of this superconnection $c_k(\mathbb{A})$. They satisfy the transgression formula (integrated form)

$$(10.1) \quad \hat{\delta}c_k^1(\mathbb{A}) = c_k(\mathbb{A}_1) - c_k(\mathbb{A}_0) - \int_0^1 \hat{d}_{\mathbb{A}}\omega_k^1(\mathbb{A})$$

or

$$(10.2) \quad \hat{\delta}c_k^1(\mathbb{A}) = \partial c_k(\mathbb{A}) - dc_k^1(\mathbb{A}) - \int_0^1 [\mathbb{A}, \omega_k^1(\mathbb{A})].$$

That is, the Chern-Simons form $c_k^1(\mathbb{A})$ defines cocycles modulo $\partial c_k(\mathbb{A}) + \hat{d}\Omega + [\Omega, \Omega]$. Since both of the last two terms are in fact supercommutators, it would be tempting to just take the supertraces from both sides. Then we would (at least formally) obtain a δ -cocycle, if the form $\partial c_k(\mathbb{A})$ would vanish, since supertraces vanish on $\hat{d}\Omega + [\Omega, \Omega]$.

However, the operators above are usually not in the trace class. So, we must use regularized traces. We consider only the regularized traces defined earlier. Let us first deal with the $\hat{d}\Omega$ part. This is done by requiring the chosen regularization of the trace, which we denote by $\overline{\mathbf{Tr}}_s$, as before, to be \hat{d} -compatible. This means following.

Definition 18. A regularized trace $\overline{\mathbf{Tr}}_s$ on the cusp calculus $\Psi_c^*(M; S \otimes E)$ whose value at the commutators $\hat{d}\Omega$ vanishes in the interior is called \hat{d} -compatible. Here, the vanishing in the interior means following. The expression $\overline{\mathbf{Tr}}_s \hat{d}\omega$ vanishes in the interior if the Wodzicki-residue term arising from the trace anomaly formula vanishes.

Remark 13. In the case of a closed manifold, this means $\overline{\mathbf{Tr}}_s$ is a closed supertrace. That is $\overline{\mathbf{Tr}}_s \hat{d}\Omega = 0$. This terminology is used in [Co] and in [Sc].

Similarly we need the compability with respect to δ .

Definition 19. A regularized trace $\overline{\mathbf{Tr}}_s$ on the cusp calculus $\Psi_c^*(M; S \otimes E)$, whose weight Q satisfies $\delta Q = 0$ is said to be δ -compatible.

We can also say $\hat{\delta}$ -compatible in the above definition.

Remark 14. Due to our conventions, the grading operator Γ needed in the definition of the supertrace is already built in the forms Ω . Thus, when we work with forms Ω , the regularized supertrace is taken to be ordinary regularized trace. We still denote this 'trace' by $\overline{\text{Tr}}_s$ when using the abstract BRST-algebra on Ω . When we insert the conventions $\hat{\delta} = \Gamma\delta$ and $\theta \rightarrow \Gamma\theta$, then we drop the subscript s from the trace. For example, the expression $\overline{\text{Tr}}_s a\theta$ means $\overline{\text{Tr}} a\Gamma\theta$. Similar conventions are used when we apply the trace anomaly formula.

Remark 15. As before, we refer the terms whose value under the regularized trace $\overline{\text{Tr}}_s$ vanishes as boundary terms with respect to $\overline{\text{Tr}}_s$. If the choice of a regularization is clear, then we drop the reference to the trace. Particularly, $\hat{d}\Omega$ terms are boundary terms for any \hat{d} -compatible regularized trace.

Example 29 (\hat{d} and δ compatible supertrace). The standard example of a compatible supertrace is already known to us. Choose the weight $Q = \sqrt{\delta_0^2}$ for the regularized trace, as before, where 0 refers to the canonical flat connection on B . Then, it follows from the trace anomaly formula that \hat{d} -compatible regularized trace. The weight Q is δ -compatible, since it does not depend on connections B .

On a manifold with boundary we construct cocycles modulo boundary terms (transgressive forms). This is because even the smoothing operators are not in the trace class in this case.

The above discussion motivates the following definitions.

Definition 20 (Eta-chain). For any choice of \hat{d} and δ compatible supertrace $\overline{\text{Tr}}_s$ and for any choice of superconnection \mathbb{A} on Ω depending on parameter $t \in [0, 1]$, we define the *eta-chain* of degree k by

$$(10.3) \quad \eta_k(\mathbb{A}) = \overline{\text{Tr}}_s c_k^1(\mathbb{A}) = \int_0^1 \overline{\text{Tr}}_s \omega_k^1(\mathbb{A}),$$

where k is a positive integer.

Definition 21 (Eta-cocycle). Let \mathbb{A} be superconnection on Ω depending on parameter $t \in [0, 1]$ and let $\eta_k(\mathbb{A})$ be any eta-chain. If $\hat{\delta}\eta_{k,[m]}(\mathbb{A})$ is a boundary term, for positive integer m , then we say that the $\eta_{k,[m]}(\mathbb{A})$ is an *eta m-cocycle*.

Note, that $\eta_{k,[m]}(\mathbb{A})$ is a boundary term, if it vanishes when restricted to vertical vector fields that vanish on the boundary of M . This can be taken as a definition of the boundary term condition in the above definition. We can also say that the eta-m-cocycle property is the condition to build transgressive differential forms over the base.

From now on, we fix the regularization to be the one in Example 29, unless stated otherwise. Now, we need only to specify the superconnection \mathbb{A} depending on parameter $t \in [0, 1]$ to define the eta-chains. Therefore, we say that the eta-chain is associated with the superconnection \mathbb{A} .

Now, let \mathbb{A} be a superconnection depending on a parameter $t \in [0, 1]$. Recall that

$$(10.4) \quad \hat{\delta}c_k(\mathbb{A}) = \partial c_k(\mathbb{A}) - \hat{d}c_k^1(\mathbb{A}) - \int_0^1 [\mathbb{A}, \omega_k^1(\mathbb{A})].$$

Thus, when constructing eta-cocycles, we only have to worry about the terms

$$(10.5) \quad \partial c_k(\mathbb{A}) = c_k(\mathbb{A}_1) - c_k(\mathbb{A}_0)$$

and

$$(10.6) \quad \int_0^1 [\mathbb{A}, \omega_k^1(\mathbb{A})].$$

The term $\partial c_k(\mathbb{A})$ in general is not vanishing. It is usually vanishing only for some ghost degrees. The term (10.6) can be made a boundary term for suitable values of k and ghost degree.

Let us give standard examples of eta-chains and eta-cocycles. Here, when we expand Chern-Simons forms in the powers of ghost we generally use $\phi_{k,m}$ to denote the corresponding normalization constants. These can be read from the examples given earlier, if needed.

Example 30. Consider the superconnection $\mathbb{A} = ta + \theta$ on Ω , where $t \in [0, 1]$. Then

$$(10.7) \quad c_{k,[m]}^1(\mathbb{A}) = \phi_{k,m} S_k(a, (a^2)^{k-m-1}, (\hat{d}\theta)^m),$$

for $0 \leq m \leq k-1$, and

$$(10.8) \quad \partial c_k(\mathbb{A}) = c_k(\mathbb{A}_1) - c_k(\mathbb{A}_0) = -(\hat{d}\theta)^k.$$

The last term is a boundary term, so we only need to consider the commutator $[\mathbb{A}, \omega_k^1(\mathbb{A})]$, to get the eta-cocycle condition.

The eta-chain is

$$(10.9) \quad \begin{aligned} \eta_k(\mathbb{A}) &= \sum_{m=0}^{k-1} \overline{\text{Tr}}_s c_{k,[m]}^1(\mathbb{A}) \\ &= \sum_{m=0}^{k-1} \phi_{k,m} \overline{\text{Tr}}_s S_k(a, (a^2)^{k-m-1}, (\hat{d}\theta)^m). \end{aligned}$$

Observe, that as a pseudodifferential operator $S_k(a, (a^2)^{k-m-1}, (\hat{d}\theta)^m)$ has order $-2k + m + 1$. Thus $c_{k,[m]}^1(\mathbb{A})$ has order $-2k + m + 1$. We compute order of the commutator $[\mathbb{A}, S_k(a, (a^2)^{k-m-1}, (\hat{d}\theta)^m)]_{[m+1]}$ to be $-2k + m - 1$.

Thus we get eta-m cocycle, using the trace anomaly formula, when $-2k + m - 1 \leq -\dim M$. This is typical what happens. We need to consider high enough degree k in Chern-Simons-forms in order to construct eta-cocycles.

Example 31. Consider the superconnection $\mathbb{A} = ta$ on Ω , where $t \in [0, 1]$, then

$$(10.10) \quad c_{k,[m]}^1(\mathbb{A}) = \phi_{k,m} S_k(a, (a^2)^{k-m-1}, (\hat{\delta}a)^m),$$

for $0 \leq m \leq k - 1$, and

$$(10.11) \quad \partial c_k(\mathbb{A}) = c_k(\mathbb{A}_1) - c_k(\mathbb{A}_0) = (\hat{\delta}a)^k.$$

Now, the last term is essentially $[F, \theta]^k$. This can be written, using supercommutator, as $\frac{1}{2}[F, F[F, \theta]^k]$. Therefore, it is a boundary term if $k \geq \dim M$. Order of $\partial c_{k,[l]}(\mathbb{A})$ is k , if $k = l$, else $\partial c_{k,[l]}(\mathbb{A})$ has order $-\infty$.

Thus, when we assume $m \leq k - 2$, we only need to consider the commutator $[\mathbb{A}, \omega_k^1(\mathbb{A})]$.

The eta-chain is, in any case,

$$(10.12) \quad \begin{aligned} \eta_k(\mathbb{A}) &= \sum_{m=0}^{k-1} \overline{\text{Tr}}_s c_{k,[m]}^1(\mathbb{A}) \\ &= \sum_{m=0}^{k-1} \phi_{k,m} \overline{\text{Tr}}_s S_k(a, (a^2)^{k-m-1}, (\hat{\delta}a)^m). \end{aligned}$$

We see at once, by comparing above, that this gives eta-m cocycle precisely when $\eta_k(ta + \theta)$ does.

Example 32. Now consider the superconnection $\mathbb{A} = t\theta$ on Ω , where $t \in [0, 1]$. Then

$$(10.13) \quad c_{k,[m]}^1(\mathbb{A}) = \phi_{k,m} S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^{2k-m-1}),$$

where $k \leq m \leq 2k - 1$, and

$$(10.14) \quad \partial c_k(\mathbb{A}) = c_k(\mathbb{A}_1) - c_k(\mathbb{A}_0) = (\hat{d}\theta)^k.$$

Again, the last term is a boundary term.

We expand

$$(10.15) \quad \begin{aligned} \eta_k(\mathbb{A}) &= \sum_{m=k}^{2k-1} \overline{\text{Tr}}_s c_{k,[m]}^1(\mathbb{A}) \\ &= \sum_{m=k}^{2k-1} \phi_{k,m} \overline{\text{Tr}}_s S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^{2k-m-1}). \end{aligned}$$

We see that $c_{k,[m]}^1(\mathbb{A})$ has order $-2k + m + 1$. It follows that the commutator $[\mathbb{A}, \omega_k^1(\mathbb{A})]_{[m+1]}$ has order $-2k + m + 1$. Therefore, $\eta_k(\mathbb{A})$ is an eta-m-cocycle, if $-2k + m + 1 \leq -\dim M$.

10.2. Homotopy invariance of eta-cocycles. We need to establish homotopy invariance of eta-cocycles as in the case of the Chern-Simons-forms. To do this we need to know what does it mean to two eta-m cocycles to be equivalent.

Definition 22. Let \mathbb{A} and \mathbb{A}' be superconnections on Ω depending on a parameter. We say that the corresponding eta-chains are equivalent if the corresponding Chern-Simons forms differ by an element of $d\Omega + [\Omega, \Omega]$. The corresponding eta-m-cocycles

are equivalent if they differ by a $\hat{\delta}$ -coboundary and a boundary term.

Remark 16. The idea in the definition is that the equivalent eta-cocycles give equivalent δ -cocycles modulo boundary terms.

Now consider the superconnections $\mathbb{A} = ta + \theta$ and $\mathbb{A}' = ta$ on Ω , where $t \in [0, 1]$ and their corresponding eta-chains. We now already that (see Corollary 3)

$$(10.16) \quad c_{k,[m]}^1(\mathbb{A}) - c_{k,[m]}^1(\mathbb{A}') = \left(\int_{I^2} d_{\mathbf{A}} \omega_k^2(\mathbf{A}) \right)_{[m]},$$

for $0 \leq m \leq k - 1$.

Recall that

$$(10.17) \quad \omega_{k,[m]}^2(\mathbf{A}) = \sum_{m_1+2m_2=m-1} \phi_{k,m_1,m_2} S_k(a, \theta, \mathbb{F}_{[0]}^{k-2-m_1-m_2}, \mathbb{F}_{[1]}^{m_1}, \mathbb{F}_{[2]}^{m_2}).$$

From this we can compute order of $\omega_{k,[m]}^2(\mathbf{A})$. To compute order recall that $\mathbb{F}_{[0]}$ has order -2 , $\mathbb{F}_{[1]}$ has order -1 and $\mathbb{F}_{[2]}$ has order 0 . This gives order

$$(10.18) \quad -1 - 2(k - 2 - m_1 - m_2) - m_1 = -2k + 3 + m_1 + 2m_2.$$

Using the constraint

$$(10.19) \quad m_1 + 2m_2 = m - 1,$$

we get $-2k + m + 2$.

Order of the commutator $[\mathbf{A}, \omega_k^2(\mathbf{A})]_{[m]}$ is easily computed as $-2k + m + 1$. We obtain that the two eta-chains are equivalent and if

$$(10.20) \quad -2k + m + 1 \leq -\dim M,$$

then the corresponding eta-m cocycles are also equivalent. We note that $c_{k,[m]}^1(ta + \theta)$ and $c_{k,[m]}^1(ta)$ are of order $-2k + m + 1$ when $0 \leq m \leq k - 1$. Thus, if $-2k + m + 1 \leq -\dim M$, then the above inequality is satisfied. Therefore, we have the following.

Proposition 35. Consider the forms

$$(10.21) \quad \hat{\omega}_{k,[m]} = b_{k,m} S_k(F - \epsilon, [(F - \epsilon)^2]^{k-1-m}, (dF)^m),$$

where

$$(10.22) \quad b_{k,m} = (-1)^{k-m-1} k \binom{k-1}{m} B(k-m, k).$$

Then the forms $\Gamma \hat{\omega}_{k,[m]}$ agree with the forms $c_{k,[m]}^1(ta)$ when restricted to vertical directions. Particularly, the expression

$$(10.23) \quad \hat{\eta}_k = \overline{\mathbf{Tr}}_s \hat{\omega}_{k,[m]},$$

defines δ -transgression forms whenever $\eta_k(ta)$ does.

Proof. Follows from above. □

Remark 17. We would like to think $\overline{\text{Tr}}_s \hat{\omega}_{k,[m]}$ as regularizations of the forms $\overline{\text{Tr}}_s F(dF)^m$, which we met in the earlier chapters. Later, we prove that these forms are equivalent modulo normalization, coboundaries and regularized traces of commutators.

We put the above observations into the following theorem.

Theorem 1 (Homotopy invariance of eta-cocycles). *Let \mathbb{A}_0 and \mathbb{A}_2 be fixed superconnections on Ω . Then the eta-chain*

$$(10.24) \quad \eta_k(\mathbb{A}_0, \mathbb{A}_2) = \overline{\text{Tr}}_s c_k^1(\mathbb{A}_0, \mathbb{A}_2),$$

is independent of the chosen path in the following sense. For another two superconnections $\mathbb{A}_1, \mathbb{A}_2$ on Ω we have

$$(10.25) \quad \eta_k(\mathbb{A}_0, \mathbb{A}_2) = \overline{\text{Tr}}_s c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2) + \overline{\text{Tr}}_s \hat{d}\Omega + \overline{\text{Tr}}_s [\Omega, \Omega],$$

and

$$(10.26) \quad \eta_k(\mathbb{A}_0, \mathbb{A}_2) = \overline{\text{Tr}}_s c_k^1(\mathbb{A}_0, \mathbb{A}_3, \mathbb{A}_2) + \overline{\text{Tr}}_s \hat{d}\Omega + \overline{\text{Tr}}_s [\Omega, \Omega].$$

Particularly, the eta-chains

$$(10.27) \quad \eta_k(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2) \equiv \overline{\text{Tr}}_s c_k^1(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2),$$

and

$$(10.28) \quad \eta_k(\mathbb{A}_0, \mathbb{A}_3, \mathbb{A}_2) \equiv \overline{\text{Tr}}_s c_k^1(\mathbb{A}_0, \mathbb{A}_3, \mathbb{A}_2),$$

satisfy

$$(10.29) \quad \begin{aligned} & \eta_k(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2) - \eta_k(\mathbb{A}_0, \mathbb{A}_3, \mathbb{A}_2) \\ &= \int_{I^2} \overline{\text{Tr}}_s d\omega_k^2(\mathbf{A}) + \int_{I^2} \overline{\text{Tr}}_s [\mathbf{A}, \omega_k^2(\mathbf{A})], \end{aligned}$$

where the notation is the same as in the proposition 34.

Furthermore, if order of $c_{k,[m]}^1(\mathbb{A}_0, \mathbb{A}_1)$ is less or equal to $-\dim M$, then the eta-cocycles $\eta_{k,[m]}(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$, $\eta_{k,[m]}(\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2)$ and $\eta_{k,[m]}(\mathbb{A}_0, \mathbb{A}_2)$ are equivalent.

Proof. This follows from the homotopy invariance of the Chern-Simons forms, proposition 34, and the following computation.

We assume $c_{k,[m]}^1(\mathbb{A}_0, \mathbb{A}_2)$ has order $l \leq -\dim M$. The goal is to show that the commutator $[\mathbf{A}, \omega_k^2(\mathbf{A})]_{[m]}$ is a boundary term.

We need to estimate the order of $\omega_{k,[m]}^2(\mathbf{A})$ when we pull it back to simplices defined by the above vertices. Note that we do not need the precise formula for the form $\omega_{k,[m]}^2(\mathbf{A})$. It is enough to study the total superconnection.

First, we note that $c_{k,[m]}^1(\mathbb{A}_0, \mathbb{A}_2)$ can be expanded in terms of

$$(10.30) \quad S_k(a, \mathbb{F}_{[2]}^{m_1}, \mathbb{F}_{[1]}^{m_2}, \mathbb{F}_{[0]}^{k-1-m_1-m_2}),$$

where $2m_1 + m_2 = m$, and

$$(10.31) \quad S_k(\theta, \mathbb{F}_{[2]}^{m'_1}, \mathbb{F}_{[1]}^{m'_2}, \mathbb{F}_{[0]}^{k-1-m'_1-m'_2}),$$

$$2m'_1 + m'_2 = m - 1.$$

Both terms are of order $-2k + m + 1 \leq -\dim M$.

Recall that $\omega_k^2(\mathbf{A})$ can be expanded as follows

$$(10.32) \quad S_k(a, \theta, \mathbb{F}_{[2]}^{m_1}, \mathbb{F}_{[1]}^{m_2}, \mathbb{F}_{[0]}^{k-2-m_1-m_2}),$$

where $2m_1 + m_2 = m - 1$. The expression above has order

$$(10.33) \quad \begin{aligned} -1 - m_2 - 2(k - 2 - m_1 - m_2) &= -2k + m + 2 \\ &= (-2k + m + 1) + 1 \leq -\dim M + 1. \end{aligned}$$

Now the order of $[\mathbb{A}_{[0]}, \omega_{k,[m]}^2(\mathbf{A})]$ is $-2k + m + 1 \leq -\dim M$. Similarly we estimate $[\mathbb{A}_{[1]}, \omega_{k,[m-1]}^2(\mathbf{A})]$.

Thus the commutator $[\mathbf{A}, \omega_k(\mathbf{A})]_{[m]}$ is always a boundary term by the trace anomaly formula. \square

10.3. Locality. After we have constructed an eta-cocycle, the next step is to find a local formula modulo boundary terms for the eta-cocycle. Recall, by local we mean a pseudodifferential operator expression that depends only on finite number of terms of its asymptotic expansion. A prime example is the Wodzicki residue of any (classical) pseudodifferential operator expression. In general, it is impossible to give an exact local formula for eta-cocycles. However, often it is possible to write local formulas modulo δ -coboundary and boundary terms. In practice, these 'local expressions' are constructed by writing a Chern-Simons form in terms of supercommutators and δ, \hat{d} coboundaries. Then we act by $\overline{\mathbf{Tr}}_s$ to get expressions in terms of Wodzicki-residues, boundary terms and δ -coboundaries.

Therefore, we need to find a way to decompose Chern-Simons forms in the form $d\Omega + [\Omega, \Omega]$. Actually, we already know, that in the even dimensional case we have to add a correction term. The decomposition of the Chern-Simons form is the topic of the next section. By byproduct we get a method how to settle the equivalence question in the above remark.

Remark 18. Some authors call forms of the type $[\Omega, \Omega]$ as local, when working on closed manifolds, because the regularized traces of commutators give local expressions. In the case of boundary this does not make sense, since, by the trace anomaly formula, we always have to consider the 'global' trace defect. In our case the terminology 'local modulo boundary terms' is a more appropriate.

11. DECOMPOSITION THEOREMS

In this section we prove that any Chern-Simons form on Ω of the form $c_k^1(0, \mathbb{A}_1, a + \theta)$, where \mathbb{A}_1 is any superconnection on Ω , can be represented in the *standard form*

$$(11.1) \quad d\Omega + [\Omega, \Omega] + \Theta,$$

where Θ denotes the linear combinations of forms of type θ^{2m+1} , where m is a non-negative integer.

Particularly, we construct an explicit algorithm, which computes the above decomposition.

We split the construction into three major pieces. First, we handle the Chern-Simons form corresponding to the superconnection $\mathbb{A} = ta + \theta$ and then the Chern-Simons form corresponding to $\mathbb{A} = t\theta$. Finally, we use the homotopy invariance of the Chern-Simons forms to handle the other cases.

In the case of the superconnection $\mathbb{A} = ta + \theta$ we decompose the corresponding Chern-Simons forms to the form

$$(11.2) \quad d\Omega + [\Omega, \Omega] + c_k^1(t\theta).$$

Therefore, by decomposing the Chern-Simons form corresponding to the superconnection $\mathbb{A} = t\theta$ and combining with the above decomposition we get the decomposition that we are after.

The existence of such decompositions were already handled in [LaMiRy]. Unfortunately, they do not give a fully constructive proof of this. Particularly, they do not give the explicit formulas for commutators, which for us is the most important part. Therefore, we have to give the proof from the scratch.

11.1. Superconnection $\mathbb{A} = ta + \theta$. Consider the superconnection $\mathbb{A} = ta + \theta$ on Ω , where $t \in [0, 1]$. The corresponding Chern-Simons form is of the form

$$(11.3) \quad c_k^1(\mathbb{A}) = \sum_{p+q+1=k} \phi_{p,q} S_k(a, (a^2)^p, (\hat{d}\theta)^q) = \sum_{p+q+1=k} \int_0^1 \phi_{p,q} \Omega_{p,q},$$

where p, q and k are non-negative integers, and where we have set

$$(11.4) \quad \begin{aligned} \Omega_{p,q} &= S_k(a, (a^2)^p, (\hat{d}\theta)^q), \\ \phi_{p,q} &= (-1)^p (p+q+1) \binom{p+q}{q} B(p+1, p+q+1) = (-1)^p \frac{(p+q)!(p+q+1)!}{(2p+q+1)!q!}. \end{aligned}$$

We begin from the expression

$$(11.5) \quad \Omega_{0,q} = S_{q+1}(a, (\hat{d}\theta)^q).$$

Note that

$$(11.6) \quad \phi_{0,q} = 1.$$

Thus

$$(11.7) \quad c_{q+1,[q]}^1(\mathbb{A}) = \phi_{0,q} \Omega_{0,q} = S_{q+1}(a, (\hat{d}\theta)^q).$$

Example 33. Assume that M is the $2n$ -dimensional flat torus. We choose $q = 2n - 1$, and consider the projected eta-chain $\eta = \overline{\mathbf{Tr}}_s \Omega_{0,2n-1}$, with $\Omega_{0,2n-1}$ as above. The first step is to show that this is an eta-cocycle. This can be seen from the descent equations or by a direct computation.

We may write (by considering orders of the operators)

$$(11.8) \quad \Omega_{0,2n-1} = a(\hat{d}\theta)^{2n-1},$$

modulo boundary terms.

We compute the coboundary of $\Omega_{0,2n-1}$

$$\begin{aligned}
(11.9) \quad \hat{\delta}a(\hat{d}\theta)^{2n-1} &= (\hat{\delta}a)(\hat{d}\theta)^{2n-1} - a\hat{\delta}(\hat{d}\theta)^{2n-1} \\
&= -\hat{d}\theta(\hat{d}\theta)^{2n-1} - [a, \theta](\hat{d}\theta)^{2n-1} - \sum_{m=0}^{2n-2} (\hat{d}\theta)^m (\hat{d}\theta^2)(\hat{d}\theta)^{2n-2-m} \\
&= -(\hat{d}\theta)^{2n} - [a, \theta](\hat{d}\theta)^{2n-1} - \sum_{m=0}^{2n-2} (\hat{d}\theta)^m [\hat{d}\theta, \theta](\hat{d}\theta)^{2n-2-m} \\
&= -\hat{d}(\theta(\hat{d}\theta)^{2n-1}) - [a, \theta](\hat{d}\theta)^{2n-1} - a[(\hat{d}\theta)^{2n-1}, \theta] \\
&= -\hat{d}(\theta(\hat{d}\theta)^{2n-1}) - [a(\hat{d}\theta)^{2n-1}, \theta].
\end{aligned}$$

Ofcourse, it is easier to use the descent equations. We see from the above that $\hat{\delta}\Omega_{0,2n-1}$ is a boundary term. Thus η is a δ -cocycle. Next, we try to decompose $\Omega_{0,2n-1}$ into the standard form : $d\Omega + [\Omega, \Omega] + \Theta$.

Compute, using integration by parts, to get

$$\begin{aligned}
(11.10) \quad a(\hat{d}\theta)^{2n-1} &= a(\hat{d}\theta)^{2n-2}[\epsilon, \theta] \\
&= [a(\hat{d}\theta)^{2n-2}\epsilon, \theta] + a[(\hat{d}\theta)^{2n-2}, \theta]\epsilon + [a, \theta](\hat{d}\theta)^{2n-2}\epsilon \\
&= [a(\hat{d}\theta)^{2n-2}\epsilon, \theta] + a\hat{\delta}(\hat{d}\theta)^{2n-2}\epsilon - \hat{\delta}a(\hat{d}\theta)^{2n-2}\epsilon - \hat{d}\theta(\hat{d}\theta)^{2n-2}\epsilon,
\end{aligned}$$

where we have used $\hat{\delta}a = -\hat{d}\theta - [a, \theta]$ and the same computation as above (remember $-\theta^2 = \hat{\delta}\theta$) in reverse order, namely

$$\begin{aligned}
(11.11) \quad [(\hat{d}\theta)^{2n-2}, \theta] &= \sum_{m=0}^{2n-2} (\hat{d}\theta)^m ([\hat{d}\theta, \theta])(\hat{d}\theta)^{2n-2-m} \\
&= \sum_{m=0}^{2n-2} (\hat{d}\theta)^m (\hat{d}\theta^2)(\hat{d}\theta)^{2n-2-m} \\
&= \sum_{m=0}^{2n-2} (\hat{d}\theta)^m (\hat{\delta}\hat{d}\theta)(\hat{d}\theta)^{2n-2-m} \\
&= \hat{\delta}(\hat{d}\theta)^{2n-2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(11.12) \quad a(\hat{d}\theta)^{2n-1} &= [a(\hat{d}\theta)^{2n-2}\epsilon, \theta] - \hat{\delta}(a(\hat{d}\theta)^{2n-2}\epsilon) - (\hat{d}\theta)^{2n-1}\epsilon \\
&= [a(\hat{d}\theta)^{2n-2}\epsilon, \theta] - \hat{\delta}(a(\hat{d}\theta)^{2n-2}\epsilon) - \hat{d}(\theta(\hat{d}\theta)^{2n-2}\epsilon) - \theta(\hat{d}\theta)^{2n-2}\hat{d}\epsilon \\
&= [a(\hat{d}\theta)^{2n-2}\epsilon, \theta] - \hat{\delta}(a(\hat{d}\theta)^{2n-2}\epsilon) - \hat{d}(\theta(\hat{d}\theta)^{2n-2}\epsilon) - 2\theta(\hat{d}\theta)^{2n-2}.
\end{aligned}$$

Thus

$$(11.13) \quad a(\hat{d}\theta)^{2n-1} = [a(\hat{d}\theta)^{2n-2}\epsilon, \theta] - 2\theta(\hat{d}\theta)^{2n-2} + d\Omega.$$

The term $\theta(\hat{d}\theta)^{2n-2}$ belongs to the descent chain coming from the superconnection $\mathbb{A} = t\theta$. The decompositions for the forms coming from this superconnection are discussed later.

Now, we look the 'local part' of the above expression, namely the commutator. We obtain from the trace anomaly formula and convention ($\theta \rightarrow \Gamma\theta$) that

$$(11.14) \quad \overline{\mathbf{Tr}}[a(\hat{d}\theta)^{2n-2}\epsilon, \Gamma\theta] = \mathbf{Wres}\Gamma[l, \theta]a(\hat{d}\theta)^{2n-1}\epsilon.$$

An elementary computation gives from the right hand side

$$(11.15) \quad \int_M \mathbf{tr}A(d\theta)^{2n-1},$$

modulo a normalization. This is one of the standard cocycle formulas. The cocycle property is also easily verified by a direct computation. This computation is essentially the same as the above computation.

Let us return to the general case. Now is better to consider the forms $\Omega_{0,q}$ in their original form. We could also use the above computation modulo commutator, but the following computation gives nicer formulas and the notation that we use in the computation is very important.

Again, the first partial integration is easy

$$(11.16) \quad \begin{aligned} \Omega_{0,q} &= S_{q+1}(a, (\hat{d}\theta)^q) \\ &= [S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + (q-1)S_{q+1}(a, (\hat{d}\theta)^{q-2}, [\hat{d}\theta, \theta], \epsilon) \\ &\quad + S_{q+1}([a, \theta], (\hat{d}\theta)^{q-1}, \epsilon) \\ &= [S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + (q-1)S_{q+1}(a, (\hat{d}\theta)^{q-2}, \hat{\delta}\hat{d}\theta, \epsilon) \\ &\quad + S_{q+1}([a, \theta], (\hat{d}\theta)^{q-1}, \epsilon). \end{aligned}$$

Here we have used

$$(11.17) \quad \begin{aligned} [S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon), \theta] &= S_{q+1}(a, (\hat{d}\theta)^{q-1}, [\epsilon, \theta]) - (q-1)S_{q+1}(a, (\hat{d}\theta)^{q-2}, [\hat{d}\theta, \theta], \epsilon) \\ &\quad - S_{q+1}([a, \theta], (\hat{d}\theta)^{q-1}, [\epsilon, \theta]), \end{aligned}$$

and the identity $[\hat{d}\theta, \theta] = \hat{\delta}\hat{d}\theta$. Next, we use similar partial integration but now with respect to $\hat{\delta}$, that is

$$(11.18) \quad \hat{\delta}S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon) = S_{q+1}(\hat{\delta}a, (\hat{d}\theta)^{q-1}, \epsilon) - (q-1)S_{q+1}(a, (\hat{d}\theta)^{q-2}, \hat{\delta}\hat{d}\theta, \epsilon).$$

This gives together with $\hat{\delta}a = -\hat{d}\theta - [a, \theta]$

$$(11.19) \quad \begin{aligned} \Omega_{0,q} &= [S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \hat{\delta}S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon) \\ &\quad + S_{q+1}(\hat{\delta}a, (\hat{d}\theta)^{q-1}, \epsilon) + S_{q+1}([a, \theta], (\hat{d}\theta)^{q-1}, \epsilon) \\ &= [S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \hat{\delta}S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon) - S_{q+1}(\hat{d}\theta, (\hat{d}\theta)^{q-1}, \epsilon). \end{aligned}$$

Finally, we integrate by parts with respect to \hat{d} to get

$$(11.20) \quad S_{q+1}(\hat{d}\theta, (\hat{d}\theta)^{q-1}, \epsilon) = \hat{d}S_{q+1}(\theta, (\hat{d}\theta)^{q-1}, \epsilon) + S_{q+1}(\theta, (\hat{d}\theta)^{q-1}, \hat{d}\epsilon).$$

Recall that $\hat{d}\epsilon = 2$. It follows from the definition of S_{q+1} that

$$(11.21) \quad S_{q+1}(\theta, (\hat{d}\theta)^{q-1}, \hat{d}\epsilon) = S_{q+1}(\theta, (\hat{d}\theta)^{q-1}, 2) = 2S_{q+1}(\theta, (\hat{d}\theta)^{q-1}, 1) = 2S_q(\theta, (\hat{d}\theta)^{q-1}).$$

Thus

$$(11.22) \quad \begin{aligned} \Omega_{0,q} &= [S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \hat{\delta}S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon) \\ &\quad + \hat{d}S_{q+1}(\theta, (\hat{d}\theta)^{q-1}, \epsilon) + S_{q+1}(\theta, (\hat{d}\theta)^{q-1}, \hat{d}\epsilon) \\ &= [S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \hat{\delta}S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon) \\ &\quad + \hat{d}S_{q+1}(\theta, (\hat{d}\theta)^{q-1}, \epsilon) + 2S_q(\theta, (\hat{d}\theta)^{q-1}). \end{aligned}$$

We obtain

$$(11.23) \quad \Omega_{0,q} = [S_{q+1}(a, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + 2S_q(\theta, (\hat{d}\theta)^{q-1}) + d\Omega.$$

Proposition 36. The Chern-Simons forms $c_{m+1,[m]}^1(ta + \theta)$ on Ω , where $t \in [0, 1]$ and $m \geq 1$ is an integer, can be brought to the form

$$(11.24) \quad c_{m+1,[m]}^1(ta + \theta) = [S_{m+1}(a, (\hat{d}\theta)^{m-1}, \epsilon), \theta] + 2c_{m,[m]}^1(t\theta),$$

modulo $\hat{\delta}$ and \hat{d} coboundaries.

Proof. □

We now consider the case of the ghost degree one. This is relevant in the computation of the chiral anomaly.

Now the relevant Chern-Simons form can be written as

$$(11.25) \quad c_{k,[1]}^1(\mathbb{A}) = \phi_{k,m} S_k(a, (a^2)^{2k-2}, \hat{d}\theta) = \phi_{k,m} \Omega_{2k-2,1}.$$

Thus

$$(11.26) \quad \Omega_{2k-2,1} = S_k(a, (a^2)^{2k-2}, \hat{d}\theta).$$

Example 34 (1-cocycle on dimension two). Assume that M is a two dimensional flat torus, and consider

$$(11.27) \quad \Omega_{0,1} = a\hat{d}\theta.$$

Insert $\theta \rightarrow \Gamma\theta$ to get

$$(11.28) \quad \Omega_{0,1} = \Gamma a\hat{d}\theta \equiv \Gamma\hat{\Omega}_{0,1}.$$

Now $\hat{d}\theta = [\epsilon, \theta]_-$ in $\hat{\Omega}_{0,1}$.

We manipulate $\hat{\Omega}_{0,1}$, with the same tricks as before, to get

$$(11.29) \quad \hat{\Omega}_{0,1} = a[\epsilon, \theta] = [a\epsilon, \theta] - [a, \theta]\epsilon.$$

Recall $\hat{d}\theta + [a, \theta] = \delta a$. It follows

$$\begin{aligned}
 \hat{\Omega}_{0,1} &= [a\epsilon, \theta] - \delta a\epsilon + \hat{d}\theta\epsilon \\
 (11.30) \quad &= [a\epsilon, \theta] - \delta a\epsilon + \hat{d}(\theta\epsilon) - \theta\hat{d}\epsilon \\
 &= [a\epsilon, \theta] - \delta(a\epsilon) + \hat{d}(\theta\epsilon) - 2\theta.
 \end{aligned}$$

Thus, modulo δ and \hat{d} coboundaries we have

$$(11.31) \quad \hat{\Omega}_{0,1} = [a\epsilon, \theta] - 2\theta.$$

Therefore the $\hat{\Omega}_{0,1}$ has been brought to the standard form. Thus, we obtain a local formula for the eta-1-cocycle if we assume $\overline{\mathbf{Tr}}_s\theta$ is a boundary term. Then

$$(11.32) \quad \eta = \overline{\mathbf{Tr}}_s\Omega_{0,1} = \overline{\mathbf{Tr}}\Gamma\hat{\Omega}_{0,1} = \mathbf{Wres}_s[l, \theta]a\epsilon,$$

modulo boundary terms and coboundaries. It is easy to compute the above residue. This gives the cocycle $\int_M \mathbf{tr}d\theta A$, modulo normalization.

Example 35 (1-cocycle on dimension four). Assume that M is the four dimensional flat torus. Then

$$(11.33) \quad \Omega_{1,1} = a^3\hat{d}\theta,$$

modulo a boundary term, which we can ignore.

Insert $\theta \rightarrow \Gamma\theta$, as in the previous example, and define

$$(11.34) \quad \hat{\Omega}_{1,1} = a^3\hat{d}\theta = a^3[\epsilon, \theta]_-.$$

Integrate by parts to get

$$(11.35) \quad \hat{\Omega}_{1,1} = a^3[\epsilon, \theta] = [a^3\epsilon, \theta] - [a^3, \theta]\epsilon.$$

Now since

$$\begin{aligned}
 \delta a^3 &= (\delta a)a^2 + a(\delta a)a + a^2(\delta a) \\
 (11.36) \quad &= (\hat{d}\theta a^2 + a\hat{d}\theta a + a^2\hat{d}\theta) + [a^3, \theta].
 \end{aligned}$$

It follows

$$(11.37) \quad -[a^3, \theta]\epsilon = (\hat{d}\theta a^2 + a\hat{d}\theta a + a^2\hat{d}\theta)\epsilon - \delta a^3\epsilon.$$

Thus we have

$$(11.38) \quad \hat{\Omega}_{1,1} = [a^3\epsilon, \theta] + (\hat{d}\theta a^2\epsilon + a\hat{d}\theta a\epsilon + a^2\hat{d}\theta\epsilon),$$

modulo δ -coboundary. Now the last term becomes

$$\begin{aligned}
& [\hat{d}\theta, a^2\epsilon] - a^2\epsilon\hat{d}\theta + [a\hat{d}\theta, a\epsilon] + a\epsilon a\hat{d}\theta - a^2\epsilon\hat{d}\theta \\
&= [\hat{d}\theta, a^2\epsilon] - 2a^2\epsilon\hat{d}\theta + [a\hat{d}\theta, a\epsilon] - a^3\hat{d}\theta - a^2\epsilon\hat{d}\theta \\
&= [\hat{d}\theta, a^2\epsilon] + [a\hat{d}\theta, a\epsilon] - 3a^2\epsilon\hat{d}\theta - \hat{\Omega}_{1,1} \\
(11.39) \quad &= [\hat{d}\theta, a^2\epsilon] + [a\hat{d}\theta, a\epsilon] - 3a^2\theta + 3\epsilon a^2\theta\epsilon - \hat{\Omega}_{1,1} \\
&= [\hat{d}\theta, a^2\epsilon] + [a\hat{d}\theta, a\epsilon] - 3a^2\theta + [3\epsilon a^2\theta, \epsilon] - 3a^2\theta - \hat{\Omega}_{1,1} \\
&= [\hat{d}\theta, a^2\epsilon] + [a\hat{d}\theta, a\epsilon] + 3\hat{d}(\epsilon a^2\theta) + 6\hat{d}a\theta - \hat{\Omega}_{1,1} \\
&= [\hat{d}\theta, a^2\epsilon] + [a\hat{d}\theta, a\epsilon] + 3\hat{d}(\epsilon a^2\theta) + 6\hat{d}(a\theta) + 6a\hat{d}\theta - \hat{\Omega}_{1,1} \\
&= [\hat{d}\theta, a^2\epsilon] + [a\hat{d}\theta, a\epsilon] + 3\hat{d}(\epsilon a^2\theta) + 6\hat{d}(a\theta) + 6\hat{\Omega}_{0,1} - \hat{\Omega}_{1,1}.
\end{aligned}$$

Thus, we get modulo \hat{d} and δ coboundaries

$$\begin{aligned}
(11.40) \quad 2\hat{\Omega}_{1,1} &= [a^3\epsilon, \theta] + [\hat{d}\theta, a^2\epsilon] + [a\hat{d}\theta, a\epsilon] + 6\hat{\Omega}_{0,1} \\
&= [a^3\epsilon, \theta] + [\hat{d}\theta, a^2\epsilon] + [a\hat{d}\theta, a\epsilon] + 6[a\epsilon, \theta] + 12\theta.
\end{aligned}$$

Assuming $\overline{\mathbf{Tr}}_s\theta = 0$, we get

$$\begin{aligned}
(11.41) \quad 2\eta &= 2\overline{\mathbf{Tr}}_s\Omega_{1,1} = 2\overline{\mathbf{Tr}}\Gamma\hat{\Omega}_{1,1} = \mathbf{Wres}_s[l, \theta]a^3\epsilon + \mathbf{Wres}_s[l, a^2\epsilon]\hat{d}\theta + \mathbf{Wres}_s[l, a\epsilon]a\hat{d}\theta \\
&\quad + 6\mathbf{Wres}_s[l, \theta]a\epsilon,
\end{aligned}$$

modulo coboundaries.

The computation of the above residues is a bit tricky task but doable. After a bit of computation one obtains $\int_M \mathbf{tr}(\frac{1}{2}d\theta A^3 + d\theta dAA)$ up to a normalization.

In the above computations we have put the Chern-Simons form to the form given usually in the finite dimensions. This is not a good idea in the more general situations, that follow. Therefore, we let the Chern-Simons forms be in their symmetrized form, then the integration by parts process, given below, becomes much more tractable. Furthermore, we begin to see how to handle the more general case.

So, consider the Chern-Simons term in the form

$$(11.42) \quad \Omega_{p,1} = S_k(a, (a^2)^p, \hat{d}\theta).$$

Then the integration by parts with respect to $[\cdot, \theta]$ reads

$$\begin{aligned}
(11.43) \quad \Omega_{p,1} &= S_k(a, (a^2)^p, \hat{d}\theta) \\
&= [S_k(a, (a^2)^p, \epsilon), \theta] + pS_k(a, (a^2)^{p-1}, [a^2, \theta], \epsilon) + S_k([a, \theta], (a^2)^p, \epsilon).
\end{aligned}$$

Now we use $[a^2, \theta] = [\hat{d}\theta, a] + \hat{\delta}a^2$ to get

$$\begin{aligned}
(11.44) \quad \Omega_{p,1} &= [S_k(a, (a^2)^p, \epsilon), \theta] + pS_k(a, (a^2)^{p-1}, [\hat{d}\theta, a], \epsilon) \\
&\quad + pS_k(a, (a^2)^{p-1}, \hat{\delta}a^2, \epsilon) + S_k([a, \theta], (a^2)^p, \epsilon).
\end{aligned}$$

Integrate by parts with respect to $[\cdot, a]$ (the second term above) and the third term above with respect to $\hat{\delta}$ to get

$$\begin{aligned}
(11.45) \quad \Omega_{p,1} &= [S_k(a, (a^2)^p, \epsilon), \theta] - p[S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon), a] + pS_k(a, (a^2)^{p-1}, \hat{d}\theta, [\epsilon, a]) \\
&\quad - pS_k([a, a], (a^2)^{p-1}, \hat{d}\theta, \epsilon) - \hat{\delta}S_k(a, (a^2)^p, \epsilon) + S_k(\hat{\delta}a, (a^2)^p, \epsilon) + S_k([a, \theta], (a^2)^p, \epsilon) \\
&= [S_k(a, (a^2)^p, \epsilon), \theta] - p[S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon), a] - pS_k(a, (a^2)^p, \hat{d}\theta) \\
&\quad - 2pS_k((a^2)^p, \hat{d}\theta, \epsilon) - \hat{\delta}S_k(a, (a^2)^p, \epsilon) - S_k(\hat{d}\theta, (a^2)^p, \epsilon) \\
&= [S_k(a, (a^2)^p, \epsilon), \theta] - p[S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon), a] - p\Omega_{p,1} \\
&\quad - (2p+1)S_k((a^2)^p, \hat{d}\theta, \epsilon) - \hat{\delta}S_k(a, (a^2)^p, \epsilon).
\end{aligned}$$

Now, we treat the term $S_k((a^2)^p, \hat{d}\theta, \epsilon)$ separately. This term is handled with identities $\hat{d}a = -a^2$, $\hat{d}a^2 = 0$, and integration by parts with respect to \hat{d}

$$\begin{aligned}
(11.46) \quad S_k((a^2)^p, \hat{d}\theta, \epsilon) &= S_k(-\hat{d}a, (a^2)^{p-1}, \hat{d}\theta, \epsilon) \\
&= -\hat{d}S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon) - S_k(a, (a^2)^{p-1}, \hat{d}\theta, \hat{d}\epsilon) \\
&= -\hat{d}S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon) - 2S_{k-1}(a, (a^2)^{p-1}, \hat{d}\theta) \\
&= -\hat{d}S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon) - 2\Omega_{p-1,1}.
\end{aligned}$$

This yields

$$\begin{aligned}
(11.47) \quad \Omega_{p,1} &= [S_k(a, (a^2)^p, \epsilon), \theta] - p[S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon), a] - p\Omega_{p,1} \\
&\quad + (2p+1)\hat{d}S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon) + 2(2p+1)\Omega_{p-1,1} - \hat{\delta}S_k(a, (a^2)^p, \epsilon).
\end{aligned}$$

That is

$$\begin{aligned}
(11.48) \quad (p+1)\Omega_{p,1} &= [S_k(a, (a^2)^p, \epsilon), \theta] - p[S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon), a] \\
&\quad + (2p+1)\hat{d}S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon) \\
&\quad + 2(2p+1)\Omega_{p-1,1}.
\end{aligned}$$

Modulo coboundaries, we can write

$$\begin{aligned}
(11.49) \quad \Omega_{p,1} &= \frac{1}{p+1}[S_k(a, (a^2)^p, \epsilon), \theta] - \frac{p}{p+1}[S_k(a, (a^2)^{p-1}, \hat{d}\theta, \epsilon), a] \\
&\quad + 2\frac{2p+1}{p+1}\Omega_{p-1,1}.
\end{aligned}$$

The normalization constant is

$$(11.50) \quad \phi_{p,1} = (-1)^p(p+2) \binom{p+1}{1} B(p+1, p+2).$$

We compute first that

$$(11.51) \quad \frac{\phi_{p,1}}{\phi_{p-1,1}} = -\frac{(p+2)(p+1)}{(2p+2)(2p+1)}.$$

We obtain a basic iteration formula, when the normalizations are inserted.

Proposition 37. Let $\mathbb{A} = ta + \theta$ be a superconnection on Ω , where $t \in [0, 1]$. Then, the Chern-Simons forms $c_{k,[1]}^1(\mathbb{A})$ satisfy the following iteration formula

$$(11.52) \quad \begin{aligned} c_{k,[1]}^1(\mathbb{A}) &= d'_{k,1}[S_k(a, (a^2)^{k-2}, \epsilon), \theta] + d''_{k,1}[S_k(a, (a^2)^{k-3}, \hat{d}\theta, \epsilon), a] \\ &\quad + d_{k,1}c_{k-1,[1]}^1(\mathbb{A}) + d\Omega, \end{aligned}$$

where $k > 1$ and

$$(11.53) \quad \begin{aligned} d'_{k,1} &= \frac{\phi_{k,1}}{k-1} = (-)^k \frac{k!(k-2)!}{(2k-2)!} \\ d''_{k,1} &= -\frac{(k-2)\phi_{k,1}}{(2k-2m)!} = -(-)^k (k-2) \frac{k!(k-2)!}{(2k-2)!} \\ d_{k,1} &= 2 \frac{(2k-3)}{k-1} \frac{\phi_{k,1}}{\phi_{k-1,1}} = -2 \frac{k}{2k-2}. \end{aligned}$$

Proof. □

In the above case, we can use the iteration above down to the forms of the type $\Omega_{0,1} = S_2(a, \hat{d}\theta)$, but we know already how to handle these forms (see Example 34).

Proposition 38. Let $\mathbb{A} = ta + \theta$ be a superconnection on Ω , where $t \in [0, 1]$. The Chern-Simons form

$$(11.54) \quad c_{k,[1]}^1(\mathbb{A}) = \int_0^1 k S_k(\dot{\mathbb{A}}, \mathbb{F}^{k-1})_{[1]},$$

has a decomposition

$$(11.55) \quad c_{k,[1]}^1(\mathbb{A}) = [\Omega, \Omega] + d\Omega + d_{k,1}^{N-1} c_{k-N,[1]}^1(\mathbb{A}),$$

for any integer N satisfying $1 < N \leq k-2$. More precisely, we have the following decomposition

$$(11.56) \quad \begin{aligned} c_{k,[1]}^1(\mathbb{A}) &= \sum_{n=0}^{N-1} d_{k,1}^n d'_{k,1}[S_k(a, (a^2)^{p-n}, \epsilon), \theta] + d_{k,1}^n d''_{k,1}[S_k(a, (a^2)^{p-n-1}, \hat{d}\theta, \epsilon), a] \\ &\quad + d_{k,1}^{N-1} c_{k-N,[1]}^1(\mathbb{A}) + d\Omega, \end{aligned}$$

where

$$(11.57) \quad \begin{aligned} d_{k,1}^n &= d_{k,1} d_{k-1,1} \cdots d_{k-n,1} \\ d_{k,1}^0 &= 1. \end{aligned}$$

Proof. Iterate, using Proposition 37. □

As we have already seen, the case of the higher degree forms in the ghost cannot be handled with only one type of iteration process.

11.2. The general case. Let $\mathbb{A} = ta + \theta$ be a superconnection on Ω , where $t \in [0, 1]$. Consider the Chern-Simons forms of a general ghost degree

$$(11.58) \quad c_{k,[q]}^1(\mathbb{A}) = \int_0^1 \omega_{k,[q]}^1(\mathbb{A}) = \phi_{k,q} \Omega_{k-q-1,q},$$

where $1 \leq q \leq k-1$. Put $p = k - q - 1$, then $\Omega_{p,q} = S_k(a, (a^2)^p, (\hat{d}\theta)^q)$.

Now

$$(11.59) \quad \begin{aligned} \Omega_{p,q} &= S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \hat{d}\theta) \\ &= [S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + (q-1)S_k(a, (a^2)^p, (\hat{d}\theta)^{q-2}, [\hat{d}\theta, \theta], \epsilon) \\ &\quad + pS_k(a, (a^2)^{p-1}, [a^2, \theta], (\hat{d}\theta)^{q-1}, \epsilon) + S_k([a, \theta], (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon). \end{aligned}$$

Next, we use the identities

$$(11.60) \quad \begin{aligned} \hat{\delta}\hat{d}\theta &= [\hat{d}\theta, \theta] \\ [a, \theta] &= -\hat{d}\theta - \hat{\delta}a \\ [a^2, \theta] &= \hat{\delta}a^2 + [\hat{d}\theta, a]. \end{aligned}$$

We obtain

$$(11.61) \quad \begin{aligned} \Omega_{p,q} &= [S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + (q-1)S_k(a, (a^2)^p, (\hat{d}\theta)^{q-2}, \hat{\delta}\hat{d}\theta, \epsilon) \\ &\quad + pS_k(a, (a^2)^{p-1}, \hat{\delta}a^2, (\hat{d}\theta)^{q-1}, \epsilon) + pS_k(a, (a^2)^{p-1}, [\hat{d}\theta, a], (\hat{d}\theta)^{q-1}, \epsilon) \\ &\quad - S_k(\hat{\delta}a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) - S_k(\hat{d}\theta, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon). \end{aligned}$$

After rearranging

$$(11.62) \quad \begin{aligned} \Omega_{p,q} &= [S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - S_k(\hat{\delta}a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) \\ &\quad + pS_k(a, (a^2)^{p-1}, \hat{\delta}a^2, (\hat{d}\theta)^{q-1}, \epsilon) + (q-1)S_k(a, (a^2)^p, (\hat{d}\theta)^{q-2}, \hat{\delta}\hat{d}\theta, \epsilon) \\ &\quad + pS_k(a, (a^2)^{p-1}, [\hat{d}\theta, a], (\hat{d}\theta)^{q-1}, \epsilon) - S_k((a^2)^p, (\hat{d}\theta)^q, \epsilon). \end{aligned}$$

We observe that

$$(11.63) \quad \begin{aligned} -\hat{\delta}S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) &= -S_k(\hat{\delta}a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) + pS_k(a, (a^2)^{p-1}, \hat{\delta}a^2, (\hat{d}\theta)^{q-1}, \epsilon) \\ &\quad + (q-1)S_k(a, (a^2)^p, (\hat{d}\theta)^{q-2}, \hat{\delta}\hat{d}\theta, \epsilon). \end{aligned}$$

Thus

$$(11.64) \quad \begin{aligned} \Omega_{p,q} &= [S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + pS_k(a, (a^2)^{p-1}, [\hat{d}\theta, a], (\hat{d}\theta)^{q-1}, \epsilon) \\ &\quad - S_k((a^2)^p, (\hat{d}\theta)^q, \epsilon) - \hat{\delta}S_k(a, (a^2)^p). \end{aligned}$$

We still need to handle the remaining two terms. These are again handled by integration by parts. We have

$$\begin{aligned}
(11.65) \quad & pS_k(a, (a^2)^{p-1}, [\hat{d}\theta, a], (\hat{d}\theta)^{q-1}, \epsilon) \\
&= -\frac{p}{q}[S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon), a] + \frac{p}{q}S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^{q-1}, \hat{d}a) \\
&\quad - \frac{p}{q}S_k([a, a], (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon),
\end{aligned}$$

where we have used integration by parts with respect to $[\cdot, a]$, and the identities

$$\begin{aligned}
(11.66) \quad & [a, a^2] = 0 \\
& [\epsilon, a] = \hat{d}a.
\end{aligned}$$

The identity $\hat{d}a = -a^2$ shows that

$$(11.67) \quad \frac{p}{q}S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^{q-1}, \hat{d}a) = -\frac{p}{q}\Omega_{p,q}.$$

The remaining term is slightly different. We use $[a, a] = 2a^2 = -2\hat{d}a$ and integration by parts with respect to \hat{d} to obtain

$$\begin{aligned}
(11.68) \quad & -\frac{p}{q}S_k([a, a], (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon) \\
&= 2\frac{p}{q}\hat{d}S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon) + 2\frac{p}{q}S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \hat{d}\epsilon) \\
&= 2\frac{p}{q}\hat{d}S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon) + 4\frac{p}{q}S_{k-1}(a, (a^2)^{p-1}, (\hat{d}\theta)^q) \\
&= 2\frac{p}{q}\hat{d}S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon) + 4\frac{p}{q}\Omega_{p-1,q},
\end{aligned}$$

where we have used $\hat{d}\epsilon = 2$, $S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, 1) = S_{k-1}(a, (a^2)^{p-1}, (\hat{d}\theta)^q)$ and the definition of $\Omega_{p-1,q}$. Therefore, we obtain

$$\begin{aligned}
(11.69) \quad & pS_k(a, (a^2)^{p-1}, [\hat{d}\theta, a], (\hat{d}\theta)^{q-1}, \epsilon) = -\frac{p}{q}[S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon), a] - \frac{p}{q}\Omega_{p,q} \\
&\quad + 2\frac{p}{q}\hat{d}S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon) + 4\frac{p}{q}\Omega_{p-1,q}.
\end{aligned}$$

We still need to handle the term $-S_k((a^2)^p, (\hat{d}\theta)^q, \epsilon)$. Integrate by parts with respect to \hat{d} to get

$$\begin{aligned}
(11.70) \quad & -S_k((a^2)^p, (\hat{d}\theta)^q, \epsilon) = S_k(\hat{d}a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon) \\
&= \hat{d}S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon) + 2S_{k-1}(a, (a^2)^{p-1}, (\hat{d}\theta)^q) \\
&= \hat{d}S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon) + 2\Omega_{p-1,q},
\end{aligned}$$

where we have used the identity $\hat{d}a = -a^2$.

Putting everything together we get

$$\begin{aligned}
(11.71) \quad \Omega_{p,q} &= [S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \frac{p}{q} [S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon), a] \\
&\quad - \frac{p}{q} \Omega_{p,q} + 4 \frac{p}{q} \Omega_{p-1,q} + 2 \Omega_{p-1,q} \\
&\quad - \hat{\delta} S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) + \frac{2p+q}{q} \hat{d} S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon).
\end{aligned}$$

That is

$$\begin{aligned}
(11.72) \quad \frac{p+q}{q} \Omega_{p,q} &= [S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \frac{p}{q} [S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon), a] \\
&\quad + 2 \frac{2p+q}{q} \Omega_{p-1,q} \\
&\quad - \hat{\delta} S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) + \frac{2p+q}{q} \hat{d} S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon).
\end{aligned}$$

Thus, we obtain the general iteration formula.

Lemma 3. The forms $\Omega_{p,q} = S_k(a, (a^2)^p, (\hat{d}\theta)^q)$ on Ω , where $k = p+q+1$, associated with the Chern-Simons form of the superconnection $\mathbb{A} = ta + \theta$ on Ω , where $t \in [0, 1]$, satisfy the following iteration formula

$$\begin{aligned}
(11.73) \quad \Omega_{p,q} &= \frac{q}{p+q} [S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \frac{p}{p+q} [S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon), a] \\
&\quad + 2 \frac{2p+q}{p+q} \Omega_{p-1,q} \\
&\quad - \frac{q}{p+q} \hat{\delta} S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) + \frac{2p+q}{p+q} \hat{d} S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon).
\end{aligned}$$

Particularly, we have modulo coboundaries that

$$\begin{aligned}
(11.74) \quad \Omega_{p,q} &= \frac{q}{p+q} [S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \frac{p}{p+q} [S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon), a] \\
&\quad + 2 \frac{2p+q}{p+q} \Omega_{p-1,q}.
\end{aligned}$$

Proof.

□

The above lemma implies a relation between the Chern-Simons forms $c_{k,[m]}^1(\mathbb{A})$ and $c_{k-1,[m]}^1(\mathbb{A})$, associated with $\mathbb{A} = ta + \theta$. We just need to insert the normalization constants $\phi_{p,q} = \phi_{k,m}$ (slight abuse of notation) of the Chern-Simons forms into the above equation

$$\begin{aligned}
(11.75) \quad \phi_{p,q} \Omega_{p,q} &= \phi_{p,q} \frac{q}{p+q} [S_k(a, (a^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \phi_{p,q} \frac{p}{p+q} [S_k(a, (a^2)^{p-1}, (\hat{d}\theta)^q, \epsilon), a] \\
&\quad + 2 \frac{\phi_{p,q}}{\phi_{p-1,q}} \frac{2p+q}{p+q} (\phi_{p,q} \Omega_{p-1,q}).
\end{aligned}$$

Recall that

$$(11.76) \quad \phi_{p,q} = (-1)^p (p+q+1) \binom{p+q}{q} B(p, p+q+1).$$

We compute first that

$$(11.77) \quad \frac{\phi_{p,q}}{\phi_{p-1,q}} = \frac{(p+q+1) \binom{p+q}{q} B(p, p+q+1)}{(p+q) \binom{p+q-1}{q} B(p-1, p+q)}.$$

This gives us the following iteration formula.

Proposition 39. Let $\mathbb{A} = ta + \theta$ be a superconnection on Ω , for $t \in [0, 1]$. Then the Chern-Simons forms $c_{k,[m]}^1(\mathbb{A})$, for $1 \leq m \leq k-1$, satisfy the following iteration formula

$$(11.78) \quad \begin{aligned} c_{k,[m]}^1(\mathbb{A}) &= d'_{k,m} [S_k(a, (a^2)^{k-m-1}, (\hat{d}\theta)^{m-1}, \epsilon), \theta] \\ &\quad + d''_{k,m} [S_k(a, (a^2)^{k-m-2}, (\hat{d}\theta)^m, \epsilon), a] \\ &\quad + d_{k,m} c_{k-1,[m]}^1(\mathbb{A}), \end{aligned}$$

modulo coboundaries. Here the constants are

$$(11.79) \quad \begin{aligned} d'_{k,m} &= \phi_{k,m} \frac{m}{k-1} = (-1)^{m-k} \frac{k!(k-2)!}{(m-1)!(2k-m-1)!} \\ d''_{k,m} &= -\phi_{k,m} \frac{k-m-1}{k-1} = -(-1)^{m-k} \frac{k!(k-2)!}{m!(2k-m-1)!} (k-m-1) \\ d_{k,m} &= 2 \frac{\phi_{k,m}}{\phi_{k-1,m}} \frac{2k-m-2}{k-1} = 2 \frac{k!(k-m-1)}{(k-2)!(2k-m-1)(2k-m-2)}. \end{aligned}$$

Proof. □

We can use the above iteration down to $p = 0$ (recall that $k=p+q+1$).

Proposition 40. The Chern-Simons forms $c_k^1(\mathbb{A})$, for the superconnection $\mathbb{A} = ta + \theta$ on Ω , where $t \in [0, 1]$, can be represented in the form

$$(11.80) \quad c_{k,[m]}^1(\mathbb{A}) = d\Omega + [\Omega, \Omega] + d_{k,m}^{N-1} c_{k-N,[m]}^1(\mathbb{A}),$$

for any integer N satisfying $0 < N \leq k-m-1$. More precisely, we have a decomposition

$$(11.81) \quad \begin{aligned} c_{k,[m]}^1(\mathbb{A}) &= \sum_{n=0}^{N-1} d_{k,m}^n d'_{k-n,m} [S_{k-n}(a, (a^2)^{k-n-m-1}, (\hat{d}\theta)^{m-1}, \epsilon), \theta] \\ &\quad + \sum_{n=0}^{N-1} d_{k,m}^m d''_{k-n,m} [S_{k-n}(a, (a^2)^{k-n-m-2}, (\hat{d}\theta)^m, \epsilon), a] \\ &\quad + d_{k,m}^{N-1} c_{k-N,[m]}^1(\mathbb{A}), \end{aligned}$$

modulo coboundaries. Here we have put

$$(11.82) \quad \begin{aligned} d_{k,m}^n &= d_{k,m} d_{k-1,m} \cdots d_{k-n,m}. \\ d_{k,m}^0 &= 1. \end{aligned}$$

Proof. Follows by iteration from Proposition 39. \square

Now we combine the propositions 36 and 40 to get the iteration formula down to the forms coming from $c_k^1(t\theta)$.

Proposition 41. The Chern-Simons form for the superconnection $\mathbb{A} = ta + \theta$ on Ω , where $t \in [0, 1]$, can be presented in the following form, for $0 \leq m \leq k - 1$,

$$(11.83) \quad \begin{aligned} c_{k,[m]}^1(ta + \theta) &= \sum_{n=0}^{k-m-1} d_{k,m}^n d'_{k-n,m} [S_{k-n}(a, (a^2)^{k-n-m-1}, (\hat{d}\theta)^{m-1}, \epsilon), \theta] \\ &+ \sum_{n=0}^{k-m-1} d_{k,m}^n d''_{k-n,m} [S_{k-n}(a, (a^2)^{k-n-m-2}, (\hat{d}\theta)^m, \epsilon), a] \\ &+ d_{k,m}^{k-1} [S_{m+1}(a, (\hat{d}\theta)^{m-1}, \epsilon), \theta] + 2d_{k,m}^{k-m-1} c_{m,[m]}^1(t\theta), \end{aligned}$$

modulo coboundaries. Here the constants $d_{k,m}$, $d'_{k,m}$ and $d''_{k,m}$ are the same as above.

Proof. \square

The next step is to decompose forms of the type $c_{k,[m]}^1(ta + \theta)$ further. This requires the study of the superconnection $\mathbb{A} = t\theta$ and its Chern-Simons forms as the above formula clearly shows.

11.3. Superconnection $\mathbb{A} = t\theta$. Now, we consider the superconnection $\mathbb{A} = t\theta$ on Ω , where $t \in [0, 1]$. The corresponding curvature is $\mathbb{F} = (t^2 - t)\theta^2 + t\hat{d}\theta$. The Chern-Simons form can be expanded as follows

$$(11.84) \quad c_k^1(t\theta) = \sum_{p+q=k-1} \phi_{k,q} S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^q).$$

Observe that the projection to the ghost degree m , where $k \leq m \leq 2k - 1$ is

$$(11.85) \quad c_{k,[m]}^1(t\theta) = \phi_{k,m} S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^{2k-m-1}).$$

As a pseudodifferential operator $c_{k,[m]}^1(t\theta)$ has order $-(2k - m - 1)$. Therefore, these forms have very limited use when considering the construction of cocycles.

There are three different type of forms of special interest. First, there are forms

$$(11.86) \quad S_k(\theta, (\theta^2)^p, \hat{d}\theta).$$

These are special, because they can be decomposed at once. More precisely the above expression is the same as

$$(11.87) \quad [S_k(\theta, (\theta^2)^p, \epsilon), \theta] + S_k([\theta, \theta], (\theta^2)^p, \epsilon),$$

since $[\theta, \theta^2] = 0$. Recall that $[\theta, \theta] = -2\hat{\delta}\theta$, which yields

$$(11.88) \quad S_k(\theta, (\theta^2)^p, (\hat{d}\theta)) = [S_k(\theta, (\theta^2)^p, \epsilon), \theta] - 2\hat{\delta}S_k(\theta, (\theta^2)^p, \epsilon).$$

Thus, any form of the above type can be decomposed to the standard form $d\Omega + [\Omega, \Omega]$. However, the above type of forms are of even degree in the ghost. So this special case comes important only when the dimension of the manifold is odd.

The second special case happens when we consider odd degree forms in the ghost. Then, there are Chern-Simons forms of the following type (these show up when $k = m' = \frac{m+1}{2}$)

$$(11.89) \quad c_{m', [m]}^1(t\theta) = \phi_{m', m} S_{m'}(\theta, (\theta^2)^{m'}).$$

These are precisely Θ -forms.

The final important special case is the type

$$(11.90) \quad c_{m, [m]}^1(t\theta) = S_m(\theta, (\hat{d}\theta)^{m-1}).$$

These are precisely the type of forms that come from our previous decomposition result, Proposition 41.

The forms $c_{m, [m]}^1(t\theta)$, however, do not have any easier decompositions than the general forms given above. Therefore, we consider the general case directly.

Put for convenience

$$(11.91) \quad \Omega_{p, q} = S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^q),$$

where $p = m - k$ and $q = 2k - m - 1$. First, integrate by parts with respect to $[\cdot, \theta]$, and use $[\hat{d}\theta, \theta] = \hat{\delta}\hat{d}\theta$ to get

$$(11.92) \quad \begin{aligned} \Omega_{p, q} &= [S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + (q-1)S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-2}, [\hat{d}\theta, \theta], \epsilon) \\ &\quad + S_k([\theta, \theta], (\theta^2)^p, (\hat{d}\theta)^{q-2}, \epsilon) \\ &= [S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + (q-1)S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-2}, \hat{\delta}\hat{d}\theta, \epsilon) \\ &\quad + 2S_k((\theta^2)^{p+1}, (\hat{d}\theta)^{q-2}, \epsilon). \end{aligned}$$

Now, integrate by parts with respect to $\hat{\delta}$, and use $\hat{\delta}\theta = -\theta^2$ to get

$$(11.93) \quad \begin{aligned} \Omega_{p, q} &= [S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \hat{\delta}S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) \\ &\quad + S_k(\hat{\delta}\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) + 2S_k((\theta^2)^{p+1}, (\hat{d}\theta)^{q-1}, \epsilon) \\ &= [S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] - \hat{\delta}S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon) + S_k((\theta^2)^{p+1}, (\hat{d}\theta)^{q-1}, \epsilon). \end{aligned}$$

Use integration by parts with respect to \hat{d} , and use $\hat{d}\theta^2 = \hat{\delta}\hat{d}\theta$ to get

$$\begin{aligned}
(11.94) \quad & S_k((\theta^2)^{p+1}, (\hat{d}\theta)^{q-1}, \epsilon) = S_k((\theta^2)^{p+1}, \hat{d}\theta, (\hat{d}\theta)^{q-2}, \epsilon) \\
& = \hat{d}S_k((\theta^2)^{p+1}, \theta, (\hat{d}\theta)^{q-2}, \epsilon) - (p+1)S_k((\theta^2)^p, \hat{d}\theta^2, \theta, (\hat{d}\theta)^{q-2}, \epsilon) \\
& + S_k((\theta^2)^{p+1}, \theta, (\hat{d}\theta)^{q-2}, \hat{d}\epsilon) \\
& = \hat{d}S_k((\theta^2)^{p+1}, \theta, (\hat{d}\theta)^{q-2}, \epsilon) - (p+1)S_k((\theta^2)^p, \hat{\delta}\hat{d}\theta, \theta, (\hat{d}\theta)^{q-2}, \epsilon) + 2\Omega_{p+1, q-2}.
\end{aligned}$$

Now, we have to integrate by parts with respect to $\hat{\delta}$. This yields

$$\begin{aligned}
(11.95) \quad & S_k((\theta^2)^{p+1}, (\hat{d}\theta)^{q-1}, \epsilon) = \hat{d}S_k((\theta^2)^{p+1}, \theta, (\hat{d}\theta)^{q-2}, \epsilon) - \frac{p+1}{q-1}\hat{\delta}S_k((\theta^2)^p, \theta, (\hat{d}\theta)^{q-1}, \epsilon) \\
& + \frac{p+1}{q-1}S_k((\theta^2)^p, \hat{\delta}\theta, (\hat{d}\theta)^{q-1}, \epsilon) + 2\Omega_{p+1, q-2} \\
& = \hat{d}S_k((\theta^2)^{p+1}, \theta, (\hat{d}\theta)^{q-2}, \epsilon) - \frac{p+1}{q-1}\hat{\delta}S_k((\theta^2)^p, \theta, (\hat{d}\theta)^{q-1}, \epsilon) \\
& - \frac{p+1}{q-1}S_k((\theta^2)^{p+1}, (\hat{d}\theta)^{q-1}, \epsilon) + 2\Omega_{p+1, q-2}.
\end{aligned}$$

Thus

$$\begin{aligned}
(11.96) \quad & \frac{q+p}{q-1}S_k((\theta^2)^{p+1}, (\hat{d}\theta)^{q-1}, \epsilon) \\
& = \hat{d}S_k((\theta^2)^{p+1}, \theta, (\hat{d}\theta)^{q-2}, \epsilon) - \frac{p+1}{q-1}\hat{\delta}S_k((\theta^2)^p, \theta, (\hat{d}\theta)^{q-1}, \epsilon) + 2\Omega_{p+1, q-2}.
\end{aligned}$$

This is the same as

$$\begin{aligned}
(11.97) \quad & S_k((\theta^2)^{p+1}, (\hat{d}\theta)^{q-1}, \epsilon) = \frac{q-1}{q+p}\hat{d}S_k((\theta^2)^{p+1}, \theta, (\hat{d}\theta)^{q-2}, \epsilon) \\
& - \frac{p+1}{q+p}\hat{\delta}S_k((\theta^2)^p, \theta, (\hat{d}\theta)^{q-1}, \epsilon) + 2\frac{q-1}{q+p}\Omega_{p+1, q-2}.
\end{aligned}$$

We finally get the general iteration formula.

Lemma 4. Let k be a positive integer, and let p, q be positive integers such that $k = p + q + 1$. Then the forms $\Omega_{p, q} = S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^q)$ on Ω , associated with the Chern-Simons forms $c_k^1(\mathbb{A})$ for the superconnection $\mathbb{A} = t\theta$ on Ω , where $t \in [0, 1]$, satisfy the following iteration formula

$$\begin{aligned}
(11.98) \quad & \Omega_{p, q} = [S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + 2\frac{q-1}{q+p}\Omega_{p+1, q-2} \\
& + \frac{q-1}{q+p}\hat{d}S_k(\theta, (\theta^2)^{p+1}, (\hat{d}\theta)^{q-2}, \epsilon) - \frac{2p+1}{q+p}\hat{\delta}S_k((\theta^2)^p, \theta, (\hat{d}\theta)^{q-1}, \epsilon).
\end{aligned}$$

Particularly, we have

$$(11.99) \quad \Omega_{p,q} = [S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + 2\frac{q-1}{q+p}\Omega_{p+1,q-2},$$

modulo coboundaries.

Proof. □

Insert normalization constants to get

$$(11.100) \quad \phi_{p,q}\Omega_{p,q} = \phi_{p,q}[S_k(\theta, (\theta^2)^p, (\hat{d}\theta)^{q-1}, \epsilon), \theta] + 2\frac{\phi_{p,q}}{\phi_{p+1,q-2}}\frac{q-1}{q+p}\phi_{p+1,q-2}\Omega_{p+1,q-2} + d\Omega.$$

Constants are

$$(11.101) \quad \phi_{p,q} = (-1)^p(p+q+1)\binom{p+q}{p}B(p+1, p+q+1) = (-1)^p\frac{(p+q+1)!(p+q)!}{(2p+q+1)!q!}$$

and

$$(11.102) \quad \frac{\phi_{p,q}}{\phi_{p+1,q-2}} = -\frac{(p+q+1)(p+q)}{q(q-1)}.$$

This gives the iteration formula for the Chern-Simons forms when we change variables with $p = m - k$ and $q = 2k - m - 1$.

Proposition 42. The Chern-Simons forms $c_k^1(\mathbb{A})$ for the superconnection $\mathbb{A} = t\theta$ on Ω satisfy the following relation, for $k \leq m \leq 2k - 1$,

$$(11.103) \quad c_{k,[m]}^1(\mathbb{A}) = e'_{k,m}[S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^{2k-m-2}, \epsilon), \theta] + e_{k,m}c_{k-1,[m]}^1(\mathbb{A}),$$

modulo coboundaries, where

$$(11.104) \quad e'_{k,m} = (-1)^{k-m}k\binom{k-1}{m-k}B(m-k+1, k) = (-1)^{m-k}\frac{k!(k-1)!}{m!(2k-m-1)!}$$

$$e_{k,m} = -2\frac{k}{2k-m-1}.$$

Proof. □

This iteration can be done down to the forms of the type $\Omega_{p+\frac{q}{2},0}$, when m is odd. The above proposition gives by iteration the decomposition result.

Proposition 43. The Chern-Simons forms $c_k^1(\mathbb{A})$ for the superconnection $\mathbb{A} = t\theta$ on Ω can be represented in the following form, for $k \leq m \leq 2k - 1$,

$$(11.105) \quad c_{k,[m]}^1(t\theta) = d\Omega + [\Omega, \Omega] + e_{k,m}^{N-1}c_{k-N,[m]}^1(t\theta)$$

for any integer N satisfying $0 < N \leq k - 1 - \frac{m-1}{2}$ and where

$$(11.106) \quad e_{k,m}^n = e_{k,m}e_{k-1,m} \cdots e_{k-n,m}$$

$$e_{k,m}^0 = 1.$$

More precisely, we have a decomposition

$$(11.107) \quad c_{k,[m]}^1(t\theta) = \sum_{n=0}^{N-1} e_{k,m}^n e'_{k-n,m} [S_{k-n}(\theta, (\theta^2)^{m-k+n}, (\hat{d}\theta)^{2k-m-2-2n}, \epsilon), \theta] \\ + e_{k,m}^{N-1} c_{k-N,[m]}^1(t\theta),$$

modulo coboundaries.

Proof. Follows by iteration from the above proposition. \square

The case $k = m$ is the most important for us. Then we have

$$(11.108) \quad c_{m,[m]}^1(t\theta) = d\Omega + [\Omega, \Omega] + e_{m,m}^{m'-1} c_{m'+1,[m]}^1(t\theta).$$

If m is even, then we can iterate down to the forms of the type $\Omega_{p+q',1}$, which were shown to decompose, where $q = 2q' + 1$. In this case we do not have Θ forms.

We now obtain the main result of this section.

Theorem 2. *Let \mathbb{A} be the superconnection $ta + \theta$ on Ω , where $t \in [0, 1]$, then the corresponding Chern-Simons forms can be decomposed to the standard form*

$$(11.109) \quad d\Omega + [\Omega, \Omega] + \Theta.$$

More precisely, we have the following decomposition, for $0 \leq m \leq k - 1$,

$$(11.110) \quad c_{k,[m]}^1(ta + \theta) = \sum_{n=0}^{k-m-2} d_{k,m}^n d'_{k-n,m} [S_{k-n}(a, (a^2)^{k-n-m-1}, (\hat{d}\theta)^{m-1}, \epsilon), \theta] \\ + \sum_{n=0}^{k-m-2} d_{k,m}^n d''_{k-n,m} [S_{k-n}(a, (a^2)^{k-n-m-2}, (\hat{d}\theta)^m, \epsilon), a] \\ + d_{k,m}^{k-m-2} [S_{m+1}(a, (\hat{d}\theta)^{m-1}, \epsilon), \theta] \\ + 2d_{k,m}^{N-1} \sum_{n=0}^{N-1} e_{m,m}^n e'_{m-n,m} [S_{m-n}(\theta, (\theta^2)^n, (\hat{d}\theta)^{m-2n}, \epsilon), \theta] \\ + 2e_{m,m}^{N-1} c_{m-N,[m]}^1(t\theta),$$

modulo coboundaries. Here, $N = \frac{m-3}{2}$ in the case m odd. In the case m even we take $N = k - \frac{m+2}{2}$. Then the final term in the above expansion is of the type $S_k(\theta, (\theta^2), \hat{d}\theta)$, which can be decomposed.

Here, the above constants are the same as in the propositions Proposition 41 and Proposition 43.

Proof. We use the proposition 41 to get a decomposition of a form $d\Omega + [\Omega, \Omega] + d_{k,m}^{N-1} 2c_{m,m}^1(t\theta)$. Now the decomposition result for $\mathbb{A} = t\theta$, proposition 43, shows that the forms $c_{m,m}^1(t\theta)$ can be decomposed. By combining these decompositions we get the decomposition in the assertion. This concludes the proof. \square

Proposition 44. Let $\mathbb{A}_0 = 0, \mathbb{A}_1 = a + \theta$ be fixed superconnections on Ω . Then the Chern-Simons form $c_k^1(\mathbb{A}_0, \mathbb{A}_1)$ can be decomposed to the standard form.

Proof. By triangle formula

$$(11.111) \quad c_k^1(0, \theta) + c_k^1(\theta, a + \theta) + c_k^1(a + \theta, 0) = d\Omega + [\Omega, \Omega].$$

That is

$$(11.112) \quad c_k^1(0, a + \theta) = c_k^1(0, \theta) + c_k^1(\theta, a + \theta) + d\Omega + [\Omega, \Omega].$$

Thus $c_k^1(\mathbb{A}_0, \mathbb{A}_1)$ can be decomposed. \square

Corollary 4. Let $\mathbb{A}_0 = 0$ and $\mathbb{A}_1 = a + \theta$ be fixed superconnections on Ω . Furthermore, let \mathbb{A}_2 be a fixed superconnection on Ω . Then the Chern-Simons form $c_k(\mathbb{A}_0, \mathbb{A}_2, \mathbb{A}_1)$ can be decomposed to the standard form.

Proof. Follows from the triangle formula, as above. \square

Corollary 5. Let $\mathbb{A} = ta$ be a superconnection on Ω , where $t \in [0, 1]$, then the corresponding Chern-Simons-forms can be decomposed to the standard form.

Proof. The Chern-Simons forms $c_{k,[m]}^1(ta)$ are part of the 'chain' $c_{k,[m]}^1(0, a, a + \theta)$, when $0 \leq m \leq k - 1$. Thus the above corollary applies. \square

12. FURTHER PROPERTIES OF ETA-COCYCLES AND COMPUTATIONS

We now apply the results of the previous section to the eta-cocycles. Most of the results follow directly from the decomposition theorem with homotopy invariance results.

12.1. Reducing eta-chains.

Definition 23. Let \mathbb{A} be a superconnection on Ω depending on a parameter $t \in [0, 1]$. We say that $\eta_k(\mathbb{A})$ -chain

$$(12.1) \quad \eta_k(\mathbb{A}) = \overline{\mathbf{Tr}}_s c_k^1(\mathbb{A}),$$

has been reduced if the Chern-Simons form $c_k^1(\mathbb{A})$ has been brought to the standard form.

That $\eta_k(\mathbb{A})$ has been reduced means that $\eta_k(\mathbb{A})$ is of (formally) a form

$$(12.2) \quad \overline{\mathbf{Tr}}_s[\Omega, \Omega] + \overline{\mathbf{Tr}}_s d\Omega + \overline{\mathbf{Tr}}_s \Theta,$$

where $\overline{\mathbf{Tr}}_s[\Omega, \Omega]$ denotes the regularized traces of (graded)commutators, $\overline{\mathbf{Tr}}_s d\Omega$ denotes the regularized traces of $\hat{\delta}$ and \hat{d} coboundaries, and $\overline{\mathbf{Tr}}_s \Theta$ denotes the regularized traces of Θ -terms. The above decomposition reads modulo boundary terms and coboundaries

$$(12.3) \quad \mathbf{Wres}_s[l, \Omega]\Omega + \overline{\mathbf{Tr}}_s \Theta,$$

where $\mathbf{Wres}_s[l, \Omega]\Omega$ denotes the Wodzicki-residues of the form $\mathbf{Wres}_s[l, \omega]\eta$, where $l = \log Q$ and $\omega, \eta \in \Omega$.

Theorem 3. Let $\mathbb{A} = ta + \theta$ be a superconnection on Ω , where $t \in [0, 1]$. Denote $l = \log Q$, where Q is the weight of the regularized trace. Consider the corresponding eta-chain

$$(12.4) \quad \eta_k(\mathbb{A}) = \overline{\mathbf{Tr}}_s c_k^1(\mathbb{A}).$$

Then $\eta_k(\mathbb{A})$ can be reduced. Particularly, it can be expressed in terms of Wodzicki-residues modulo coboundaries, boundary terms and $\overline{\mathbf{Tr}}_s \Theta$ -terms. More precisely, we have modulo boundary terms and coboundaries, for $0 \leq m \leq k - 1$,

$$(12.5) \quad \begin{aligned} \eta_{k,[m]}(\mathbb{A}) &= \sum_{n=0}^{k-m-2} d_{k,m}^n d'_{k-n,m} \mathbf{Wres}_s[l, \theta] S_{k-n}(a, (a^2)^{k-n-m-1}, (\hat{d}\theta)^{m-1}, \epsilon) \\ &+ \sum_{n=0}^{k-m-2} d_{k,m}^n d''_{k-n,m} \mathbf{Wres}_s[l, a] S_{k-n}(a, (a^2)^{k-n-m-2}, (\hat{d}\theta)^m, \epsilon) \\ &+ d_{k,m}^{k-m-2} \mathbf{Wres}_s[l, \theta] S_{m+1}(a, (\hat{d}\theta)^{m-1}, \epsilon) \\ &+ 2d_{k,m}^{N-1} \sum_{n=0}^{N-1} e_{m,m}^n e'_{m-n,m} \mathbf{Wres}_s[l, \theta] S_{m-n}(\theta, (\theta^2)^n, (\hat{d}\theta)^{m-2n}, \epsilon) \\ &+ 2e_{m,m}^{N-1} \overline{\mathbf{Tr}}_s c_{m-N,[m]}^1(t\theta). \end{aligned}$$

Here $N = m - 1 - \frac{m-1}{2}$ and the constants can be read from Proposition 41 and Proposition 43.

Proof. This follows directly from Theorem 2 and the trace anomaly formula. \square

Theorem 4. Let $\mathbb{A} = t\theta$ be a superconnection on Ω , where $t \in [0, 1]$. Denote $l = \log Q$, where Q is the weight of the regularized trace. Consider the corresponding eta-chain

$$(12.6) \quad \eta_k(\mathbb{A}) = \overline{\mathbf{Tr}}_s c_k^1(\mathbb{A}).$$

Then $\eta_k(\mathbb{A})$ can be reduced. Particularly, it can be expressed in terms of Wodzicki-residues modulo coboundaries, boundary terms and $\overline{\mathbf{Tr}}_s \Theta$. More precisely, we have modulo boundary terms and coboundaries, for $k \leq m \leq 2k - 1$,

$$(12.7) \quad \begin{aligned} \eta_{k,[m]}(\mathbb{A}) &= \sum_{n=0}^{N-1} e_{k,m}^n e'_{k-n,m} \mathbf{Wres}_s[l, \theta] S_{k-n}(\theta, (\theta^2)^{m-k+n}, (\hat{d}\theta)^{2k-m-2-2n}, \epsilon) \\ &+ e_{k,m}^{N-1} \overline{\mathbf{Tr}}_s c_{k-N,[m]}^1(\mathbb{A}), \end{aligned}$$

where $N = k - 1 - \frac{m-1}{2}$.

Proof. Follows from Proposition 43 and the trace anomaly formula. \square

Corollary 6. Let $\mathbb{A} = ta + \theta$ be a superconnection on Ω , where $t \in [0, 1]$. Assume $\eta_k(\mathbb{A}) = \overline{\mathbf{Tr}}_s c_k^1(\mathbb{A})$ is an eta-m-cocycle. Then δ -coboundary of $\eta_{k,[m]}(\mathbb{A})$ lies on the boundary. Furthermore, these boundary-cocycles are local modulo δ -coboundaries on the boundary.

Proof. That $\delta\eta_{k,[m]}(\mathbb{A})$ lies on the boundary, follows from the definition. We may assume $\eta_{k,[m]}(\mathbb{A})$ is reduced. Now, all the boundary terms arising from regularized traces of commutators in the reduced form of $\eta_{k,[m]}(\mathbb{A})$ become trivial δ -coboundaries on the boundary, when we take δ coboundary. Next, we consider the Θ -terms. Write $m = 2m' + 1$, then

$$(12.8) \quad \delta\theta^{2m'+1} = \Gamma\hat{\delta}\theta(-\hat{\delta}\theta)^{m'} = -\Gamma\theta^{2m'+2} = \frac{1}{2}\Gamma[\theta, \theta^{2m'+1}].$$

Consider $\overline{\mathbf{Tr}}\Gamma[\theta, \theta^{2m'+1}]$. The terms coming from the boundary term in the trace anomaly formula are vanishing. This follows from the product geometry assumption (vertical vector fields are constant in x near the collar neighborhood of the boundary). Thus, by our assumption, the local formula for $\delta\eta_k(\mathbb{A})$ comes from the δ -coboundaries of the Wodzicki-residues and the Wodzicki residue coming from the δ -coboundary of Θ -term. Together these local expressions must lie on the boundary or vanish identically. \square

Example 36. Consider the superconnections $\mathbb{A} = ta$ and $\mathbb{A}' = ta + \theta$ on Ω and their corresponding Chern-Simons forms when $0 \leq m \leq k - 1$.

Recall that (see Corollary 3)

$$(12.9) \quad c_{k,[m]}^1(\mathbb{A}) - c_{k,[m]}^1(\mathbb{A}') = \left(\int_{I^2} d_{\mathbb{A}}\omega_k^2(\mathbb{A}) \right)_{[m]},$$

where I^2 denotes the unit square.

The above formula gives an explicit way to reduce the eta-chain corresponding to ta by using results on eta-chain coming from $ta + \theta$. Thus, we can also reduce the forms $F(dF)^m$, when restricted to vertical directions.

13. A REGULARIZATION OF ETA-COCYCLES

We now discuss the following problem. Fix an odd integer m and let $\mathbb{A} = ta + \theta$ be a superconnection on Ω , where $t \in [0, 1]$. Consider its eta- m chain

$$(13.1) \quad \eta_{k,[m]}(\mathbb{A}) = \overline{\mathbf{Tr}}_s c_{k,[m]}^1(\mathbb{A}),$$

where $k > m$.

Suppose the eta- m -cocycle condition is not satisfied. We wish to regularize this expression by raising the k so that the eta- m -cocycle property is satisfied. This is indeed possible. We also want the following normalization condition. Namely, if $\eta_{k,[m]}$ would be an eta- m cocycle, then we would want $\eta_{k,[m]} = \eta_{k+l,[m]}$ modulo δ coboundaries and boundary terms.

This, however, does not happen. We must introduce a normalization constant $g_{l,m}$, which is to assure that $\eta_{k,[m]}$ and $g_{l,m}\eta_{k+l,[m]}$ are equivalent. This constant is found by using the decomposition results.

From the decomposition result, Proposition 36, we obtain

$$(13.2) \quad \eta_{k+l,[m]}(\mathbb{A}) = d_{k,m}^{l-1}\eta_{k,[m]}(\mathbb{A}) + \overline{\mathbf{Tr}}_s d\Omega + \overline{\mathbf{Tr}}_s[\Omega, \Omega].$$

Therefore, we define the l -step regularization of $\eta_{k+l,[m]}$ by $\frac{1}{d_{k+l,m}^{l-1}}\eta_{k+l,[m]}$.

Definition 24. Let $\mathbb{A} = ta + \theta$ be a superconnection on Ω , where $t \in [0, 1]$. Then the l -step regularization of

$$(13.3) \quad \eta_{k,[m]}(\mathbb{A}) = \overline{\mathbf{Tr}}_s c_{k,[m]}^1(\mathbb{A})$$

is

$$(13.4) \quad R_l \eta_{k,[m]}(\mathbb{A}) = \frac{1}{d_{k+l,m}^{l-1}} \overline{\mathbf{Tr}}_s c_{k+l,[m]}^1(\mathbb{A}),$$

where the normalization constant $d_{k+l,m}^{l-1}$ can be read from Proposition 36.

Proposition 45. Let $\mathbb{A} = ta + \theta$ be a superconnection on Ω , where $t \in [0, 1]$. Fix positive integers k and m such that $0 \leq m \leq k-1$. Then, the eta-chain $\eta_{k,[m]}(\mathbb{A})$ can be regularized by $R_l \eta_{k,[m]}(\mathbb{A})$ by choosing l high enough such that the eta- m -cocycle condition is satisfied. Furthermore, if $l' > l$, where $R_{l'} \eta_{k,[m]}(\mathbb{A})$ is eta- m -cocycle and l' is a positive integer, then

$$(13.5) \quad R_l \eta_{k,[m]}(\mathbb{A}) = R_{l'} \eta_{k,[m]}(\mathbb{A}),$$

modulo boundary terms and coboundaries.

Proof. The eta-cocycle condition can be checked by the computation of orders. Recall that order of $c_{k,[m]}^1(\mathbb{A})$ is $-2k + m - 1$. Now, if we choose a positive integer l , such that $-2(k+l) + m - 1 \leq -\dim M$, then $R_l \eta_{k,[m]}(\mathbb{A})$ satisfies the eta- m -cocycle property (see Example 30). Latter assertion follows from the definition and Proposition 36 by considering the orders of the operators. \square

We finally get the regularizations of all the forms of the type $F(dF)^m$.

Proposition 46. Consider again the forms from Proposition 35

$$(13.6) \quad \hat{\omega}_{k,[m]} = b_{k,m} S_k(F - \epsilon, [(F - \epsilon)^2]^{k-1-m}, (dF)^m),$$

where $0 \leq m \leq k-1$ and

$$(13.7) \quad b_{k,m} = (-1)^{k-m-1} k \binom{k-1}{m} B(k-m, k).$$

Then the following expressions yield regularizations of the above type forms

$$(13.8) \quad R_l \hat{\eta}_k \equiv \frac{1}{d_{k+l,m}^{l-1}} \overline{\mathbf{Tr}}_s \hat{\omega}_{k+l,[m]}.$$

Here the $d_{k+l,m}^{l-1}$ comes from the iteration formula for the Chern-Simons form for the superconnection $\mathbb{A} = ta + \theta$ on Ω (see Proposition 36). Particularly

$$(13.9) \quad R_l \hat{\eta}_{m+1} = \frac{1}{d_{m+1+l,m}^{l-1}} \overline{\mathbf{Tr}}_s \hat{\omega}_{m+1+l,[m]},$$

gives regularizations of the forms of the type $F(dF)^m$.

Proof. Recall that the forms $\Gamma\hat{\omega}_{k,[m]}$ agree with the forms $c_{k,[m]}^1(ta)$ when restricted to vertical directions.

Now, the proposition follows from the homotopy invariance (Proposition 3) and the decomposition result, Proposition 36, by considering orders of operators. \square

13.1. The counterterm regularization of Mickelsson and Paycha. We now discuss yet another way to construct δ -cocycles, introduced in [MP]. This method is best described when the boundary is empty, which we now assume. In this method we use forms of the type

$$(13.10) \quad b_{m+1,m}F(dF)^m,$$

where constants $b_{m+1,m}$ are as above and m is a positive integer.

The above forms are equivalent to forms

$$(13.11) \quad \hat{\eta}_{m+1,[m]}(ta) = \overline{\mathbf{Tr}}_s b_{m+1,m} S_{m+1}(a, (dF)^m),$$

modulo coboundaries.

Now the obstruction being a cocycle is

$$(13.12) \quad \Lambda = \frac{1}{2} b_{m+1,m} \overline{\mathbf{Tr}}_s [F(dF)^{m+1}, F].$$

That is

$$(13.13) \quad \Lambda = \frac{1}{2} b_{m+1,m} \mathbf{Wres}_s [l, F] F(dF)^{m+1}.$$

Since the Wodzicki residue is a local, closed differential form on a contractible space B , we can find a m -form λ (this is an example of *counterterm*) on B such that

$$(13.14) \quad d\lambda = \Lambda.$$

Now the form

$$(13.15) \quad \eta_{m+1,[m]}(ta; \lambda) = \hat{\eta}_{m+1,[m]}(ta) - \lambda.$$

has the cocycle property.

Recall that $\eta_{m+1,[m]}(ta)$ is $\Gamma\hat{\eta}_{m+1,[m]}(ta)$ when restricted to the vertical directions. This gives us an alternative way to reduce and regularize the chain $\eta_{m+1,[m]}(ta)$. Now, we must keep track of the commutators coming from

$$(13.16) \quad \eta_{m+1,[m]}(ta) - \eta_{m+1,[m]}(ta + \theta) = \left(\int_{I^2} d_{\mathbf{A}} \omega_{m+1}^2 \right)_{[m]},$$

when using the reduction results. Here, the benefit is that we do not have to use the complicated iteration formula for $c_k^1(ta + \theta)$ in full generality. This makes the proof of the decomposition result for $\eta_{m+1,[m]}(ta; \lambda)$ easier. Also, we see directly that $\eta_{m+1,[m]}(ta; \lambda)$ really gives regularizations.

However, the construction of the counterterm and the computation of the residues coming from the homotopy formula is, in general, a very difficult task. Thus, the computation of the local formulas modulo Θ -terms is not any easier than working directly with the regularizations coming from $c_k^1(ta + \theta)$ for a suitable k .

See also appendix, for the direct decomposition result.

In any case, we can regularize any $\hat{\eta}_{k,[m]}(ta)$ by the above counterterm regularization. In this generality the counterterms get more complicated as k gets higher. When the manifold has a boundary, then we cannot find λ as above. However, we expect to find such λ with property that $d\lambda = \Lambda$ modulo boundary terms (by ignoring the boundary terms from the trace anomaly formula). This is good enough for our purposes.

14. COMPUTATIONS OF ETA-COCYCLES

After an eta- m cocycle is reduced, then the next step is the computation of the residues. Here, we assume the case of a closed manifold. We let M^n be a torus of dimension n (even) equipped with the standard flat metric. In the computations that follow, we discard the Θ -terms and trivial δ -coboundaries.

Actually, in the flat case, the regularized traces of Θ -terms are vanishing, if the weight is chosen carefully. Our standard choice in Example 29 works. Now in order to obtain a nontrivial result from $\overline{\mathbf{Tr}}_s \theta^m$, where m is a positive odd integer, we have to be able to choose a term of Clifford order n from the complex power of the chosen weight Q^{-z} . However, there is no such term in the flat case. Thus $\overline{\mathbf{Tr}}_s \theta^m$ vanishes identically. This argument fails in the non-flat case.

14.1. Computations in the case $\mathbb{A} = t\theta$. Consider the superconnection $\mathbb{A} = t\theta$ on Ω , where $t \in [0, 1]$, and the corresponding eta-chain

$$(14.1) \quad \eta_{k,[m]}(t\theta) = \overline{\mathbf{Tr}}_s c_{k,[m]}^1(t\theta) = \phi_{k,m} \overline{\mathbf{Tr}}_s S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^{2k-m-1}).$$

We first observe that if $2k - m - 1 < n$, then all the residues in the decomposition formula for $\eta_{k,[m]}(t\theta)$ vanish. This follows from the Clifford-algebra.

Next consider

$$(14.2) \quad \eta_{n+1,[n]}(t\theta) = \overline{\mathbf{Tr}}_s c_{n+1,[n]}^1(t\theta) = \overline{\mathbf{Tr}}_s S_{n+1}(\theta, (\hat{d}\theta)^n).$$

Then the only residue that survives is

$$(14.3) \quad \mathbf{Wres}_s[l, \theta] S_{n+1}(\theta, (\hat{d}\theta)^{n-1}, \epsilon).$$

This follows again by the Clifford algebra. The residue is

$$(14.4) \quad \int_{M^n} \mathbf{tr} S_{n+1}(\theta, (d\theta)^n) = \int_{M^n} \mathbf{tr} \theta (d\theta)^n,$$

modulo a normalization. See also [La2], for slightly different approach.

More generally, consider $2k - m - 1 = n$, then $k = \frac{n+m+1}{2}$ (m has to be odd)

$$(14.5) \quad \eta_{k,[m]}(t\theta) = \overline{\mathbf{Tr}}_s c_{k,[m]}^1 = \phi_{k,m} \overline{\mathbf{Tr}}_s S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^n).$$

Then we obtain from Proposition 40 that this is a sum of residues modulo boundary and Θ -terms. Again, only one residue term survives, and we have

$$(14.6) \quad \eta_{k,[m]}(t\theta) = \phi_{k,m} \mathbf{Wres}_s[l, \theta] S_k(\theta, (\theta^2)^{m-k}, (\hat{d}\theta)^{n-1}, \epsilon).$$

The above residue is

$$(14.7) \quad \int_{M^n} \mathbf{Str}(\theta, (\theta^2)^{m-k}, (d\theta)^n),$$

modulo a normalization.

Now the only remaining case is $2k - m - 1 > n$. Then $c_{k,[m]}^1(\mathbb{A})$ is in the trace class. Using the iteration formula from Proposition 42, we end up in the situation above. We collect the results in the following proposition.

Proposition 47. On the n -dimensional flat torus (n even) M^n , for $n = 2k - m - 1$, where k and m are positive integers, we have modulo δ -coboundaries that

$$(14.8) \quad \eta_{k,[m]}(t\theta) = N_{k,m} \int_{M^n} \mathbf{Str}(\theta, (\theta^2)^{m-k}, (d\theta)^n),$$

where \mathbf{Str} is the symmetrized trace and $N_{k,m}$ is the normalization constant. Furthermore, if $2k - m - 1 < n$, then $\eta_{k,[m]} = 0$ modulo δ -coboundaries and if $2k - m - 1 > n$, then the formula above applies modulo a normalization.

Proof. □

Remark 19. It is an instructive computation to calculate δ -coboundary from the expression $\mathbf{Str}(\theta, (\theta^2)^{m-k}, (d\theta)^n)$. We get that is d -exact.

14.2. Computations in the case $\mathbb{A} = ta + \theta$. Consider the superconnection $\mathbb{A} = ta + \theta$ on Ω , where $t \in [0, 1]$. Recall that

$$(14.9) \quad c_{k,[m]}^1(\mathbb{A}) = \phi_{k,m} S_k(a, (a^2)^{k-m-1}, (\hat{d}\theta)^m),$$

where $\phi_{k,m}$ are the normalization constants.

The easiest case is when $k = m + 1$, then

$$(14.10) \quad \eta_{m+1,[m]}(\mathbb{A}) = \overline{\mathbf{Tr}}_s S_{m+1}(a, (\hat{d}\theta)^m).$$

These are the forms that we considered earlier in example 33. First, use Proposition 36 to get

$$(14.11) \quad \eta_{m+1,[m]}(\mathbb{A}) = \overline{\mathbf{Tr}}_s [S_{m+1}(a, (\hat{d}\theta)^{m-1}, \epsilon), \theta] + 2\overline{\mathbf{Tr}}_s c_{m,[m]}^1(t\theta).$$

We are interested in the cases where $n \leq m + 1$, then the term $\overline{\mathbf{Tr}}_s c_{m,[m]}^1(t\theta)$ gives only δ -coboundaries and a trace of Θ -term (by the computations above), which we can ignore. Therefore, we only need to consider the term

$$(14.12) \quad \overline{\mathbf{Tr}}_s [S_{m+1}(a, (\hat{d}\theta)^{m-1}, \epsilon), \theta] = \mathbf{Wres}_s [l, \theta] S_{m+1}(a, (\hat{d}\theta)^{m-1}, \epsilon).$$

The above residue never vanishes on dimensions where $n \leq m + 1$ by trivial reasons. The case that can be calculated at once happens when $n = m + 1$. Then we get

$$(14.13) \quad \eta_{m+1,[m]}(\mathbb{A}) = \int_{M^n} \mathbf{Str}(A, (d\theta)^m),$$

modulo normalization. This is a δ -cocycle. Particularly, δ -coboundary of the expression $\mathbf{Str}(A, (d\theta)^m)$ is d -exact.

Now, by using Corollary 3 we obtain also a local formula for the forms $F[F, \theta]^m$.

Next we consider

$$(14.14) \quad c_{m+2, [m]}^1(\mathbb{A}) = \phi_{m+2, m} S_{m+2}(a, (a^2)^1, (\hat{d}\theta)^m).$$

By Example 30 the above Chern-Simons form yields eta- m -cocycles when

$$(14.15) \quad -(m+3) \leq -n.$$

We have the representation (modulo δ -coboundaries)

$$(14.16) \quad c_{m+2, [m]}^1(\mathbb{A}) = d'_{m+2, m} [S_{m+2}(a, a^2, (\hat{d}\theta)^{m-1}, \epsilon), \theta] + d''_{m+2, m} [S_{m+2}(a, (\hat{d}\theta)^m, \epsilon), a] \\ + d_{m+2, m} c_{m+1, m}^1(\mathbb{A}).$$

Thus

$$(14.17) \quad \eta_{m+2, [m]}(\mathbb{A}) \\ = d'_{m+2, m} \mathbf{Wres}_s[l, \theta] S_{m+2}(a, a^2, (\hat{d}\theta)^{m-1}, \epsilon) + d''_{m+2, m} \mathbf{Wres}_s[l, a] S_{m+2}(a, (\hat{d}\theta)^m, \epsilon) \\ + d_{m+2, m} \overline{\mathbf{Tr}}_s c_{m+1, [m]}^1(\mathbb{A}),$$

modulo δ -coboundaries. Now, using the knowledge of earlier results, we may write (modulo δ -coboundaries)

$$(14.18) \quad \eta_{m+2, [m]}(\mathbb{A}) \\ = d'_{m+2, m} \mathbf{Wres}_s[l, \theta] S_{m+2}(a, a^2, (\hat{d}\theta)^{m-1}, \epsilon) + d''_{m+2, m} \mathbf{Wres}_s[l, a] S_{m+2}(a, (\hat{d}\theta)^m, \epsilon) \\ + d_{m+2, m} d'_{m+1, m} \mathbf{Wres}_s[l, \theta] S_{m+1}(a, (\hat{d}\theta)^{m-1}, \epsilon).$$

Example 37 (1-cocycle up to dimension 4). For $m = 1$ the formula (14.18) yields

$$(14.19) \quad \eta_{3, [1]}(\mathbb{A}) = d'_{3, 1} \mathbf{Wres}_s[l, \theta] S_3(a, a^2, \epsilon) + d''_{3, 1} \mathbf{Wres}_s[l, a] S_3(a, \hat{d}\theta, \epsilon) \\ + d_{3, 1} d'_{2, 1} \mathbf{Wres}_s[l, \theta] S_2(a, \epsilon).$$

The cocycle property follows from the eta-cocycle condition (14.15).

Example 38 (3-cocycle up to dimension 6). For $m = 3$ the formula (14.18) yields

$$(14.20) \quad \eta_{5, [3]}(\mathbb{A}) = d'_{5, 3} \mathbf{Wres}_s[l, \theta] S_5(a, a^2, (\hat{d}\theta)^2, \epsilon) + d''_{5, 3} \mathbf{Wres}_s[l, a] S_5(a, (\hat{d}\theta)^3, \epsilon) \\ + d_{5, 3} d'_{4, 3} \mathbf{Wres}_s[l, \theta] S_4(a, (\hat{d}\theta)^2, \epsilon).$$

The cocycle property follows as above.

Remark 20. It is difficult to compute the above forms, even in the case of the flat metric. However, if we restrict B to flat connections, then we can compute every form. This follows, again, by analysing the Clifford-algebra as above.

15. THE ODD CASE

We extend the formalism to the odd-dimensional manifolds, following the approach in [LaMiRy]. We take the same assumptions as in Section 6.

Now there is no grading operator Γ . We introduce the grading operator by hand. This is done by doubling the Hilbert space \mathcal{H} , where the operators act. We consider $\mathcal{H} \otimes \mathbb{C}^2$ and introduce two odd variables σ_1 and σ_3 (Pauli-matrices), that anticommute with each other, square to one and commute with everything else. They act only on the second factor of the doubled Hilbert space.

Now we define

$$\begin{aligned}
 (15.1) \quad v &= \theta \otimes \sigma_1 \\
 s &= \delta \otimes \sigma_1 \\
 \bar{\epsilon} &= \epsilon \otimes \sigma_3 \\
 \bar{F} &= F \otimes \sigma_3 \\
 \bar{a} &= \bar{F} - \bar{\epsilon} = (F - \epsilon) \otimes \sigma_3.
 \end{aligned}$$

We think F , a and ϵ as even objects. Operators v , s , \bar{F} , \bar{a} and $\bar{\epsilon}$ become odd objects, since they contain odd number of sigmas. We define the odd noncommutative BRST-complex Ω_σ using the above variables, as generators. Any element of Ω_σ can be written in the form $\omega \otimes e$, where $\omega \in \Omega$ and e is an element generated by σ_1 and σ_3 . Here Ω is defined as before except we do not use the grading operator in front of θ .

We can now define the noncommutative exterior derivative on Ω_σ as the graded commutator $\hat{d} = [\bar{\epsilon}, \cdot]$. However, now the grading comes, essentially, from the sigmas. More precisely, if $\omega_\sigma = \omega \otimes e$ and $\omega'_\sigma = \omega' \otimes e'$ are elements of Ω_σ , then

$$(15.2) \quad [\omega_\sigma, \omega'_\sigma] = \omega_\sigma \omega'_\sigma - (-1)^{s'} \omega'_\sigma \omega_\sigma,$$

where

$$(15.3) \quad s' = p(\omega_\sigma) p(\omega'_\sigma).$$

Here, the parity p is defined as the number of the sigmas. That is, $p(\omega_\sigma)$ is the number of the sigmas in e . Observe, that by definition

$$(15.4) \quad p(\omega_\sigma) = p(\omega) + \partial\omega,$$

where $p(\omega)$ stands for the parity of ω and $\partial\omega$ is the the form degree as in (6.4).

It follows that

$$\begin{aligned}
 (15.5) \quad [\omega_\sigma, \omega'_\sigma] &= (\omega \otimes e) \cdot (\omega' \otimes e') - (-1)^{s'} (\omega' \otimes e') \cdot (\omega \otimes e) \\
 &= \omega \omega' \otimes ee' - (-1)^{s'} \omega' \omega \otimes e' e \\
 &= (\omega \omega' - (-1)^{s' + p(\omega) \partial\omega' + \partial\omega p(\omega)} \omega' \omega) \otimes ee' \\
 &= (\omega \omega' - (-1)^{p(\omega) p(\omega') + \partial\omega \partial\omega'} \omega' \omega) \otimes ee'.
 \end{aligned}$$

Thus, we can use the above calculation to get back to supercommutators given by the sign rule (6.4).

We have, for example,

$$(15.6) \quad \hat{d}v = [\bar{\epsilon}, v] = \bar{\epsilon}v + v\bar{\epsilon} = \epsilon\sigma_3\theta\sigma_1 + \theta\sigma_1\epsilon\sigma_3 = (\epsilon\theta - \theta\epsilon)\sigma_3\sigma_1 = [\epsilon, \theta]\sigma_3\sigma_1.$$

Similarly we have

$$(15.7) \quad \hat{d}\bar{a} = [\bar{\epsilon}, \bar{a}] = \bar{\epsilon}v + v\bar{a} = \epsilon\sigma_3a\sigma_3 + a\sigma_3\epsilon\sigma_3 = (\epsilon a + a\epsilon)\sigma_3\sigma_3 = [\epsilon, a]_+1.$$

Let $\Omega_\sigma = \Omega_{\sigma_+} \oplus \Omega_{\sigma_-}$ with the \mathbb{Z}_2 -grading given by the sigmas as above. Note that s and \hat{d} are odd maps in Ω_σ and they anticommute with each other. Hence we get a bicomplex with differentials s, \hat{d} and the total differential $d = \hat{d} + s$. The BRST-algebra relations are the same for s, \hat{d}, v, \bar{a} as before (for old 'variables') in the even dimensional case. Also, the graded-symmetrization operator is defined as before.

Example 39. Let us compute sa . We have

$$(15.8) \quad sa = \delta a \otimes \sigma_3 = ([\epsilon, \theta] + [a, \theta]) \otimes \sigma_3.$$

Example 40. Now, we compute $s\bar{a}$. We obtain

$$(15.9) \quad s\bar{a} = \delta a \otimes \sigma_1\sigma_3 = -([\epsilon, \theta] + [a, \theta]) \otimes \sigma_3\sigma_1 = -[\bar{\epsilon}, v] - [\bar{a}, v].$$

The regularized traces are extended to the doubled Hilbert space in the obvious way. The supertrace is, however, defined as follows (as in [LaMiRy]).

Definition 25. We define the regularized supertrace $\overline{\mathbf{Tr}}_s$ in odd dimensions with σ_3 taking the place of Γ .

Remark 21. Now we do not replace θ by $\Gamma\theta$ or s by Γs !

The superconnections are defined as before. For example, the total superconnection is

$$(15.10) \quad \begin{aligned} \mathbf{A} &= t_1\bar{a} + t_2v \\ &= t_1a\sigma_3 + t_2\theta\sigma_1, \end{aligned}$$

where $t_1, t_2 \in [0, 1]$.

The corresponding curvature is

$$(15.11) \quad \begin{aligned} \mathbf{F} &= d\mathbf{A} + \mathbf{A}^2 \\ &= dt_1\bar{a} + dt_2v \\ &+ (t_1^2 - t_1)\bar{a}^2 + t_1s\bar{a} + t_1t_2[\bar{a}, v] + t_2\hat{d}v + (t_2^2 - t_2)v^2 \\ &= dt_1\bar{a} + dt_2v \\ &+ (t_1^2 - t_1)a^2 \otimes 1 + (-t_1\delta a + t_1t_2[a, \theta] + t_2[\epsilon, \theta]) \otimes \sigma_3\sigma_1 + (t_2^2 - t_2)\theta^2 \\ &= d_t\mathbf{A} + \mathbb{F}. \end{aligned}$$

Here we have denoted

$$\begin{aligned}
\mathbb{A} &= t_1 \bar{a} + t_2 v \\
&= t_1 a \sigma_3 + t_2 \theta \sigma_1 \\
(15.12) \quad \mathbb{F} &= d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}] \\
&= (t_1^2 - t_1) \bar{a}^2 + t_1 s \bar{a} + t_1 t_2 [\bar{a}, v] + t_2 \hat{d}v + (t_2^2 - t_2) v^2 \\
&= (t_1^2 - t_1) a^2 \otimes 1 + (-t_1 \delta a + t_1 t_2 [a, \theta] + t_2 [\epsilon, \theta]) \otimes \sigma_3 \sigma_1 + (t_2^2 - t_2) \theta^2.
\end{aligned}$$

The Chern forms, Chern-Simons forms and their generalizations are defined in the same way as in the even case.

Example 41. Let $\mathbb{A} = t\bar{a} + v$ be a superconnection on Ω_σ , for $t \in [0, 1]$. Then

$$(15.13) \quad c_3^1(\mathbb{A}) = 3 \int_0^1 S_3(\bar{a}, \mathbb{F}^2).$$

Therefore

$$\begin{aligned}
(15.14) \quad c_{3,[1]}^1(\mathbb{A}) &= \phi_{3,1} S_3(\bar{a}, \bar{a}^2, \hat{d}v) \\
&= \phi_{3,1} S_3(a, a^2, [\epsilon, \theta]) \sigma_1,
\end{aligned}$$

and

$$\begin{aligned}
(15.15) \quad c_{3,[2]}^1(\mathbb{A}) &= S_3(\bar{a}, \hat{d}v, \hat{d}v) = S_3(a, [\epsilon, \theta], [\epsilon, \theta]) \\
&= -S_3(a, [\epsilon, \theta], [\epsilon, \theta]) \sigma_3.
\end{aligned}$$

Lemma 5. If the weight Q in the regularization of the trace satisfies $\delta Q = 0$, then the following identity holds

$$(15.16) \quad \delta \overline{\mathbf{Tr}}_s b = \overline{\mathbf{Tr}}_s \sigma_1 s b,$$

where b is any element of Ω_σ .

Proof. □

In contrast to the even dimensional case we consider even forms in the ghost degree.

Example 42. Consider the superconnection $\mathbb{A} = t\bar{a} + v$ on Ω_σ , where $t \in [0, 1]$.

Now we have

$$\begin{aligned}
(15.17) \quad \eta_{3,[2]}(\mathbb{A}) &= -\overline{\mathbf{Tr}}_s c_{3,[1]}^1(\mathbb{A}) = \overline{\mathbf{Tr}}_s S_3(a, [\epsilon, \theta], [\epsilon, \theta]) \sigma_3 \\
&= -\overline{\mathbf{Tr}} S_3(a, [\epsilon, \theta], [\epsilon, \theta]),
\end{aligned}$$

and

$$\begin{aligned}
(15.18) \quad \eta_{3,[1]}(\mathbb{A}) &= \overline{\mathbf{Tr}}_s c_{3,[1]}^1(\mathbb{A}) = \overline{\mathbf{Tr}} S_3(a, a^2, [\epsilon, \theta]) \sigma_3 \sigma_1 \\
&= 0.
\end{aligned}$$

15.1. Local formulas. The decomposition methods in Section 11 used, essentially, only the abstract BRST-algebra. It follows that the results there, can be used in the odd case. Especially, the reduction theorems hold directly for the eta-chains. In fact the results are even simpler, because now we do not have Θ -terms. This means that we obtain local formulas modulo boundary terms and coboundaries for eta-chains.

15.2. Mickelsson-Faddeev-Shatasvili-cocycle. We now assume $M = M^3$ is a three dimensional flat torus. As an example, we discuss one of the most important anomaly expressions that arises from quantum field theory. It is a special case of the so-called Schwinger-term. The classical expression for MFS-anomaly is usually written [LaMi] as

$$(15.19) \quad c_{MFS}(A; X, Y) = N \int_M \mathbf{tr} A[dX, dY],$$

where X and Y are vertical vector fields, A an element of B and N is the normalization constant

$$(15.20) \quad N = \frac{i}{24\pi^2}.$$

The Schwinger term coming from the commutator anomaly comes from the expression [MR], [St]

$$(15.21) \quad c_{MR} = -\frac{1}{8} \mathbf{Tr}_C(F - \epsilon)[[\epsilon, X], [\epsilon, Y]],$$

where F and ϵ have the usual meaning and \mathbf{Tr}_C denotes the conditional trace. The conditional trace is defined as

$$(15.22) \quad \mathbf{Tr}_C A = \frac{1}{2} \mathbf{Tr}(A + \epsilon A \epsilon).$$

The conditional trace agrees with the ordinary L^2 trace, when A is a trace class operator. The commutator anomaly c_{MR} can be given in terms of the regularized trace $\overline{\mathbf{Tr}}$, which we prefer.

The expression c_{MR} is far from local, but it is well known that c_{MR} and c_{MFS} are in fact equivalent δ -cocycles, that is, they differ by a δ -coboundary [LaMi]. There is a yet another form for the Schwinger-term. This is

$$(15.23) \quad c_{MP} = \frac{1}{8} \overline{\mathbf{Tr}} F[[F, X], [F, Y]].$$

The above form was studied, for example, in [MP]. We see later that c_{MP} too is equivalent to the MFS-cocycle.

First, we prove the locality of c_{MR} cocycle. We begin from

$$(15.24) \quad \omega = \bar{a}dv dv.$$

We put ω to the standard form $d\Omega_\sigma + [\Omega_\sigma, \Omega_\sigma]$. This proves the locality, when we take the (compatible) regularized trace from ω . Compute, as before, to get

$$\begin{aligned}
\bar{a}dv dv &= [\bar{a}dv\bar{\epsilon}, v] + \bar{a}[dv, v]\bar{\epsilon} + [\bar{a}, v]dv\bar{\epsilon} \\
&= [\bar{a}dv\bar{\epsilon}, v] + \bar{a}sdv\bar{\epsilon} - s\bar{a}dv\bar{\epsilon} + dv dv\bar{\epsilon} \\
(15.25) \quad &= [\bar{a}dv\bar{\epsilon}, v] - s(\bar{a}dv\bar{\epsilon}) + d(vdv\bar{\epsilon}) + 2vdv \\
&= [\bar{a}dv\bar{\epsilon}, v] - s(\bar{a}dv\bar{\epsilon}) + d(vdv\bar{\epsilon}) + 2[v\bar{\epsilon}, v] + 2[v, v]\bar{\epsilon} \\
&= [\bar{a}dv\bar{\epsilon}, v] + 2[v\bar{\epsilon}, v] - s(\bar{a}dv\bar{\epsilon}) + d(vdv\bar{\epsilon}) - 4(sv\bar{\epsilon}) \\
&= [\bar{a}dv\bar{\epsilon}, v] + 2[v\bar{\epsilon}, v] - s(\bar{a}dv\bar{\epsilon} + 4v\bar{\epsilon}) + d(vdv\bar{\epsilon}).
\end{aligned}$$

Thus

$$(15.26) \quad \omega = [\bar{a}dv\bar{\epsilon}, v] + 2[v\bar{\epsilon}, v] = -[a[\epsilon, \theta], \theta]\sigma_3 - 2[\theta\epsilon, \theta]\sigma_3,$$

modulo coboundaries. Therefore, we have

$$\begin{aligned}
(15.27) \quad \overline{\mathbf{Tr}}_s \omega &= -\overline{\mathbf{Tr}}[a[\epsilon, \theta], \theta] - 2\overline{\mathbf{Tr}}[\theta\epsilon, \theta] \\
&= -\mathbf{Wres}[l, \theta]a[\epsilon, \theta] - 2\mathbf{Wres}[l, \theta]\theta\epsilon,
\end{aligned}$$

modulo coboundaries. This proves the locality, and a computation gives

$$(15.28) \quad \mathbf{Wres}[l, \theta]\theta\epsilon = 0.$$

Moreover, an elementary computation yields

$$(15.29) \quad -\mathbf{Wres}[l, \theta]a[\epsilon, \theta] = N \int_M \mathbf{tr} Ad\theta d\theta.$$

The above expression was also derived in [LaMiRy]. Evaluating this expression with respect to vertical vector fields X and Y gives

$$(15.30) \quad c_{MR} = c_{MFS},$$

modulo δ -coboundaries.

We, however, prefer to begin from

$$(15.31) \quad c_3^1(t\mathbb{A})_{[2]} = S_3(\bar{a}, (dv)^2) = \Omega_{0,2},$$

since this is how the Schwinger term comes from our machinery. Note that in dimension 3 this is equivalent to $\bar{a}(dv)^2$ modulo boundary terms.

Now we prove that c_{MP} and c_{MR} are equivalent. This follows, almost immediately, from the homotopy invariance (Corollary 3). We have

$$(15.32) \quad c_3^1(t\bar{a} + \theta)_{[2]} - c_3^1(t\bar{a})_{[2]} = \left(\int_{I^2} d_{\mathbf{A}} \omega_3^2 \right)_{[2]},$$

where

$$\begin{aligned}
(15.33) \quad \int_{I^2} (d_{\mathbf{A}}\omega_3^2)_{[2]} = & s \int_{I^2} 6S_3(\bar{a}, v, (t_1^2 - t_1)\bar{a}^2) \\
& + \hat{d} \int_{I^2} 6S_3(\bar{a}, v, t_1 t_2 [\bar{a}, v] + t_2 \hat{d}v) \\
& + \int_{I^2} 6[t_1 \bar{a}, S_3(\bar{a}, v, t_1 t_2 [\bar{a}, v] + t_2 \hat{d}v)] \\
& + \int_{I^2} 6[t_2 v, S_3(\bar{a}, v, (t_1^2 - t_1)\bar{a}^2)].
\end{aligned}$$

By the trace anomaly formula we can write

$$(15.34) \quad 6 \int_{I^2} (t_1^2 - t_1)\bar{a}^3 v = 6 \frac{1}{3} \bar{a}^3 v = 2\bar{a}^3 v,$$

modulo boundary terms. Therefore, we obtain

$$\begin{aligned}
(15.35) \quad \overline{\mathbf{Tr}}_s S_3(\bar{a}, dv^2) = & \overline{\mathbf{Tr}}_s S_3(\bar{a}, s\bar{a}, s\bar{a}) + 2\overline{\mathbf{Tr}}_s s\bar{a}^3 v \\
= & \overline{\mathbf{Tr}}_s \bar{F} s\bar{a} s\bar{a} + \overline{\mathbf{Tr}}_s s\bar{e}\bar{a} s\bar{a} + 2\overline{\mathbf{Tr}}_s s\bar{a}^3 v.
\end{aligned}$$

In other words

$$(15.36) \quad \overline{\mathbf{Tr}}(F - \epsilon)[\epsilon, \theta][\epsilon, \theta] = \overline{\mathbf{Tr}}F[F, \theta][F, \theta] + \delta\overline{\mathbf{Tr}}\epsilon F[F, \theta] + 2\delta\overline{\mathbf{Tr}}(F - \epsilon)^3\theta.$$

Contracting with vertical vector fields X and Y gives

$$(15.37) \quad \frac{1}{2}\overline{\mathbf{Tr}}(F - \epsilon)[[\epsilon, X], [\epsilon, Y]] = \frac{1}{2}\overline{\mathbf{Tr}}(F - \epsilon)[[F, X], [F, Y]] + \delta b(X, Y),$$

where

$$(15.38) \quad b(X) = \overline{\mathbf{Tr}}\epsilon F[F, X] + 2\overline{\mathbf{Tr}}(F - \epsilon)^3 X.$$

Thus c_{MR} and c_{MP} are equivalent.

15.3. Schwinger term in dimension 5. We construct a 2-cocycle in the case of 5-dimensional flat torus.

Now, we cannot use the same expression as above, since the cocycle property reads

$$(15.39) \quad \delta\overline{\mathbf{Tr}}_s c_3^1(t\bar{a} + \theta) = - \int_0^1 \overline{\mathbf{Tr}}_s (d_{\mathbb{A}} S_3(\bar{a}, \mathbb{F}^2))_{[3]} = 0,$$

which is not satisfied.

Therefore, we need to regularize the original expression. We consider the Chern-Simons form for $\mathbb{A} = t\bar{a} + v$

$$(15.40) \quad c_{k,[2]}^1(\mathbb{A}) = \phi_{k,2} S_k(a, (a^2)^{k-3}, (\hat{d}v)^2).$$

We choose $k = 4$, then $\overline{\mathbf{Tr}}_s c_k^1(\mathbb{A})_{[2]}$ defines a cocycle (see Example 30). Thus, we consider the following expression

$$(15.41) \quad \Omega_{1,2} = S_4(\bar{a}, \bar{a}^2, (\hat{d}v)^2).$$

We can now use the results from Section 11 or compute directly. We choose to compute. First integrate by parts with respect to $[\cdot, v]$, to get

$$\begin{aligned}
(15.42) \quad \Omega_{1,2} &= [S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon}), v] + S_4(\bar{a}, \bar{a}^2, [\hat{d}v, v], \bar{\epsilon}) + S_4(\bar{a}, [\bar{a}^2, v], \hat{d}v, \bar{\epsilon}) \\
&\quad + S_4([\bar{a}, v], \bar{a}^2, \hat{d}v, \bar{\epsilon}) \\
&= [S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon}), v] + S_4(\bar{a}, \bar{a}^2, s\hat{d}v, \bar{\epsilon}) + S_4(\bar{a}, s\bar{a}^2, \hat{d}v, \bar{\epsilon}) + S_4(\bar{a}, [\hat{d}v, \bar{a}], \hat{d}v, \bar{\epsilon}) \\
&\quad - S_4(s\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon}) - S_4(\hat{d}v, \bar{a}^2, \hat{d}v, \bar{\epsilon}),
\end{aligned}$$

where we have used the identities

$$\begin{aligned}
(15.43) \quad [\hat{d}v, v] &= s\hat{d}v \\
[a^2, v] &= s\bar{a}^2 + [\hat{d}v, \bar{a}] \\
s\bar{a} &= -\hat{d}v - [\bar{a}, v].
\end{aligned}$$

We can identify an s -coboundary in the above expression. This allows us to write

$$(15.44) \quad \Omega_{1,2} = [S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon}), v] - s(S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon})) + S_4(\bar{a}, [dv, \bar{a}], \hat{d}v, \bar{\epsilon}) - S_4(\bar{a}^2, (\hat{d}v)^2, \bar{\epsilon}).$$

Now, we integrate by parts with respect to $[\cdot, a]$, to get

$$\begin{aligned}
(15.45) \quad \Omega_{1,2} &= [S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon}), v] - s(S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon})) \\
&\quad - \frac{1}{2}[S_4(\bar{a}, (\hat{d}v)^2, \bar{\epsilon}), \bar{a}] + \frac{1}{2}S_4(\bar{a}, (\hat{d}v)^2, d\bar{a}) \\
&\quad - \frac{1}{2}S_4([\bar{a}, \bar{a}], (\hat{d}v)^2, \bar{\epsilon}) - S_4(\bar{a}^2, (\hat{d}v)^2, \bar{\epsilon}) \\
&= [S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon}), v] - s(S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon})) \\
&\quad - \frac{1}{2}[S_4(\bar{a}, (\hat{d}v)^2, \bar{\epsilon}), \bar{a}] - \frac{1}{2}S_4(\bar{a}, \bar{a}^2, (\hat{d}v)^2) + 2S_4(\hat{d}\bar{a}, (dv)^2, \bar{\epsilon}) \\
&= [S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon}), v] - s(S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon})) \\
&\quad - \frac{1}{2}[S_4(\bar{a}, (\hat{d}v)^2, \bar{\epsilon}), \bar{a}] - \frac{1}{2}\Omega_{1,2} + 2S_4(\hat{d}\bar{a}, (dv)^2, \bar{\epsilon}),
\end{aligned}$$

where we have used $\hat{d}\bar{a} = -\bar{a}^2$, $[\bar{a}, \bar{a}] = 2\bar{a}^2$ and the definition of $\Omega_{1,2}$. Integrate by parts with respect to \hat{d} to get

$$\begin{aligned}
(15.46) \quad \Omega_{1,2} &= [S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon}), v] - s(S_4(\bar{a}, \bar{a}^2, \hat{d}v, \bar{\epsilon})) \\
&\quad - \frac{1}{2}[S_4(\bar{a}, (\hat{d}v)^2, \bar{\epsilon}), \bar{a}] - \frac{1}{2}\Omega_{1,2} + 2\hat{d}S_4(\bar{a}, (dv)^2, \bar{\epsilon}) + 4S_3(\bar{a}, (\hat{d}v)^2),
\end{aligned}$$

where we have used $\hat{d}\bar{\epsilon} = 2$ and $S_4(\bar{a}, (\hat{d}v)^2, 1) = S_3(\bar{a}, (\hat{d}v)^2)$. We see the familiar term $S_3(\bar{a}, (dv)^2)$. We put it in the form given below (this representation follows

using similar tricks as above)

$$\begin{aligned}
(15.47) \quad S_3(\bar{a}, (dv)^2) &= \Omega_{0,2} \\
&= [S_3(\bar{a}, dv, \bar{\epsilon}), v] - sS_3(\bar{a}, dv, \bar{\epsilon}) - \hat{d}S_3(v, dv, \bar{\epsilon}) - 2S_2(v, dv) \\
&= [S_3(\bar{a}, dv, \bar{\epsilon}), v] - sS_3(\bar{a}, dv, \bar{\epsilon}) - \hat{d}S_3(v, dv, \bar{\epsilon}) - 2[S_2(v, \bar{\epsilon}), v] \\
&\quad + 2sS_2(v, \bar{\epsilon}).
\end{aligned}$$

Putting all together yields

$$\begin{aligned}
(15.48) \quad \Omega_{1,2} &= [S_4(\bar{a}, \bar{a}^2, dv, \bar{\epsilon}), v] - \frac{1}{2}[S_4(\bar{a}, (dv)^2, \bar{\epsilon}), \bar{a}] \\
&\quad + 4[S_3(\bar{a}, dv, \bar{\epsilon}), v] - 8[S_2(v, \bar{\epsilon}), v] \\
&\quad - s(S_4(\bar{a}, \bar{a}^2, dv, \bar{\epsilon})) - 4sS_3(\bar{a}, dv, \bar{\epsilon}) + 8sS_2(v, \bar{\epsilon}) \\
&\quad + 2\hat{d}S_4(\bar{a}, (dv)^2, \bar{\epsilon}) - 4\hat{d}S_3(v, dv, \bar{\epsilon}) \\
&\quad - \frac{1}{2}\Omega_{1,2}.
\end{aligned}$$

Finally, we get a representation in terms of commutators and coboundaries

$$\begin{aligned}
(15.49) \quad \Omega_{1,1} &= \frac{2}{3}[S_4(\bar{a}, \bar{a}^2, dv, \bar{\epsilon}), v] - \frac{1}{3}[S_4(\bar{a}, (dv)^2, \bar{\epsilon}), \bar{a}] \\
&\quad - \frac{8}{3}[S_3(\bar{a}, dv, \bar{\epsilon}), v] + \frac{16}{3}[S_2(v, \bar{\epsilon}), v] \\
&\quad - \frac{2}{3}s(S_4(\bar{a}, \bar{a}^2, dv, \bar{\epsilon})) - \frac{8}{3}sS_3(\bar{a}, dv, \bar{\epsilon}) + 8sS_2(v, \bar{\epsilon}) \\
&\quad + \frac{4}{3}\hat{d}S_4(\bar{a}, (dv)^2, \bar{\epsilon}) - \frac{8}{3}\hat{d}S_3(v, dv, \bar{\epsilon}).
\end{aligned}$$

The normalization constant is

$$(15.50) \quad \phi_{1,2} = -\frac{3}{5}.$$

Therefore we have

$$(15.51) \quad \eta_{4,[2]} = -\frac{3}{5}\overline{\mathbf{Tr}}_s \Omega_{1,1}.$$

15.4. The general Schwinger term. Let us return to the original set up in Section 6. We still assume, however, that M is odd dimensional.

We now consider the general case

$$(15.52) \quad \omega_k = S_k(\bar{a}, (\bar{a}^2)^{k-3}, (\hat{d}v)^2).$$

Compute as before. First, integrate by parts with respect to $[\cdot, v]$ and use the above identities (15.43) to get

$$\begin{aligned}
\omega_k &= [S_k(\bar{a}, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon}), v] + S_k(\bar{a}, (\bar{a}^2)^{k-3}, [dv, v], \bar{\epsilon}) \\
&\quad + (k-3)S_k(\bar{a}, (\bar{a}^2)^{k-4}, [\bar{a}^2, v], dv, \bar{\epsilon}) + S_k([\bar{a}, v], \bar{a}^2, dv, \bar{\epsilon}) \\
(15.53) \quad &= [S_k(\bar{a}, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon}), v] + S_k(\bar{a}, (\bar{a}^2)^{k-3}, s dv, \bar{\epsilon}) \\
&\quad + (k-3)S_k(\bar{a}, (\bar{a}^2)^{k-4}, s\bar{a}^2, dv, \bar{\epsilon}) + (k-3)S_k(\bar{a}, (\bar{a}^2)^{k-4}, [dv, \bar{a}], dv, \bar{\epsilon}) \\
&\quad - S_k(s\bar{a}, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon}) - S_k(dv, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon}).
\end{aligned}$$

Observe that

$$\begin{aligned}
(15.54) \quad &s(S_k(\bar{a}, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon})) = -S_k(s\bar{a}, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon}) + (k-3)S_k(\bar{a}, (\bar{a}^2)^{k-4}, s\bar{a}^2, dv, \bar{\epsilon}) \\
&\quad + S_k(\bar{a}, (\bar{a}^2)^{k-3}, s dv, \bar{\epsilon}).
\end{aligned}$$

This yields, together with integration by parts with respect to \hat{d} in the term $S_k((\bar{a}^2)^{k-3}, (dv)^2, \bar{\epsilon})$

$$\begin{aligned}
\omega_k &= [S_k(\bar{a}, \bar{a}^2, dv, \bar{\epsilon}), v] + s(S_k(\bar{a}, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon})) \\
&\quad + (k-3)S_k(\bar{a}, (\bar{a}^2)^{k-4}, [dv, \bar{a}], dv, \bar{\epsilon}) - \hat{d}S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}) \\
(15.55) \quad &+ S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \hat{d}\bar{\epsilon}) \\
&= [S_k(\bar{a}, \bar{a}^2, dv, \bar{\epsilon}), v] + s(S_k(\bar{a}, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon})) \\
&\quad + (k-3)S_k(\bar{a}, (\bar{a}^2)^{k-4}, [dv, \bar{a}], dv, \bar{\epsilon}) - \hat{d}S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}) + 2\omega_{k-1},
\end{aligned}$$

where we have used $\hat{d}\bar{\epsilon} = 2$ and

$$(15.56) \quad S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, 1) = S_{k-1}(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2) = \omega_{k-1}.$$

Integrate by parts with respect to $[\cdot, \bar{a}]$ to get

$$\begin{aligned}
(15.57) \quad &2S_k(\bar{a}, (\bar{a}^2)^{k-4}, [dv, \bar{a}], dv, \bar{\epsilon}) = -[S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}), \bar{a}] + S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, d\bar{a}) \\
&\quad - S_k([\bar{a}, \bar{a}], (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}) \\
&= -[S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}), \bar{a}] - S_k(\bar{a}, (\bar{a}^2)^{k-3}, (dv)^2) \\
&\quad + 2S_k(d\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}),
\end{aligned}$$

where we have used $\hat{d}\bar{a} = [\bar{\epsilon}, \bar{a}]$ and $[\bar{a}, \bar{a}] = 2\bar{a}^2 = -2\hat{d}\bar{a}$. Now, use integration by parts with respect to \hat{d} and the definition of ω_k to get

$$\begin{aligned}
(15.58) \quad &2S_k(\bar{a}, (\bar{a}^2)^{k-4}, [dv, \bar{a}], dv, \bar{\epsilon}) = -[S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}), \bar{a}] - \omega_k \\
&\quad + 2\hat{d}S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}) + 4\omega_{k-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
(k-3)S_k(\bar{a}, (\bar{a}^2)^{k-4}, [dv, \bar{a}], dv, \bar{\epsilon}) &= -\frac{k-3}{2}[S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}), \bar{a}] \\
(15.59) \qquad \qquad \qquad &+ (k-3)\hat{d}S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}) \\
&- \frac{k-3}{2}\omega_k + 2(k-3)\omega_{k-1}.
\end{aligned}$$

Combine everything to get

$$\begin{aligned}
\omega_k &= \frac{2}{k-1}[S_k(\bar{a}, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon}), v] - \frac{k-3}{k-1}[S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}), \bar{a}] \\
(15.60) \qquad \qquad \qquad &+ \frac{2}{k-1}s(S_k(\bar{a}, (\bar{a}^2)^{k-3}, dv, \bar{\epsilon})) + 4\frac{k-2}{k-1}\hat{d}S_k(\bar{a}, (\bar{a}^2)^{k-4}, (dv)^2, \bar{\epsilon}) \\
&+ 4\frac{k-2}{k-1}\omega_{k-1}.
\end{aligned}$$

This gives by iteration

$$\begin{aligned}
\omega_k &= \sum_{p=0}^{k-4} 4^p \frac{2}{k-1} [S_{k-p}(\bar{a}, (\bar{a}^2)^{k-p-3}, dv, \bar{\epsilon}), v] \\
&- \sum_{p=0}^{k-4} 4^p \frac{k-(3+p)}{k-1} [S_{k-p}(\bar{a}, (\bar{a}^2)^{k-p-4}, (dv)^2, \bar{\epsilon}), \bar{a}] \\
(15.61) \qquad \qquad \qquad &+ \sum_{p=0}^{k-4} 4^p \frac{2}{k-1} s(S_{k-p}(\bar{a}, (\bar{a}^2)^{k-p-3}, dv, \bar{\epsilon})) \\
&+ \sum_{p=0}^{k-4} 4^{p+1} \frac{k-(2+p)}{k-1} \hat{d}S_{k-p}(\bar{a}, (\bar{a}^2)^{k-p-4}, (dv)^2, \bar{\epsilon}) \\
&+ 4^{k-3} \frac{2}{k-1} \omega_3.
\end{aligned}$$

Note that

$$(15.62) \qquad \qquad \qquad \omega_3 = \bar{a}(\hat{d}v)^2,$$

modulo commutators. From the above formula for ω_k we can find the normalization for the Schwinger term. The standard Schwinger term has the normalization $\frac{1}{8}$ [MR]. So we normalize accordingly. That is, we choose

$$(15.63) \qquad \qquad \qquad N_k = \frac{1}{16} \frac{k-1}{4^{k-3}},$$

as the normalization.

It follows from above that the forms $N_k \overline{\mathbf{Tr}}_s \omega_k$ for $k \geq 4$ are equivalent to the standard Schwinger term modulo δ -coboundaries, when we restrict to the three dimensional case. We also see immediately the local formula for the Schwinger term by the use of the trace anomaly formula and the earlier computation for $\Omega_{0,2}$.

16. ZERO MODES

We briefly discuss the case where the Dirac family has zero modes. Here, the idea is to perturb the Dirac family invertible. Particularly, we have to perturb the family to fully elliptic. The main technicality is the non-compactness of the parameter space. Therefore, we cannot use the powerful stabilization results from [MePi], [MePi2], [MeRo] and [MeRo2], directly, that perturb the Dirac family to a fully elliptic family. Instead, we have to apply the localization argument described below.

For explicit constructions of the boundary perturbations and relation to boundary value problems, see the above references. See also [Mo], for an explicit construction of the perturbations in the non-families case.

16.1. The fully elliptic case. We begin with the odd dimensional case. Assume the full ellipticity for the moment. Therefore the Dirac operator $\mathfrak{D}_{b,E}$ is a Fredholm operator with respect to the natural Sobolev spaces for any $b \in B$.

Using full ellipticity, we can define $\Pi_b \in x^\infty \Psi^{-\infty}(M; S \otimes E)$ as an orthogonal projection to the kernel of $\mathfrak{D}_{E,b}$, for each $b \in B$. Then $D_b = \mathfrak{D}_{E,b} + \Pi_b$, defines an invertible and gauge covariant operator, acting on \mathcal{H} , for each $b \in B$. We put $F_b = \frac{D_b}{|D_b|}$, then F_b is a fully elliptic cusp operator of order 0 acting on \mathcal{H} , satisfying $F_b^2 = 1$, for each $b \in B$.

In general, the map $b \rightarrow F_b$, where $b \in B$, is not even continuous. However, it is always continuous to gauge directions. Therefore, we have to restrict dF to the vertical directions. This is not really a restriction for us.

We have defined the grading operator F , acting on \mathcal{H} , satisfying $F^2 = 1$ and $dF|_{\text{vert}} = [F, \theta]$. This is all that is needed to construct the noncommutative BRST-complex.

Now, we consider the even dimensional case. Assume that the index of $\mathfrak{D}_{E,b}^+ : H_c^1(M; S^+ \otimes E) \rightarrow L_c^2(M; S^- \otimes E)$ is vanishing for a $b \in B$, hence for all $b \in B$. It follows that the dimensions of the kernels of $\mathfrak{D}_{E,b}^+$ and $\mathfrak{D}_{E,b}^-$ are the same. Therefore, for any b in B , there exists a smoothing operator $T_b \in x^\infty \Psi_c^{-\infty}(M; S \otimes E)$ such that $\Gamma T_b + T_b \Gamma = 0$ and $T_b^2 = 1$, when T_b is restricted to the kernel of $\mathfrak{D}_{E,b}$. The operator T_b is constructed by using an identification between the above kernels.

We define $D_b = \mathfrak{D}_{E,b} + T_b$, which is gauge covariant, then $F_b = \frac{D_b}{|D_b|} \in \Psi_c^0(M; S \otimes E)$ defines the grading operator F , acting on \mathcal{H} , and satisfying $\Gamma F + F \Gamma = 0$, $F^2 = 1$. Again, we have to restrict to the vertical directions as above. In any case, the grading operator F , needed in the definition of the noncommutative BRST-complex, is now constructed.

If the index above is not vanishing, then we cannot define the grading operator above. There is, however, a trick to overcome this difficulty. This is the so-called doubling trick. This is extensively used in noncommutative geometry. We refer to [Co] for details for this approach. The doubling, in this context, refers to the doubling of the Hilbert-space \mathcal{H} . See also [LaMiRy] and [La], where the grading operator F is defined directly via the spectral mapping theorem.

We, however, shall restrict to the index zero case.

16.2. The general case. Now, we no longer assume that $\tilde{\mathfrak{D}}_E \in \Psi_c^1(\mathbb{M}/B; S \otimes E)$ to be fully elliptic. The idea, now, is to get to the fully elliptic case. To this end, define the open sets on B by

$$(16.1) \quad U_\lambda = \{z \in B \mid \lambda \notin \text{spec}(\tilde{\mathfrak{D}}_\partial)\},$$

where λ is any non-negative real number and $\tilde{\mathfrak{D}}_\partial$ denotes the boundary (total) Dirac operator.

First assume that M is even dimensional. Then the boundary of M is odd dimensional. It is proved in [MePi] that on U_λ , the Dirac family $\tilde{\mathfrak{D}}_\partial$ has a spectral section P_λ . This spectral section is an orthogonal projection to the eigenspaces of $\tilde{\mathfrak{D}}_\partial$ with eigenvalues greater than λ .

Moreover, there exists a perturbation $A_{b,P_\lambda} \in \Psi_c^{-\infty}(M; S \otimes E)$ associated with the spectral section P_λ such that $D'_b = \tilde{\mathfrak{D}}_{E,b} + A_{b,P_\lambda}$ is fully elliptic, for any b in U_λ . Furthermore, the family D'_b over U_λ is smooth and gauge covariant. See [MePi] for details of this construction. The precise form for the perturbation is not important for us. We note that the perturbation associated with the spectral section is not canonical. However, the index (in \mathbb{Z}_2 -graded sense) of the operator $D'_b : H_c^1(M; S \otimes E) \rightarrow L_c^2(M; S \otimes E)$, for any b in U_λ , is independent of the choice of the perturbation A_{b,P_λ} .

Thus, we are back in the fully elliptic case. Now, we can apply the above arguments to the perturbed Dirac family D' over U_λ . If we assume the index zero condition, then we can construct the grading operators $F_{\lambda,b}$ defined for b in U_λ by the above argument. If the index is not vanishing, then we have to use, for example, the doubling trick construction.

The odd dimensional case is similar. We localize over the open sets U_λ of B as above. Also, the boundary Dirac family over U_λ has a $\text{Cl}(1)$ -spectral section P_λ [MePi2]. This spectral section can be taken as an orthogonal projection to the eigenspaces of $\tilde{\mathfrak{D}}_\partial$ with eigenvalues greater than λ . As before, the Dirac family is perturbed with an operator $A_{b,P_\lambda} \in \Psi_c^{-\infty}(M; S \otimes E)$, such that $D'_b = \tilde{\mathfrak{D}}_{E,b} + A_{b,P_\lambda}$ is fully elliptic and gauge covariant, for any b in U_λ (details in [MePi2]). Now, we can apply the above arguments to define the grading operators F over U_λ .

In even and odd dimensional case, the open sets U_λ of the above form cover B . Particularly, on the overlaps $U_{\lambda\lambda'} = U_\lambda \cap U_{\lambda'}$ the grading operators differ by smoothing operators. It follows that the Wodzicki-residue parts of the eta-cocycles are consistent on overlaps $U_{\lambda\lambda'}$, since the Wodzicki-residue is vanishing on smoothing perturbations. Therefore, the BRST-cohomology classes induced by the eta-cocycles are well-defined on the whole parameter space B .

Note that we can use the same weight in the regularization of the trace on each set U_λ . Namely, we can take a modification of the standard weight $Q = \sqrt{\tilde{\mathfrak{D}}_0^2}$ in Example 29, as the weight. Note that, strictly speaking, the operator Q is not a weight, if the Dirac operator is not invertible. However, we can modify Q by a smoothing operator such that Q becomes a weight. For the explicit construction

of this perturbation see [Mo]. The regularized trace associated with this perturbed weight is \hat{d} - and δ -compatible.

If we also assume that the kernels of the perturbed Dirac families D' over each set U_λ have constant rank. Then the corresponding grading operator families become smooth on each set U_λ . Particularly, the forms dF_λ on U_λ are defined.

17. SUMMARY

We have carried out the construction of the δ -transgressive differential forms on manifolds with boundary. This rather direct approach to the construction of such forms appears to be new. Usually, the construction of these differential forms uses the families index theorem. See for example [M2], [CW] (for the boundary case) and [AS2] (for the manifold without boundary).

The noncommutative framework that was used to construct the transgression forms is due to E. Langmann [La2]. In this framework, the transgression forms were constructed, essentially, from the Chern-Simons-forms.

We have proved that these Chern-Simons forms are path independent in the sense discussed earlier (see Proposition 34). Furthermore, the dependence of the path is described, essentially, by the triangle formula (see Proposition 33). In the framework that we have used, the triangle formula and the homotopy invariance of Chern-Simons forms that were proven are new.

The construction of the local representations of the induced cocycles on the boundary of the transgression forms was based on decomposing Chern-Simons forms to the standard form, that is, in terms of commutators, coboundaries and Θ -terms. We have constructed an explicit algorithm (see Lemma 3, Lemma 4, Proposition 39 and Proposition 42) which allows us to compute this decomposition. This algorithm is new.

The above algorithm was used together with the trace-anomaly formula to construct the local representations, given by Wodzicki-residues, for the coboundaries of the transgressions forms (see Theorem 3, Theorem 4 and Corollary 6). These representations are new.

If the boundary is empty, then we obtained local formulas for the eta-cocycles, given by the Wodzicki-residues modulo δ -coboundaries and possibly regularized traces of Θ -terms. These formulas (see Theorem 3 and Theorem 4), in full generality, are new. Particularly, we have obtained a local representation formula for the Schwinger term, in any odd dimension (see the equation 15.61). Even this case seems to be new. Actually, already the local formula in dimension 5 (see the equation 15.49) seems to be new. However, the three dimensional case is well-known and it is due to Mickelsson and Langmann [LaMi].

Using the above representations, we gave some explicit examples of these cocycles that are standard. We obtained the standard cocycle formulas given by differential polynomials. For example, we calculated all the δ -cocycles given by the superconnection $\mathbb{A} = t\theta$, on a flat torus (see Proposition 47). These formulas are mostly well

known. Our approach to the computation of such forms seems to be partly new (but see [La2]).

We also discussed, on a closed manifold, the construction of δ -cocycles via the counterterm regularization (see Section 13.1 and Appendix), introduced in [MP]. This method yields new formulas for the δ -cocycles. We generalized the results in [MP], giving an explicit recipe that allows us to compute the cocycles in terms of Wodzicki-residues (see Appendix). In even dimensions we have to also consider Θ -terms. Here the treatment of the even-dimensional case is new. The recipe how to obtain representations of the cocycles in terms of Wodzicki-residues, in the general ghost degree, is new.

17.1. Open problems. There are still some open problems related to the construction of δ -transgression forms in the framework that we have discussed. Here is some of them.

One problem is the actual computation of the local representatives (the Wodzicki-residues coming from the trace anomalies) of such forms in the general case. This is, however, most likely impossible to do in the general case by the techniques that we have discussed. Particularly, if we consider non-flat metrics, then the computation of the Wodzicki residues becomes extremely difficult. The difficulties are of the similar kind as trying to prove the local index theorem from the commutator anomaly (Fedosov/Calderon-formula) (see for example [Fe], [Fe2]) but our case is a lot harder. Similar problems also appear in the index computation of Mickelsson and Paycha in [MP2].

However, one special case of physical interest is the computation of the Schwinger term in the dimension 5. In the case of the flat torus, the computation from the 'local representation' should be doable. The task would be to calculate all the Wodzicki-residues coming from the regularized traces of commutators in (15.49), explicitly. The difficulty here is that we need more terms from the asymptotic expansion of $F - \epsilon$ than just the leading part. Therefore, the symbol calculus needed to carry out the computation becomes more complicated than in dimension three.

Computations in more general cases would need detailed study of the symbol calculus, Clifford algebra and combinatorics of such local representations. This problem might be very difficult. However, if we restrict the parameter space B to flat connections and assume that the manifold has a flat metric, then we can easily compute the local representations explicitly. Therefore, the problem is how to treat the curvature corrections.

Another problem, that should be doable, is to obtain local representatives for the differentials of the eta-cocycles (transgression forms), on the boundary directly. This could be done, for example, using the trace-anomaly formula for $\overline{\mathbf{Tr}}_{\partial}$. The strategy would be the same as before. Namely, we try to manipulate the expression under the 'trace' $\overline{\mathbf{Tr}}_{\partial}$ in terms of commutators and coboundaries (we have to restrict to the vertical directions). This would give us a sort of non-commutative Stokes theorem, since we would have an identity between coboundaries of bulk residues from theorem

3 and the commutator anomalies on the boundary, modulo δ -coboundaries on the boundary.

18. APPENDIX : DECOMPOSING THE FORMS $F(dF)^m$

We give a direct way to decompose forms $F(dF)^m$ into standard form and give an application of this decomposition. Here, we use the same assumptions as in Section 6 : Forms associated with families of Dirac operators. Also, we use the graded commutator with the sign rule (6.4).

Proposition 48. Let $\omega_k = F(dF)^{k-1}$ be a $k - 1$ form on B , where $k > 1$ is an integer. In the case where k is even and ω_k is restricted to vertical directions, ω_k can be written in terms of commutators and δ -coboundaries. In the case k is odd, a term proportional to θ^{k-1} must be added to the commutator expansion.

Proof. First, restrict to the vertical directions to get

$$(18.1) \quad \omega_k = S_k(F, [F, \theta]^{k-1}).$$

Next, integrate by parts with respect to $[F, \cdot]$

$$(18.2) \quad \begin{aligned} \omega_k &= -[F, S_k(F, \theta, [F, \theta]^{k-2})] + S_k([F, F], \theta, [F, \theta]^{k-1}) \\ &= -[F, S_k(F, \theta, [F, \theta]^{k-2})] + 2S_{k-1}(\theta, [F, \theta]^{k-1}), \end{aligned}$$

where we have used $[F, F] = 2$ and the definition of the graded symmetrizer. Now, the claim follows, when we apply the lemma below to the term $S_k(\theta, [F, \theta]^{k-1})$. This lemma can be used to calculate the commutator expansion explicitly by iteration. \square

Lemma 6. Let $\omega_{p,q}$ be the following $2p + q + 1$ form on B

$$(18.3) \quad \omega_{p,q} = S_k(\theta, [\theta, \theta]^p, [F, \theta]^q),$$

where $k = p + q + 1$, p and q are positive integers. Then the forms $\omega_{p,q}$ satisfy the following iteration formula

$$(18.4) \quad \begin{aligned} \omega_{p,q} &= \frac{p+1}{1-p-2q} [\theta, S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1})] - \frac{q-1}{1-p-2q} [F, S_k(\theta, [\theta, \theta]^{p+1}, F, [F, \theta]^{q-2})] \\ &\quad - \delta \left(2 \frac{p+q}{1-p-2q} S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1}) \right) - 2 \frac{q-1}{1-p-2q} \omega_{p+1, q-2}. \end{aligned}$$

Particularly, modulo coboundaries

$$(18.5) \quad \begin{aligned} \omega_{p,q} &= \frac{p+1}{1-p-2q} [\theta, S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1})] - \frac{q-1}{1-p-2q} [F, S_k(\theta, [\theta, \theta]^{p+1}, F, [F, \theta]^{q-2})] \\ &\quad - 2 \frac{q-1}{1-p-2q} \omega_{p+1, q-2}. \end{aligned}$$

Proof. First, we integrate by parts with respect to $[\theta, \cdot]$, to get

$$(18.6) \quad \begin{aligned} \omega_{p,q} &= [\theta, S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1})] - S_k([\theta, \theta], [\theta, \theta]^p, F, [F, \theta]^{q-1}) \\ &\quad + (q-1)S_k(\theta, [\theta, \theta]^p, F, [F, [\theta, \theta]], [F, \theta]^{q-2}), \end{aligned}$$

where we have used $[[F, \theta], \theta] = [F, [\theta, \theta]]$.

The final term given above is handled with integration by parts with respect to $[F, \cdot]$

$$(18.7) \quad \begin{aligned} &(q-1)S_k(\theta, [\theta, \theta]^p, F, [F, [\theta, \theta]], [F, \theta]^{q-2}) \\ &= -\frac{q-1}{p+1}[F, S_k(\theta, [\theta, \theta]^{p+1}, F, [F, \theta]^{q-2})] + \frac{q-1}{p+1}S_k([F, \theta], [\theta, \theta]^{p+1}, F, [F, \theta]^{q-2}) \\ &\quad - \frac{q-1}{p+1}S_k(\theta, [\theta, \theta]^{p+1}, [F, F], [F, \theta]^{q-2}) \\ &= -\frac{q-1}{p+1}[F, S_k(\theta, [\theta, \theta]^{p+1}, F, [F, \theta]^{q-2})] - \frac{q-1}{p+1}S_k([\theta, \theta]^{p+1}, F, [F, \theta]^{q-1}) \\ &\quad - 2\frac{q-1}{p+1}S_{k-1}(\theta, [\theta, \theta]^{p+1}, [F, \theta]^{q-2}) \\ &= -\frac{q-1}{p+1}[F, S_k(\theta, [\theta, \theta]^{p+1}, F, [F, \theta]^{q-2})] - \frac{q-1}{p+1}S_k([\theta, \theta]^{p+1}, F, [F, \theta]^{q-1}) \\ &\quad - 2\frac{q-1}{p+1}\omega_{p+1,q-2}, \end{aligned}$$

where we have used $[F, F] = 2$, the definition of S_k and the definition of the forms $\omega_{p+1,q-2}$. Therefore

$$(18.8) \quad \begin{aligned} \omega_{p,q} &= [\theta, S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1})] - \frac{q-1}{p+1}[F, S_k(\theta, [\theta, \theta]^{p+1}, F, [F, \theta]^{q-2})] \\ &\quad - \frac{p+q}{p+1}S_k([\theta, \theta], [\theta, \theta]^p, F, [F, \theta]^{q-1}) - 2\frac{q-1}{p+1}\omega_{p+1,q-2}. \end{aligned}$$

Now, we handle the third term above. We integrate by parts with respect to δ to get

$$(18.9) \quad \begin{aligned} -S_k([\theta, \theta], [\theta, \theta]^p, F, [F, \theta]^{q-1}) &= 2S_k(\delta\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1}) \\ &= 2\delta S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1}) + 2S_k(\theta, [\theta, \theta]^p, \delta F, [F, \theta]^{q-1}) \\ &= 2\delta S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1}) + 2S_k(\theta, [\theta, \theta]^p, [F, \theta]^q) \\ &= 2\delta S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1}) + 2\omega_{p,q}, \end{aligned}$$

where we have used $\delta F = [F, \theta]$, $[\theta, \theta] = -2\delta\theta$ and the definition of the form $\omega_{p,q}$. Combine everything to get

$$(18.10) \quad \begin{aligned} \omega_{p,q} &= [\theta, S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1})] - \frac{q-1}{p+1} [F, S_k(\theta, [\theta, \theta]^{p+1}, F, [F, \theta]^{q-2})] \\ &\quad - 2\delta \frac{p+q}{p+1} S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1}) + 2 \frac{p+q}{p+1} \omega_{p,q} - 2 \frac{q-1}{p+1} \omega_{p+1,q-2}. \end{aligned}$$

Finally

$$(18.11) \quad \begin{aligned} \omega_{p,q} &= \frac{p+1}{1-p-2q} [\theta, S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1})] - \frac{q-1}{1-p-2q} [F, S_k(\theta, [\theta, \theta]^{p+1}, F, [F, \theta]^{q-2})] \\ &\quad - \delta \left(2 \frac{p+q}{1-p-2q} S_k(\theta, [\theta, \theta]^p, F, [F, \theta]^{q-1}) \right) - 2 \frac{q-1}{1-p-2q} \omega_{p+1,q-2}. \end{aligned}$$

□

Corollary 7 (Standard form of the eta-k form). The eta-k form

$$(18.12) \quad \eta_k = \overline{\mathbf{Tr}} F (dF)^{k-1},$$

when restricted to vertical directions, can be written in terms of the regularized traces of commutators and δ -coboundaries. Furthermore, if the manifold M is even dimensional, then we have to add a term proportional to $\overline{\mathbf{Tr}}_s \theta^{k-1}$.

Proof.

□

We give an application of the above results. This is a generalization of a result in [MP]. First, we recall the set up in that paper.

Let M be an odd dimensional closed spin manifold and let E be a trivial complex vector bundle over M . Denote by B the Hermitean connections on E . Furthermore, let $\tilde{\mathfrak{D}}_b$ denote the Dirac operator coupled to connection b in B and acting on square-integrable spinor fields. Define the following open sets

$$(18.13) \quad U_\lambda = \{z \in B \mid \lambda \notin \text{spec}(\tilde{\mathfrak{D}}_b)\},$$

where λ is a real number. Then the grading operator $F_{\lambda,b} = \frac{\tilde{\mathfrak{D}}_b - \lambda}{|\tilde{\mathfrak{D}}_b - \lambda|}$, for $b \in U_\lambda$ is well defined. Furthermore, the family so defined is smooth with respect to the parameter b in U_λ .

Moreover, the forms

$$(18.14) \quad \hat{\omega}_{k,\lambda} = F_\lambda (dF_\lambda)^{k-1}$$

are well defined over U_λ .

We let $\overline{\mathbf{Tr}}$ denote the standard regularized trace as above (we may have to project out the zero-mode sector in the weight). Recall, that for k even

$$(18.15) \quad d\hat{\omega}_{k,\lambda} = (dF_\lambda)^k = \frac{1}{2} [F_\lambda (dF_\lambda)^k, F_\lambda].$$

Thus $\overline{\text{Tr}}d\hat{\omega}_{k,\lambda}$ is local, given by the Wodzicki residue. Therefore on U_λ there exists a local counter-term C_λ such that $dC_\lambda = -\overline{\text{Tr}}d\hat{\omega}_{k,\lambda} = d\overline{\text{Tr}}\hat{\omega}_{k,\lambda}$ (see [MP], for the precise formula for the counterterm).

Proposition 49. Consider the forms $\hat{\omega}_{k,\lambda}$ and their counterterms C_λ as above. Put

$$(18.16) \quad \hat{\eta}_{k,\lambda} = \overline{\text{Tr}}\hat{\omega}_{k,\lambda}$$

and

$$(18.17) \quad \hat{\eta}_{k,\lambda,\text{ren}} = \overline{\text{Tr}}\hat{\omega}_{k,\lambda} + C_\lambda.$$

Then the form $\hat{\eta}_{k,\lambda,\text{ren}}$ defines a closed form for any integer $k \geq 1$ on U_λ . Furthermore, if we restrict the form $\hat{\eta}_{k,\lambda,\text{ren}}$ to vertical directions, then this form is given by the Wodzicki residues, modulo δ -coboundaries.

Proof. That the form is closed, follows directly from definitions. We still have to check the locality, but this follows from Proposition 48. \square

REFERENCES

- [ALNV] B. AMMANN, R. LAUTER, V. NISTOR and A. VASY: Complex powers and non-compact manifolds. - Comm. Partial Differential Equations 29 (2004), no. 5-6, 671-705.
- [AS] M. ATIYAH and I. SINGER: The index of elliptic operators IV. - Ann. of Math. 93 (1971) 119-138
- [AS2] M. ATIYAH and I. SINGER: Dirac Operators Coupled To Vector Potentials. - Proc. Natl. Acad. Sci (USA) 81, 2597 (1981)
- [B] R. BERTLMANN: Anomalies In Quantum Field Theory. - Clarendon Press 1996
- [BGV] N. BERLINE, E. GETZLER and M. VERGNE: Heat Kernels and Dirac Operators. - Springer (2004)
- [Bu] B. BUCICOVSKI: An extension of the work of V. Guillemin on complex powers and zeta functions of elliptic pseudodifferential operators. - Proc. Amer. Math. Soc. 127, (1999), 3081-3090. arXiv:dg-ga/9705006v2
- [CDP] A. CARDONA, C. DUCOURTIOUX and S. PAYCHA: From tracial anomalies to anomalies in Quantum Field Theory. - CMP 242 31 (2003)
- [CM] A. CAREY and J. MICKELSSON: The universal gerbe, Dixmier-Douady class, and gauge theory. - Lett. Math. Phys. 59 (2002) 47-60
- [CMM] A. CAREY, J. MICKELSSON and M. MURRAY: Index Theory, Gerbes, and Hamiltonian Quantization. - Commun. Math. Phys. 183 (1997) 707-722
- [Co] A. CONNES: Noncommutative Geometry. - Academic Press (1994)
- [CW] A. CAREY and B.L. WANG: On the relationship of gerbes to the odd families index theorem. - J. Geom. Phys., 57 (2006) 23-38
- [Fe] B. FEDOSOV: Analytic formulas for the index of elliptic operators. - Trans. Moscow Math. Soc. Vol. 30 (1974)
- [Fe2] B. FEDOSOV: Deformation Quantization and Index Theory. - Akad. Verl., 1.ed Berlin (1996)
- [Gr] D. GRIESER: Basics of the b-calculus. - <http://arxiv.org/abs/math/0010314v1>
- [Hou] BO-YU HOU and BO-YUAN HOU: Differential Geometry For Physicists. - World Scientific (1999)
- [H] L. HÖRMANDER: The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators. - Springer (2007)

- [L] J. LOTT: Higher degree analogs of the determinant bundle. - Commun. Math. Phys. 230 (2002) 41-69
- [La] E. LANGMANN: Descent equations of Yang-Mills anomalies in noncommutative geometry. - J.Geom.Phys. 22 (1997) 259-279, arXiv:hep-th/9508003v2
- [La2] E. LANGMANN: Noncommutative Integration Calculus. - Jour. Math. Phys. 36 (1995) 3822-3835
- [LM] H. LAWSON and M. MICHELSON: Spin Geometry, Princeton University Press (1990)
- [LaMi] E. LANGMANN and J. MICKELSSON: (3+1) Dimensional Schwinger Terms And Non-Commutative Geometry. - Phys. Lett. B 338 (1994) 241-248, hep-th/9407193v1
- [LaMiRy] E. LANGMANN, J. MICKELSSON and S. RYDH: Anomalies And Schwinger Terms In NCG Field Theory Models. - J. Math. Phys. 42 (2001) 4779-4801, hep-th/0103006v1
- [LaMonNi] R. LAUTER, B. MONTHUBERT and V. NISTOR: Spectral invariance for certain algebras of pseudodifferential operators. - J. Inst. Math. Jussieu 4, (2005), Issue 03, 405-442.
- [Lo] P. LOYA: Tempered operators and the heat kernel and complex powers of elliptic pseudodifferential operators. - Comm. Partial Differential Equations 26 (2001), 1253-1321
- [LoMoPa] P. LOYA, S. MOROIANU and J. PARK: Regularity of the eta function on manifolds with cusps. - Mathematische Zeitschrift 269 (2011), no. 3-4, 955-975.
- [LP] M. LESCH and M.F. PFLAUM: Traces on algebras of parameter dependent pseudodifferential operators and eta-invariant. - Trans. Amer. Math. Soc. , 352: (11) : 4911 - 4936, 2000
- [M] J. MICKELSSON: Current Algebras And Groups. - Plenum Press, London and New York (1989)
- [M2] J. MICKELSSON: Boundary Currents And Hamiltonian Quantization Of Fermions In Background Fields. - Phys. Lett. B 456 (1999) 124-128, hep-th/9903071
- [MP] J. MICKELSSON and S. PAYCHA: Renormalized Chern-Weil forms associated with families of Dirac operators. - Journ. Geom. Phys. 57 p. 1789-1814 (2007), math.dg/0607148
- [MP2] J. MICKELSSON and S. PAYCHA: The logarithmic residue density of a generalised Laplacian. - Journal of the Australian Mathematical Society, Vol 90, Issue 01, Feb. 2011, p. 53-80
- [MR] J. MICKELSSON and S. RAJEEV: Current algebras in (d+1)-dimensions and determinant bundles over infinite-dimensional Grassmannians. - CMP vol. 116, No. 3 (1988), p. 365-400
- [Me] R. MELROSE: Fibrations, compactifications and algebras of pseudodifferential operators. - Partial Differential Equations and Mathematical Physics. The Danish-Swedish Analysis Seminar, 1995 Birkhauser, 1996, pp.246-261
- [Me2] R. MELROSE: The eta invariant and families of pseudodifferential operators. - Math. Res. Letters 2 (1995), 541-561
- [Me3] R. MELROSE: The Atiyah-Patodi-Singer Theorem (Research Notes In Mathematics). - A K Peters/CRC Press (31 Mar 1993)
- [Me4] R. MELROSE: Differential Analysis On Manifolds With Corners. - <http://www-math.mit.edu/~rbm/book.html>
- [MeMa] R. MELROSE and R. MAZZEO: Pseudodifferential operators on manifolds with fibred boundaries. - Asian J. Math., 2(4):833-866, 1998. Mikio Sato: a great Japanese mathematician of the twentieth century
- [MeNi] R. MELROSE and V. NISTOR: Homology of pseudodifferential operators I : manifolds with boundary. - arXiv:funct-an/9606005v2
- [MePi] R. MELROSE and P. PIAZZA: Families of Dirac operators, boundaries and b-calculus. - J. Differential Geom. Volume 46, Number 1 (1997), 99-180
- [MePi2] R. MELROSE and P. PIAZZA: An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary. - J. Diff. Geom. 146, 287-334 (1997)

- [MeRo] R. MELROSE and F. ROCHON: Families index for pseudodifferential operators on manifolds with boundary. - Int. Math. Res. Not. 2004 nr. 22, 1115-1141.
- [MeRo2] R. MELROSE and F. ROCHON: Index in K-theory for families of fibred cusp operators. - K-theory 37 (2006), 25-104
- [MeRo3] R. MELROSE and F. ROCHON: Eta forms and the odd pseudodifferential families index. - arXiv:0905.0150v2
- [Mo] S. MOROIANU: Cusp geometry and cobordism invariance of the index. - Advances Math. 194 (2005), 504-519
- [Mo2] S. MOROIANU: K-theory of suspended operators. - K-Theory 28 (2003)
- [MoLa] S. MOROIANU and R. LAUTER: The index of cusp operators on manifolds with corners. - Ann. Global Anal. Geom. 21 (2002), 31-49
- [MoNi] S. MOROIANU and V. NISTOR: Index and homology and of pseudodifferential operators on manifolds with boundary. - Perspectives Oper. Algebras Math. Phys., Theta (2008), 123-148
- [P] S. PAYCHA: Chern-Weil calculus extended to a class of infinite dimensional manifolds, arXiv:0706.2554 (June 2007)
- [Pi] P. PIAZZA: Determinant bundles, manifolds with boundary and surgery. - CMP, vol. 176 no. 3 (1997), 597-626
- [Pi2] P. PIAZZA: K-Theory and Index Theory on Manifolds with Boundary. - Thesis (Ph. D.) -Massachusetts Institute of Technology, Dept. of Mathematics, 1991
- [PR] S. PAYCHA and S. ROSENBERG: Curvature on Determinant Bundles And First Chern Forms. - Journal of Geometry and Physics 45 p.393-429, 2003, arXiv.org:math/0009172
- [Sc] S. SCOTT: Traces and Determinants of Pseudodifferential Operators - Oxford University Press (2010)
- [S1] R. STORA: Algebraic structure and topological origin of anomalies. - Recent progress in gauge theories. 1983 Cargese Lectures, NATO ASI series, Plenum Press, New York
- [S2] R. STORA: Algebraic structure of chiral anomalies. - New perspectives in quantum field theory. 1985 GIFT lectures, World Scientific, Singapore
- [Se] R. T. SEELEY: Complex powers of elliptic operators. - Proc. Symp. Pure Math. 10 (1967), 288-307
- [St] D. STEVENSON: Geometry of infinite dimensional Grassmannians and the Mickelsson-Rajeev cocycle. - J. Geom. Phys. 60, No. 4, 664-677 (2010), arXiv:0802.3608v1 [math.DG]
- [TJZW] S. TREIMAN, R. JACKIW, E. WITTEN and B. ZUMINO: Current Algebras And Anomalies. - Princeton University Press (1985)
- [W] M. WODZICKI: Noncommutative Residue. - Lecture Notes In Math, vol. 1289, Springer-Verlag (1987)
- [Q] D. QUILLEN: Superconnections and the Chern Character. - Topology 24, 89. (1985)
- [Q2] D. QUILLEN: Superconnection character forms and the Cayley transform. - Topology 27 (1988), no. 2, 211-238. [23]

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