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ELENA RAVAIOLI

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Editor: Olli Martio
Department of Mathematics and Statistics
P.O. Box 68
FI-00014 University of Helsinki
Finland

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ELENA RAVAIOLI

University of Helsinki, Department of Mathematics and Statistics

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Helsinki, 10 November 2005

Elena Ravaioli

Contents

Introduction

Let G be a Lie group, and let M and N be real analytic G-manifolds. We denote by $C^{G,\infty}_{vSW}(M,N)$ the set of G-equivariant, C^∞ maps from M to N, endowed with the very-strong-weak topology. This topology was introduced by S. Illman in [I3], and its name is due to the fact that the idea behind it is to "mix" together the very-strong and the weak topologies on $C^{\infty,G}(M,N)$. In this way one obtains a topology that should be regarded as the most appropriate for the set $C^{\infty,G}(M,N)$, because it allows to avoid some pathological situations that may occur in the very-strong (and even in the strong C^∞) case. For this reason it is natural and important to try to investigate a classic approximation problem in transformation groups in the context of the very-strong-weak topology. In fact, this Thesis is concerned with the following question: is it possible to approximate a map $f \in C^{G,\infty}_{vSW}(M,N)$ with a real analytic, G-equivariant map? Our main result is Theorem A below, see Theorem 4.2.2.

Theorem A *Let G be a good Lie group (i.e., G can be embedded as a closed subgroup in a Lie group with only finitely many connected components), and let M and N be real analytic, proper G-manifolds. Then $C^{\omega,G}(M,N)$ is dense in $C^{\infty,G}_{vSW}(M,N)$.*

A fundamental step in the proof of Theorem A is to prove the approximation result for the compact case. Note that in this case the very-strong-weak topology coincides with the very-strong one, that is, $C^{G,\infty}_{vSW}(M,N) = C^{G,\infty}_{vS}(M,N)$ when G is compact. Thus we prove the following result, see Corollary 3.3.2.

Theorem B *Let K be a compact Lie group, and let M and N be two real analytic K-manifolds. Then $C^{\omega,K}(M,N)$ is dense in $C^{\infty,K}_{vS}(M,N)$.*

A similar approximation result was previously proven by Illman under the additional assumption that the number of K-isotropy types in N is finite (see Theorem 7.2 in [I2]). We generalize Illman's result following the work of F. Kutzschebauch in the case of the strong C^∞ topology. Using tools from Riemannian differential geometry we construct a map $\mathcal{C}$ ("center"), which allows to average maps in $C^\infty(M,N)$ which are suitably close to a K-equivariant map. We call these maps "almost K-equivariant". In fact we prove the following result, see Theorem 3.3.1.

Theorem C *Let K be a compact Lie group, and let M and N be two real analytic K-manifolds. There exists in $C^\infty_{vS}(M,N)$ an open neighborhood $\mathcal{M} \supset C^{\infty,K}_{vS}(M,N)$ of almost K-equivariant maps and a continuous map*

$$\mathcal{C} : \mathcal{M} \to C^{\infty,K}_{vS}(M,N),$$

which is a retraction and preserves real analyticity.

By means of the "non-linear" average $\mathcal{C}$ we are then able to prove Theorem B without having to turn to embedding results, for which the assumption on the number of K-isotropy types in N would have been strictly necessary.

We complete the proof of Theorem A basicly following the work presented by Illman and Kankaanrinta in [I-Ka1] and [I-Ka2] for the case of the strong-weak topology. A crucial result is Theorem D below, which describes the behavior of the very-strong-weak topology with respect to the twisted product (see Theorem 4.1.4).

Theorem D *Let G be a Lie group, and let $H < G$ be a closed subgroup. Assume M is a smooth H-manifold, and N a smooth G-manifold. Then there exists a homeomorphism*

$$\mu : C^{\infty,H}_{vSW}(M,N) \to C^{\infty,G}_{vSW}(G \times_H M, N), \quad f \mapsto \mu(f).$$

In the proof of Theorem A an important role is also played by Theorem 1.2.4, the real analytic version of Abels' Theorem, due to Kutzschebauch (see [Ku1]; see also [H-H-Ku]).

In Chapter 1 we set some notation and give a review of the basic tools from differential geometry and transformation groups that will be involved in this work. In particular we describe Riemannian structures of G-manifolds.

In Chapter 2 we discuss the possible choices for a topology on the sets $C^\infty(M,N)$ and $C^{\infty,G}(M,N)$. We also give some technical results and prove some basic properties of the very-strong-weak topology.

Chapter 3 is concerned with the compact case: we construct a continuous, "non-linear" average for smooth, almost G-equivariant maps between two G-manifolds, and thus prove the approximation result in Theorem B.

Finally, in Chapter 4 we prove Theorem D and Theorem A.

Chapter 1. G-manifolds

In this Chapter we introduce most of the tools and basic facts that are going to be used in this Thesis. In fact, we give a short review of some fundamental results involving actions of Lie groups on manifolds and describe Riemannian structures on such manifolds.

1.1. Basic definitions

In this section we set some notation and establish some basic facts that are going to be needed further on. The reader is assumed to be familiar with the definitions and results given in the following; for further details, see for example [W] and [Kaw]. We will assume all topological spaces to be Hausdorff, and manifolds will be always taken to be second countable and connected. Furthermore, the word "manifold" will mean "manifold without boundary", unless something else is explicitely specified. In this work we will mainly deal with smooth (C^∞) and real analytic (C^ω) manifolds. Recall that if Ω is an open subset of $\mathbb{R}^m$, a map $f : \Omega \to \mathbb{R}$ is said to be real analytic if for any $a \in \Omega$ there exists a power series that converges to $f(x)$, for every x in a neighborhood of a. A vector-valued real function is real analytic if each of its components is C^ω. By an abuse of notation we will use the expression "$r \leq \omega$", meaning that "$r \leq \infty$, or $r = \omega$". If $0 \leq r \leq \omega$, and M and N are two C^r-manifolds, we will denote by $C^r(M, N)$ the set of all C^r maps from M to N.

It can be useful to recall the following basic result about manifolds:

Theorem 1.1.1. *Let M be a manifold. Then*

(i) *There exists an exhaustion of M, i.e. a countable open cover $\{V_j\}_{j\in\mathbb{N}}$ of M such that $\overline{V}_j$ is compact and $\overline{V}_j \subset V_{j+1}$, for every $j \in \mathbb{N}$.*
(ii) *M is paracompact. In fact, each open cover of M has a countable, locally finite refinement consisting of open sets with compact closures.*

Proof. See [W], Lemma 1.9. □

We say that a continuous map $f : X \to Y$ between two topological spaces is *of finite type* if for each locally finite family $\{A_i\}_{i\in\Lambda}$ of subsets of X, the family $\{f(A_i)\}_{i\in\Lambda}$ is locally finite in Y, see [I-Ka1], page 145. Furthermore, f is said to be *proper* if $f^{-1}(B) \subset X$ is compact, for every compact subset $B \subset Y$.

Lemma 1.1.2. *Let X and Y be topological spaces, and assume that Y is locally compact. Then every proper map $f : X \to Y$ is closed and of finite type.*

Proof. The first claim is a standard fact, for the second one see Lemma 1.8 in [I-Ka1]. □

By a Lie group G we mean a topological group which is also a real analytic manifold, and such that the multiplication map $G \times G \to G$, $(g, \tilde{g}) \mapsto g\tilde{g}$, and the map $G \to G$, $g \mapsto g^{-1}$, are real analytic. A map $f : G_1 \to G_2$ between two Lie groups is called a homomorphism of Lie groups if f is a real analytic map and a group homomorphism. Recall that every closed subgroup of a Lie group is a Lie group (see for example [Hel], Theorem II.2.3, or [Kaw], Theorem 3.36). Furthermore, if H is a closed, normal subgroup of a Lie group G, then the quotient space G/H is also a Lie group, and the projection $\pi : G \to G/H$ is a homomorphism of Lie groups (see [Kaw], Corollary 3.41). If G is a Lie group with only finitely many connected components, then there exists a maximal compact subgroup K of G. Any two maximal compact subgroups are conjugate in G (see [Ho], Theorem XV.3.1). A Lie group will be called "good" if it can be embedded as a closed subgroup in a Lie group with only finitely many connected components (see [I-Ka2], page 169).

A (continuous) action of a Lie group G on a manifold M is a continuous map

$$\Psi : G \times M \to M, \quad (g, x) \mapsto gx = \Psi(g, x),$$

such that

1. $ex = x$, for all $x \in M$, where e is the identity element of G.
2. $\tilde{g}(gx) = (\tilde{g}g)x$, for all $g, \tilde{g} \in G$ and $x \in M$.

We can now give the following:

Definition 1.1.3. Let G be a Lie group. A *smooth (real analytic) G-manifold* is a smooth (real analytic) manifold M on which G acts by a smooth (real analytic) action.

Definition 1.1.4. Let M and N be smooth G-manifolds, where G is a Lie group. A map $f : M \to N$ is called *G-equivariant* if $f(gx) = gf(x)$, for all $g \in G$ and $x \in M$. For $0 \leq r \leq \omega$, we denote the set of all C^r G-equivariant maps from M to N by $C^{r,G}(M, N)$.

Given a positive integer n, we denote by $GL(n, \mathbb{R})$ the set of all real, invertible square matrices of order n. A *representation* of a Lie group G is a continuous homomorphism $\rho : G \to GL(n, \mathbb{R})$, for some $n \in \mathbb{N}$, $n \geq 1$. We denote by $\mathbb{R}^n(\rho)$ the corresponding linear representation space, i.e. the Euclidean space $\mathbb{R}^n$ on which G acts by the following action:

$$\Phi_\rho : G \times \mathbb{R}^n \to \mathbb{R}^n, \quad (g, x) \mapsto gx = \rho(g)x.$$

By a well-known theorem every continuous homomorphism between Lie groups is real analytic (see e.g. [Hel], Theorem II.2.6). Thus, Φ_ρ is real analytic since ρ itself is. Theorem 1.1.5 below states that by means of the Haar integral it is possible, in the compact case, to average maps which take their values in a linear representation space (see [Kaw], Section 2.7):

Theorem 1.1.5. *Let K be a compact Lie group, and let M be a smooth (real analytic) K-manifold. Let $\rho : K \to GL(n, \mathbb{R})$ be a representation of K for some $n \in \mathbb{N}$, $n \geq 1$, and $f : M \to \mathbb{R}^n(\rho)$ a smooth (real analytic) map. Then the map*

$$A(f) : M \to \mathbb{R}^n(\rho), \quad x \mapsto \int_K k f(k^{-1}x)\, dk,$$

is a K-equivariant smooth (real analytic) map.

Proof. See [Ka], Theorem 1.16. □

Let now G be a Lie group, and let M be a smooth G-manifold. For every $g \in G$ the map

$$\Psi_g : M \to M, \quad x \mapsto gx,$$

is a diffeomorphism of M (see [Kaw], Lemma 1.29 and Section 3.2). If $x \in M$, the *isotropy subgroup* of x is the closed subgroup of G

$$G_x = \{g \in G \mid gx = x\}.$$

Note that if G is compact, then G_x is compact for all $x \in M$. If H is a closed subgroup of G, we call its conjugacy class $[H]$ a *G-isotropy type*, and we say that a G-isotropy type $[H]$ appears in M if there exists $x \in M$ such that $[G_x] = [H]$.

The *orbit* of $x \in M$ is the (G-invariant) subset of M

$$Gx = \{gx \in M \mid g \in G\}.$$

We denote by M/G the set of all the orbits of G on M, and by π the natural projection $\pi : M \to M/G$, $x \mapsto Gx$. Then M/G endowed with the quotient topology is called the *orbit space* of M, and π is a continuous, open map. Assume that G is compact: then M/G is a Hausdorff, locally compact space, and the projection $\pi : M \to M/G$ is a proper map (see [Kaw], Proposition 1.58). Thus, by Lemma 1.1.2, if G is compact the projection π is a closed map of finite type.

Let G be a Lie group, and let $H \subset G$ be a closed subgroup. Given a smooth (real analytic) H-manifold M, it is always possible to construct an induced G-manifold as follows: let H act on $G \times M$ by

$$H \times (G \times M) \to (G \times M), \quad (h, (g, x)) \mapsto (gh^{-1}, hx). \tag{1}$$

Then the *twisted product* $G \times_H M$ is defined as the orbit space of the action (28). We denote by

$$p : G \times M \to G \times_H M, \quad (g, x) \mapsto [g, x],$$

the natural projection. Recall that under the assumptions above $G \times_H M$ is a smooth (real analytic) manifold (see [I1], Section 4). Now, an action of G on $G \times_H M$ is naturally obtained by

$$G \times (G \times_H M) \to G \times_H M, \quad (\bar{g}, [g, x]) \mapsto [\bar{g}g, x],$$

and this action is smooth (real analytic) (see again [I1], Section 4).

If N is a smooth (C^ω) G-manifold, and $f : M \to N$ is a smooth (C^ω), H-equivariant map, then the map

$$\mu(f) : G \times_H M \to N, \quad [g, x] \mapsto gf(x),$$

is smooth (C^ω) and G-equivariant (see [I1], Lemma 4.1). On the other hand, consider the closed embedding

$$i : M \to G \times_H M, \quad x \mapsto [e, x].$$

Then, if $z \in C^{\infty,G}(G \times_H M, N)$ we have that $\mu^{-1}(z) = z \circ i$. Thus we have a canonical bijection

$$\mu : C^{\infty,H}(M,N) \to C^{\infty,G}(G \times_H M, N), \quad f \mapsto \mu(f),$$

which preserves real analyticity, that is,

$$\mu(C^{\omega,H}(M,N)) = C^{\omega,G}(G \times_H M, N).$$

Later on we will need the following result:

Lemma 1.1.6. *Let H be a closed subgroup of a Lie group G, and assume M is an H-manifold. Then the natural inclusion $i : M \to G \times_H M$, $x \mapsto [e,x]$, induces a homeomorphism*

$$\bar{i} : M/H \to (G \times_H M)/G.$$

Proof. See [Kaw], Proposition 1.90. □

1.2. Proper actions and slices

Let G be a Lie group and let M be a smooth G-manifold. If B is a subset of M, we denote

$$G_{[B]} = \{g \in G \mid gB \cap B \neq \emptyset\}.$$

Definition 1.2.1. The action of G on M is called *proper* if $G_{[B]}$ is compact for every compact subset B of M.

It is well known that Definition 1.2.1 is equivalent to the following definition:

Definition 1.2.2. The action $\Phi : G \times M \to M$, $(g,x) \mapsto gx$, is called *proper* if the map

$$\Phi^* : G \times M \to M \times M, \quad (g,x) \mapsto (gx, x),$$

is a proper map.

Proper actions form a special and very important class of actions. Clearly every compact action (that is, every action of a compact group) is proper; the converse is not true, but proper actions share with the compact case some fundamental properties that make the theory of G-manifolds (and, more generally, of G-spaces) much more interesting and rich. We review in the following some of these properties. Fore more details and proofs the reader is referred to [Pa].

Let then M be a proper G-manifold. Then for each $x \in M$ the isotropy subgroup $G_x \subset G$ is compact, and the orbit Gx is closed in M. Furthermore, the orbit space M/G is Hausdorff and locally compact. If the action of G on M is proper and free, then M/G can be given a smooth manifold-structure so that the projection $\pi : M \to M/G$ is a smooth, principal G-bundle.

Proposition 1.2.3. *Let G be a Lie group, and let the closed subgroup $H \subset G$ act on the smooth manifold M by a proper action. Then the action of G on $G \times_H M$ is proper.*

Proof. See [I-Ka1], Lemma 3.10. □

We end this section with a fundamental result about proper actions, namely the real analytic version of Abels' theorem, Theorem 1.2.4 below. This theorem is due to Kutzschebauch (see [Ku1], Satz 2.5.3; see also [H-H-Ku]). The smooth version of Theorem 1.2.4 was proved by Abels in [A]. First we recall the notion of "slice": let then H be a closed subgroup of a Lie group G and, for $0 \leq r \leq \omega$, let M be a C^r G-manifold and S an H-invariant, C^r submanifold of M. We say that S is a C^r *H-slice* in M if GS is open in M and the map

$$\mu : G \times_H S \to GS, \quad [g, x] \mapsto gx,$$

is a G-equivariant C^r diffeomorphism (note that, in general, μ is a C^r, G-equivariant, surjective map). If $x \in S$ and $H = G_x$ we call S a *slice at x*. If $GS = M$ we call S a *global slice*.

The above definition of slice is the same used by Illman (see for example [I1]), and in [Br] and [Kaw] for the compact case. The definition given by Palais in [Pa] is slightly different, but the two definitions are equivalent (see [I1], Lemma 5.2).

Theorem 1.2.4. *Let J be a Lie group with only finitely many connected components, and let K be a maximal compact subgroup of J. Let M be a real analytic, proper J-manifold. Then there exists a global C^ω K-slice in M.*

Proof. See [Ku1], Satz 2.5.3. □

1.3. Riemannian G-manifolds

We will now recall some basic results in differential geometry: for more details and proofs the reader is referred for example to [K-N] and [W]. Throughout this section let N be a n-dimensional smooth manifold. We will denote by T_pN the tangent space at $p \in N$ to N, and by $TN = \cup_{p \in N} T_pN$ the tangent bundle of N. Let M be a smooth manifold, and let $\phi : M \to N$ be a C^∞ map: we will denote by $\phi_* : T_qM \to T_{\phi(q)}N$ the *differential* of ϕ at $q \in M$. If $\phi_* : T_qM \to T_{\phi(q)}N$ is a monomorphism for each $q \in M$, then ϕ is called an *immersion*. If ϕ is also a homeomorphism onto its image $\phi(M)$ then ϕ is called an *embedding*. Note that a subset $M \subset N$ is a smooth submanifold if and only if it is the image of a smooth embedding (see [H], Theorem 1.3.1).

It is a well-known fact that N can be given a smooth Riemannian metric g; then g induces a distance function d on N, and the topology of the metric space (N, d) is the same as the manifold topology (see [K-N], Vol. I, Proposition IV.3.5).

Let $\Xi(N)$ be the algebra of smooth vector fields on N. To a Riemannian manifold (N, g) is associated a unique Riemannian connection ∇; then the (Riemannian) curvature tensor field on N is the (3-linear) map

$$R : \Xi(N) \times \Xi(N) \times \Xi(N) \to \Xi(N)$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Let $p \in N$, let P be a two-dimensional subspace of T_pN and let v, $w \in P$ be linearly independent; denote by

$$|v \wedge w| = \sqrt{g(v,v)_p^2 g(w,w)_p^2 - g(v,w)_p^2} \neq 0$$

the area of the parallelogram spanned by v and w. The real number

$$\mathcal{K}(P) = \mathcal{K}(v,w) = \frac{g(R(v,w)w,v)_p}{|v \wedge w|^2}$$

is independent of the choice of the basis of P, and it is called the *sectional curvature* of P at p (see for example [Hel], Theorem I.12.2).

Recall that given $p \in N$ and $v \in T_pN$ there exists a unique *maximal geodesic* $\gamma^v : I_v \to N$ such that $\gamma^v(0) = p$ and $\dot{\gamma}_0^v = v$. Denote by $[0, \ell_v)$ the non-negative part of the maximal interval I_v where γ^v is defined, and let $E \subset TN$ be the set of vectors $v \in TN$ such that $\ell_v > 1$, i.e. such that γ^v is defined for every $t \in [0,1]$. The *exponential map* is then defined as follows:

$$\exp : E \to N, \quad v \mapsto \exp(v) = \gamma^v(1).$$

For $p \in N$, the exponential map at p is the map $\exp_p := \exp|_{E_p}$, where $E_p = E \cap T_pN$. The exponential map is a real analytic map and, for every $p \in N$, $\exp_p$ is a diffeomorphism in a neighborhood U of $\underline{0} \in T_pN$ onto a neighborhood V of p in N: in this case V is called a *normal neighborhood* of p. Sufficiently small balls around p are normal: in fact it is possible to find $r \in \mathbb{R}_+$ such that the restriction

$$\exp_p : E_r(\underline{0}) \to B_r(p) = \{q \in N \mid d(p,q) < r\}$$

is a diffeomorphism (here $E_r(\underline{0}) = \{v \in T_pN \mid |v| < r\}$ denotes the usual euclidean ball of center $\underline{0} \in T_pN$ and radius r). If V is a normal neighborhood of p, we can get coordinate functions on V in the following way: let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pN, and let θ be the naturally induced isomorphism $\theta : \mathbb{R}^n \to T_pN$, $\theta(t_1, ..., t_n) = \sum_{i=1}^n t_i e_i$. Then we can define $\psi : V \to \mathbb{R}^n$ by $\psi = \theta^{-1} \circ (\exp_p)^{-1}$, that is $\psi(\exp_p(\sum_{i=1}^n x_i e_i)) = (x_1, ..., x_n) \in \mathbb{R}^n$; then ψ is clearly a diffeomorphism onto an open subset of $\mathbb{R}^n$. The pair (V, ψ) is called a *normal chart at* $p \in N$, and $(x_1, ..., x_n) \in \mathbb{R}^n$ are called *normal coordinates* of the point $x = \exp_p(\sum_{i=1}^n x_i e_i)$.

Let (N, g) be a smooth Riemannian manifold, and let d be the distance function induced by g on N. The following, crucial result states the existence of what we will call a "convexity function" on N:

Proposition 1.3.1. *There exists a continuous function $r : N \to \mathbb{R}_+$ such that for every $p \in N$ the ball*

$$B := B_{r(p)}(p) = \{q \in N : d(q,p) < r(p)\}$$

is geodesically convex, that is,

1. *B is a normal neighborhood of each of its points, i.e. for every $q \in B$ the exponential map $\exp_q : T_qN \to N$ gives a diffeomorphism from a neighbourhood of $\underline{0}$ in T_qN onto B.*

2. *Any two points of B can be joined by a unique minimizing geodesic, and this is the unique geodesic joining the two points and lying in B.*

Proof. For each $p \in N$ there exists a positive number $a \in \mathbb{R}$ such that, if $0 < r < a$, the conditions 1 and 2 above are satisfied (see [K-N], Vol. I, Theorem III.8.7 and Theorem IV.3.6). Then we can follow the proof given in [K-N], Vol.I, Lemma on page 174: for each $p \in N$ let $r(p) > 0$ be the supremum of $r > 0$ for which the conditions 1 and 2 are true. If $r(p) = \infty$ for some $p \in N$, then $r(q) = \infty$ for every $q \in N$ (see Theorem 1.4.1), and any positive continuous function on N (for example a constant function) has the properties required in the proposition. Assume that $r(p) < \infty$ for every $p \in N$. We claim that the function $r : p \mapsto r(p)$, $p \in N$, is continuous: it will be enough to show that $|r(p) - r(q)| \leq d(p, q)$, for all p, $q \in N$. Without loss of generality, we may assume that $r(p) > r(q)$. If $d(p, q) \geq r(p)$, then clearly $|r(p) - r(q)| < d(p, q)$. If $d(p, q) < r(p)$, then $B_{r'}(q) \subset B_{r(p)}(p)$, where $r' = r(p) - d(p, q)$, is geodesically convex. Hence $r(q) \geq r(p) - d(p, q)$, that is, $|r(p) - r(q)| \leq d(p, q)$, and so the claim is proved. □

Remark 1.3.2. Let $p \in N$, r and $B = B_{r(p)}(p)$ be as in Proposition 1.3.1. Then, for every $\rho \in \mathbb{R}$ such that $0 < \rho \leq r(p)$ the ball $B_\rho(p)$ satisfies the conditions 1. and 2. in Proposition 1.3.1. This fact will allow us to replace a fixed convexity function $r : N \to \mathbb{R}_+$ with a smaller one, when needed.

In Chapter 3 we will use the following result:

Proposition 1.3.3. *Let (N, g) be a real analytic Riemannian manifold, and let d be the distance function induced by g on N. Let r be like in Proposition* 1.3.1*. Then the restriction*

$$d^2| : B_{r(y)}(y) \times B_{r(y)}(y) \to \mathbb{R}_{\geq 0}$$

is real analytic, for all $y \in N$.

Proof. See [K-N], Vol. I, Theorem IV.3.6. □

A diffeomorphism $f : N \to N$ is an *isometry* if

$$g(f_* v, f_* w)_{f(p)} = g(v, w)_p, \quad \text{for every} \quad v,\, w \in T_p N, \quad p \in N.$$

In this case, f clearly preserves distances, i.e. $d(f(p), f(q)) = d(p, q)$, for every p, $q \in N$.

Now, let K be a compact Lie group, and let (N, g) be a smooth (real analytic) Riemannian K-manifold. Each $k \in K$ can be considered as a smooth (real analytic) diffeomorphism $k : N \to N$, $p \mapsto k(p) = kp$; thus we denote with $k_* : T_p N \to T_{kp} N$ the differential of k at $p \in N$.

Definition 1.3.4. We say that the Riemannian metric g is *K-invariant* if

$$g(k_* v, k_* w)_{kp} = g(v, w)_p,$$

for every $p \in N$, $k \in K$ and v, $w \in T_p N$.

Remark 1.3.5. If the Riemannian metric g is K-invariant, then clearly for every $k \in K$ the map $k : N \to N$ is an isometry: then we say that K is a *group of isometries on* N. As already remarked, in this case the induced distance function d is also K-invariant, i.e.

$$d(p, q) = d(kp, kq), \quad \forall\, k \in K, \quad \forall\, p,\, q \in N.$$

The invariance of the Riemannian distance d on N implies that if $\gamma : [a, b] \to N$ is a geodesic, then the curve $k\gamma : [a, b] \to N$, $t \mapsto k \cdot \gamma(t)$, is also a geodesic (see [Kaw], Section 4.2). Furthermore, the fact that K is a group of isometries on N also implies that the sectional curvatures at a point $p \in N$ are the same as at the point $kp \in N$, for each $k \in K$.

In the next Proposition we show that, if K is a group of isometries on N, then the convexity function constructed in Proposition 1.3.1 can be assumed to be K-invariant:

Proposition 1.3.6. *Let K be a compact Lie group, and let (N, g) be a smooth Riemannian K-manifold. Furthermore, assume that the metric g is K-invariant. Then there exists a continuous, K-invariant function $r : N \to \mathbb{R}_+$ such that the ball $B_{r(p)}(p)$ is geodesically convex, for every $p \in N$.*

Proof. By Remark 1.3.5, the image $k \cdot B$ of a convex ball $B \subset N$ under the isometry $k \in K$ is still convex. This fact implies that if we fix on N a function $r : N \to \mathbb{R}_+$ like in Proposition 1.3.1, we can always assume r to be K-invariant: in fact, let $p \in N$ and $k \in K$, and assume that $r(q) < \infty$ for every $q \in N$ (otherwise we can choose r to be constant, and hence K-invariant). From the construction of r in the proof of Proposition 1.3.1, we have that the convexity of $k \cdot B_{r(p)}(p) = B_{r(p)}(kp)$ implies that $r(p) \leq r(kp)$. On the other hand, the convexity of $k^{-1} \cdot B_{r(kp)}(kp) = B_{r(kp)}(p)$ implies $r(kp) \leq r(p)$, and we are done. $\square$

We will end this section by recalling a few basic facts about *tubular neighborhoods* (see [H], Section 4.5; see also [Kaw] or [B] for the equivariant case). Let $M \subset N$ be a submanifold. The *normal* (*vector*) *bundle* of M in N is the quotient bundle

$$\nu = (TN\,|_M)/TM.$$

If N is Riemannian (i.e., we have a metric on TN), TM has an orthogonal complement $TM^{\perp}$ in $TN\,|_M$, and $TM^{\perp}$ is isomorphic to ν (see [Kaw], Theorem 2.40 and Corollary 2.41, equivariant case). Thus, ν is canonically endowed with a metric; if $\rho : M \to \mathbb{R}_+$ is continuous we set:

$$D(\nu, \rho) = \{v \in \nu \mid v \in \nu_x, \|v\| \leq \rho(x), x \in M\}.$$

A tubular neighborhood of M in N is a pair (ϕ, ν), where $\nu = (\pi, E, M)$ is the normal bundle of M and $\phi : E \to N$ is an embedding, such that the following two conditions are satisfied:

1. $\phi|_M = id_M$, where M is identified with the zero-section of ν.

2. $\phi(E)$ is an open neighborhood of M in N.

A *closed tubular neighborhood* of radius $\rho \in C(M, \mathbb{R}_+)$ of $M \subset N$ is then an embedding $D(\nu, \rho) \to N$ which is the restriction of a tubular neighborhood (ϕ, ν) of M. Thus, in this case, each fiber $D(\nu, \rho)_x$, $x \in M$, is homeomorphic to the closed unit disc $D^{n-m} \subset \mathbb{R}^{n-m}$, and hence compact. Moreover, the projection $\pi| : D(\nu, \rho) \to M$ is a proper map. In fact, let $F \subset M$ be compact: then there exist bundle charts (U_1, φ_1),...,(U_s, φ_s), $s \in \mathbb{N}$, and compact subsets $F_j \subset U_j$, $1 \leq j \leq s$, such that $F = \cup_{j=1}^s F_j \subset \cup_{j=1}^s U_j$. Thus,

$$\pi^{-1}(F) = \cup_{j=1}^s \pi^{-1}(F_j) \cong \cup_{j=1}^s (F_j \times D^n),$$

and $\cup_{j=1}^s (F_j \times D^n)$ is compact. Later on it will be convenient to refer to the open set $T = \phi(E)$ as a tubular neighborhood of M. Thus we will have a retraction $p : T \to M$ associated to T, such that (p, T, M) is a vector bundle whose zero-section is the inclusion $M \to T$.

The following is a fundamental result in differential topology:

Theorem 1.3.7. *Let $M \subset N$ be a submanifold. Then there exists a tubular neighborhood of M in N, and this is unique up to isotopy.*

Proof. See [H], Theorems 4.5.2 and 4.5.3. □

1.4. A complete, invariant Riemannian metric

A Riemannian manifold N (or a Riemannian metric g on N) is said to be (geodesically) complete if every maximal geodesic is defined for all $t \in \mathbb{R}$. We have the following important result:

Theorem 1.4.1. *Let (N, g) be a Riemannian manifold. The following are equivalent:*

1. *N is geodesically complete.*
2. *N is a complete metric space with respect to the distance function induced by g.*
3. *Every bounded subset of N (with respect to d) is relatively compact.*
4. *There exists $p \in N$ such that $\exp_p$ is defined on all of T_pN.*
5. *For every $p \in N$ $\exp_p$ is defined on all of T_pN.*

Moreover, any of the above conditions implies:

If $p, q \in N$, then there exists a geodesic γ from p to q with $\ell(\gamma) = d(p, q)$.

Proof. See [Hel], Theorems I.10.3 and I.10.4, and following remark. □

We are interested in working on a real analytic K-manifold (K compact Lie group) with a C^ω, K-invariant Riemannian metric which is also *complete*, thus we would like to be able to guarantee the existence of such a metric. We have the following fundamental result:

Theorem 1.4.2. *Let N be a real analytic K-manifold, where K is a compact Lie group. Then there exists on N a real analytic, K-invariant Riemannian metric g.*

Proof. The existence of the metric g is proved, for example, in [Ka], Theorem 1.17. We would only like to sketch here the idea of the proof: first of all, by Grauert's embedding theorem (see [G], Theorem 3), there exists a proper real analytic embedding $\varphi : N \to \mathbb{R}^n$ for some $n \in \mathbb{N}$. The usual real analytic Riemannian metric of the euclidean space $\mathbb{R}^n$ induces a real analytic metric g'' on $\varphi(N)$ in the obvious way: then, the "pull back" g' of g'' through φ is a real analytic Riemannian metric on N. Averaging g' over the compact group K with the Haar integral, one obtains a real analytic, K-invariant Riemannian metric g on N. □

Thus, we are only left to obtain the completeness of the metric. In the smooth case without any group action we have the following proposition, which states that it is always possible to construct on a smooth Riemannian manifold (N, g) a complete Riemannian metric by a suitable conformal change of g, i.e., by multiplication of g by a positive differentiable function:

Proposition 1.4.3. *Let (N, g) be a C^∞ Riemannian manifold. Then there exists a positive C^∞ function $f : N \to \mathbb{R}_+$ such that the metric $\tilde{g} := f \cdot g$ is complete.*

Proof. Since N is Hausdorff, locally compact and second countable, there exists an exhaustion of N, i.e. a family $\{U_i\}_{i\in\mathbb{N}}$ of open subsets of N such that:

1. $\overline{U}_i$ is compact, $\forall\, i \in \mathbb{N}$.
2. $\overline{U}_i \subset U_{i+1}, \quad \forall\, i \in \mathbb{N}$.
3. $N = \bigcup_{i\in\mathbb{N}} U_i$.

(Convention: $U_0 = \emptyset$). Let d be the distance function induced by g on N, and let $n > 0$. Clearly we have that, for every $p \in \overline{U}_n$, $d(p, U_{n+1}^c) > 0$, and hence by the compactness of $\overline{U}_n$ we can define

$$d_n := d(\overline{U}_n, U_{n+1}^c) > 0.$$

Now, using partition of unity it is possible to construct a differentiable, positive function $f : N \to \mathbb{R}_+$ such that

$$f \mid (\overline{U}_{n+1} \setminus U_n) > \frac{1}{d_n^2}, \quad \text{for all } n > 0.$$

Then we can define a new Riemannian metric on N by

$$\tilde{g}(v, w)_p := f(p) g(v, w)_p \quad \forall\, p \in N, \ \forall\, v, w \in T_pN.$$

Let $\tilde{d}$ and $\tilde{l}$ be the distance and length induced on N by $\tilde{g}$.
Claim: $(N, \tilde{g})$ is complete.
Proof of the claim: First we notice that for every $n > 0$ the distance between $\overline{U}_n$ and U_{n+1}^c in the new metric $\tilde{g}$ is bigger than 1. In fact, let $p \in \overline{U}_n$ and $q \in U_{n+1}^c$,

and let $\gamma \in \Gamma_{p,q}$, $\gamma : [a,b] \to N$. Then there exist $c, d \in [a,b]$ such that $\gamma_1 = \gamma \mid [c,d] \subset \overline{U}_{n+1} \setminus U_n$, and $\gamma(c) \in \partial U_n$, $\gamma(d) \in \partial U_{n+1}$. Then we have:

$$\tilde{\ell}(\gamma) \geq \tilde{\ell}(\gamma_1) = \int_c^d \tilde{g}(\dot{\gamma}_t, \dot{\gamma}_t)^{\frac{1}{2}}\, dt = \int_c^d f(\gamma_t)^{\frac{1}{2}} g(\dot{\gamma}_t, \dot{\gamma}_t)^{\frac{1}{2}}\, dt >$$

$$> \int_c^d \left(\frac{1}{d_n^2}\right)^{\frac{1}{2}} g(\dot{\gamma}_t, \dot{\gamma}_t)^{\frac{1}{2}}\, dt = \frac{1}{d_n} \int_c^d g(\dot{\gamma}_t, \dot{\gamma}_t)^{\frac{1}{2}}\, dt = \frac{1}{d_n} \ell(\gamma_1) \geq \frac{1}{d_n} \cdot d_n = 1.$$

Now we will prove the completeness of $(N, \tilde{g})$ by showing that balls in the new metric are relatively compact: by Theorem 1.4.1 this will be enough to show that the metric $\tilde{g}$ is complete. Thus, let $B := B_r(x) = \{y \in N : \tilde{d}(x,y) < r\}$, where $x \in N$ and $r \in \mathbb{R}_+$. Compactness of $\overline{B}$ will follow if we show that there exists $\bar{n} \in \mathbb{N}^*$ such that $B \subset U_{\bar{n}} \subset \overline{U}_{\bar{n}}$.
Let $m \in \mathbb{N}^*$ be such that $x \in U_m \setminus U_{m-1}$, and take $\bar{n} = m + [r] + 1$, where $[r]$ is a positive integer such that $[r] \leq r < [r] + 1$. Suppose on the contrary that there exists $y \in B$ such that $y \in U_{\bar{n}}^c$, and let $\gamma : [a,b] \to N$ be a piecewise regular curve such that $\gamma(a) = x$ and $\gamma(b) = y$. Take $a_0, a_1, ..., a_{r+1} \in [a,b]$ such that $a < a_0 < ... < a_{r+1} \leq b$, and, for every $i = 0, ..., \bar{n}$, a_i is the largest real number in $[a,b]$ such that $\gamma(a_i) \in \overline{U}_{m+i}$. Denote $\gamma_i := \gamma \mid [a_i, a_{i+1}]$, $\quad i = 0, ..., r$ and $\gamma_a := \gamma \mid [a, a_0]$, $\gamma_b := \gamma \mid [a_{r+1}, b]$. Then, for every $i = 0, ..., r$, we have that $\tilde{\ell}(\gamma_i) \geq \tilde{d}(a_i, a_{i+1}) \geq \tilde{d}(\overline{U}_i, U_{i+1}^c) > 1$, and therefore:

$$\tilde{\ell}(\gamma) = \tilde{\ell}(\gamma_a) + \sum_{i=0}^{r} \tilde{\ell}(\gamma_i) + \tilde{\ell}(\gamma_b) > \tilde{\ell}(\gamma_a) + \tilde{\ell}(\gamma_b) + r > r.$$

Since the above inequality is true for every $\gamma \in \Gamma_{x,y}$, we can write

$$\tilde{d}(x,y) = \inf_{\gamma \in \Gamma x,y} \tilde{\ell}(\gamma) \geq r,$$

and we get a contradiction since we assumed $y \in B$. $\square$

Now, in the real analytic case one can not use partitions of unity; nevertheless, we have the following important theorem, which was proved by Grauert ([G]):

Theorem 1.4.4. *Let M be a real analytic manifold, $\{V_i\}_{i \in \mathbb{N}}$ and $\{W_i\}_{i \in \mathbb{N}}$ open locally finite coverings of M, such that $W_i \subset\subset V_i$, and let $x_1^i, ..., x_n^i$ be real analytic coordinates in V_i. Then, if $f : M \to \mathbb{R}$ is a C^s function and $\{c_i\}_{i \in \mathbb{N}}$ is a sequence of positive numbers, there exists a real analytic function $\tilde{f}(x)$ in M which satisfies the inequalities:*

$$\mid \partial^{|\alpha|}(\tilde{f}(x) - f(x))/\partial^i x_1^{\alpha_1} \cdots \partial^i x_n^{\alpha_n} \mid < c_i \quad in \quad W_i$$

(where $0 \leq |\alpha| = \alpha_1 + ... + \alpha_n \leq s$, $0 \leq s < \infty$).

Proof. See [G], Proposition 8. $\square$

We are now able to prove the following result:

Theorem 1.4.5. *Let K be a compact Lie group, and let N be a real analytic K-manifold. Then there exists on N a real analytic, K-invariant, complete Riemannian metric.*

Proof. Fix on N a real analytic, K-invariant Riemannian metric g (which exists by Theorem 1.4.2). We know by Proposition 1.4.3 that there exists a positive function $f : N \to \mathbb{R}_+$ such that the metric fg is complete (but, in general, neither real analytic nor K-invariant). Without loss of generality, we can choose in the construction of f an exhaustion of N by K-invariant subsets: for example, given any exhaustion $\{A_i\}_{i\in\mathbb{N}}$ of N, it will be enough to consider the family $\{U_i := KA_i\}_{i\in\mathbb{N}}$. Let then $f : N \to \mathbb{R}_+$ be a differentiable function such that for every $n > 0$ we have:

$$f \mid (\overline{U}_{n+1} \setminus U_n) > \frac{1}{d_n^2}\,,$$

where $d_n := d(\overline{U}_n, U_{n+1}^c)$. Now f is in particular a continuous function, hence we can apply Theorem 1.4.4 with $s = 0$ to our N and f: it is clear that it is possible to choose the sequence $\{c_i\}_{i\in\mathbb{N}}$ and the coverings $\{W_i\}_{i\in\mathbb{N}}$, $\{V_i\}_{i\in\mathbb{N}}$ in such a way that $\tilde{f}$ is a positive real analytic function with the property:

$$\tilde{f} \mid (\overline{U}_{i+1} \setminus U_i) > \frac{1}{d_i^2}\,, \quad \text{for all } i \in \mathbb{N}^*. \tag{2}$$

(In fact, since for every $n > 0$ the set $\overline{U}_{n+1} \setminus U_n$ is compact, it is possible to cover it with a finite number of open subsets of N).
Next step is to average $\tilde{f}$ over the compact group K. Define

$$h : N \to \mathbb{R}\,; \quad h(x) = \int_K \tilde{f}(kx)\, dk,$$

where $\int_K$ denotes the Haar integral on K. By the properties of the Haar integral the function h is positive, real analytic and K-invariant (i.e. $h(kx) = h(x)$, for all $x \in N$ and $k \in K$). Furthermore, for every $n \in \mathbb{N}$ and $x \in \overline{U}_{n+1} \setminus U_n$, we have:

$$h(x) = \int_K \tilde{f}(kx)\, dk > \int_K \frac{1}{d_n^2}\, dk = \frac{1}{d_n^2} \tag{3}$$

(by (2) since, by the K-invariance of the sets U_i, $x \in \overline{U}_{n+1} \setminus U_n$ implies $kx \in \overline{U}_{n+1} \setminus U_n$).
Thus, if we multiply g by h, we obtain a Riemannian metric $\tilde{g} := h \cdot g$ which is still real analytic and K-invariant, that is

$$\tilde{g}((\Phi_k)_* X, (\Phi_k)_* Y)_{kx} = h(kx) g((\Phi_k)_* X, (\Phi_k)_* Y)_{kx}) = h(x) g(X,Y)_x = \tilde{g}(X,Y)_x$$

for every $x \in N$, $k \in K$ and X, $Y \in TN$. Moreover, using (3), we can show the completeness of $\tilde{g}$ by repeating the proof of the "claim" in the last part of the proof of Proposition 1.4.3. □

Chapter 2. The very-strong-weak topology

In this Chapter we describe some of the possible topologies for the space $C^\infty(M,N)$ and its subset $C^{\infty,G}(M,N)$. In particular we introduce the very-strong-weak topology on $C^{\infty,G}(M,N)$ and discuss some of its properties.

2.1. Topologies on $C^\infty(M,N)$

Let M and N be two real analytic manifolds of dimension m and n, respectively, and consider the set $C^\infty(M,N)$ of all C^∞ maps from M to N. We will describe in this section two of the possible choices for a topology on $C^\infty(M,N)$, namely the so called "strong C^∞ topology" and the "very-strong topology". We will need the following two definitions:

Definition 2.1.1. Let $f \in C^\infty(U,\mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^m$, and let $B \subset U$ be compact. If r is a non-negative integer, the C^r*-norm* of f on B is defined as

$$\|f\|_B^r := \max\{|D^\alpha f_j(x)| \mid x \in B, \quad 1 \le j \le n,\ 0 \le |\alpha| \le r\},$$

where $\alpha = (\alpha_1,...,\alpha_m)$ is a multi-index of length $|\alpha| = \alpha_1 + ... + \alpha_m$, and $D^\alpha f_j(x) = \frac{\partial^{|\alpha|} f_j(x)}{\partial x_1^{\alpha_1} ... \partial x_m^{\alpha_m}}$.

Definition 2.1.2. Let r be an integer, $0 \le r < \infty$, suppose $f \in C^\infty(M,N)$, and let $\varepsilon > 0$ be a real number (or $\varepsilon = \infty$). Let (U,φ) and (V,ψ) be charts in M and N, respectively, and let B be a compact subset of U such that $f(B) \subset V$. An *elementary* C^r *neighborhood* of f in $C^\infty(M,N)$ is a set of the form

$$\mathcal{N}^r(f;B,(U,\varphi),(V,\psi),\varepsilon) =$$

$$= \{h \in C^\infty(M,N) \mid h(B) \subset V, \quad \|\psi \circ f \circ (\varphi)^{-1} - \psi \circ h \circ (\varphi)^{-1}\|^r_{\varphi(B)} < \varepsilon\}.$$

From now on we will always use the notation $\mathcal{N}^r(f;B,(U,\varphi),(V,\psi),\varepsilon)$ to indicate a set of the form given in 2.1.2. In the particular case when $N = \mathbb{R}^n$ and $(V,\psi) = (\mathbb{R}^n, id)$, we will use the simpler notation $\mathcal{N}^r(f;B,(U,\varphi),\varepsilon)$.
Now consider the family of all sets of the form

$$\mathcal{S}^r = \bigcap_{i\in\Lambda} \mathcal{N}^r(f;B_i,(U_i,\varphi_i),(V_i,\psi_i),\varepsilon_i),$$

where $f \in C^\infty(M,N)$ and the family $\{B_i\}_{i\in\Lambda}$ is locally finite in M. It is easy to verify that this family is a basis for a topology on $C^\infty(M,N)$, which is called the "strong C^r topology"; a set like $\mathcal{S}^r$ is called a "*basic C^r-strong neighborhood of f*". The strong C^r topology is the topology usually used on $C^r(M,N)$, and the strong C^r topology on $C^\infty(M,N)$ is the relative topology from $C^r(M,N)$.

The strong C^∞ topology, introduced by Mather in [Ma], Section 2, is then defined as follows (see also [H], Section 2.1):

Definition 2.1.3. The *strong C^∞ topology* on $C^\infty(M,N)$ is the topology which has as a basis the union of all strong C^r topologies on $C^\infty(M,N)$, for $0 \leq r < \infty$. We denote by $C^\infty_S(M,N)$ the set $C^\infty(M,N)$ endowed with this topology.

Now, for $r < \infty$ the C^r strong topology is naturally the most appropriate topology to use on the space $C^r(M,N)$. The case $r = \infty$ is instead much more complex. Clearly, for each $s < \infty$ the C^s strong topology is often inadequate for the space $C^\infty(M,N)$. Nevertheless, even the C^∞ strong topology, which in fact is not a "genuine" topology on $C^\infty(M,N)$, seems to have some limitations. Before bringing this discussion any further, let us first give the definition of *very-strong topology*. This was first introduced by Cerf in [C], Definition I.4.3.1; here we will use the (equivalent) definition given in Definition 1.1 of [I2]:

Definition 2.1.4. The *very-strong topology* on $C^\infty(M,N)$ is the topology which has as a basis the family of all sets of the form

$$\mathcal{N}_{vS} = \bigcap_{i\in\Lambda} \mathcal{N}^{r_i}(f; B_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i), \tag{4}$$

where $f \in C^\infty(M,N)$, $0 \leq r_i < \infty$, $i \in \Lambda$, and the family $\{B_i\}_\imath$ is locally finite in M. A set like $\mathcal{U}$ is called a *basic very-strong neighborhood* of f. We denote by $C^\infty_{vS}(M,N)$ the set $C^\infty(M,N)$ endowed with the very-strong topology.

Remark 2.1.5. Note that Definitions 2.1.1, 2.1.3 and 2.1.4 can be given also in the case in which M and N are assumed to be smooth manifolds with boundary (see [H], Chapter 2).

Clearly, the very-strong topology on $C^\infty(M,N)$ is always at least as fine as the strong C^∞ topology, i.e. the map

$$id : C^\infty_{vS}(M,N) \to C^\infty_S(M,N)$$

is continuous. In fact, the crucial difference between the C^∞-strong and the very strong topology on $C^\infty(M,N)$ (when M is not compact) is that in (4) one can have

$$\sup_{i\in\Lambda}\{r_i\} = \infty.$$

Thus, if we consider for example the classical result by Whitney concerning approximation of C^∞ maps by C^ω maps (see [W], Lemma 6), we see that only in terms of very-strong topology we have the right means to express the involvement of partial derivatives of increasingly high order as one approaches the boundary of the maps' domain. In fact, Whitney's result contains deeper information than the C^∞-strong topology can capture. There are other reasons to believe that the very-strong topology is a better topology for the set $C^\infty(M,N)$ than the C^∞-strong is: for example, in order to continuously glue together two maps in $C^\infty(M,N)$ one has to use the very-strong topology (see [I2], Lemma B and sections 8 and 9).

However, for a more exhaustive discussion on the role of the very-strong topology and its relation to the strong C^∞ topology the reader is referred to [I2]. We would only like to take into consideration here approximation results in the two topologies.

First of all, we remark that, in terms of the strong C^∞ topology , Theorem 1.4.4 states that any smooth map in $C^\infty_S(M, N)$ can be approximated arbitrarily well with a real analytic one (see also [H], Theorem 2.5.1).

Furthermore, an analogous result was proved by Illman in the case of the very-strong topology: in fact, we have the following

Theorem 2.1.6. *Let M and N be real analytic manifolds. Then $C^\omega(M, N)$ is dense in $C^\infty_{vS}(M, N)$.*

Proof. See [I2], Theorem 4.4. □

Both Grauert's and Illman's result were proved using the Grauert-Morrey embedding theorem for real analytic manifolds (see [G], Theorem 3).

2.2. Elementary neighborhoods

In this section we present some technical results which will be needed later. Lemma 2.2.1 and Lemma 2.2.3 below are in fact given by Lemma 2.1 in [I-Ka1] and Lemma 2.1 in [I2], respectively. They both involve elementary neighborhoods, and they are very useful when working with any of the usual topologies on $C^\infty(M, N)$ and its subsets. In particular, Corollary 2.2.2 allows us to replace an elementary neighborhood with neighborhoods whose charts satisfy specific conditions.

Lemma 2.2.1. *Let M, N and P be C^∞-manifolds, and let $h : N \to P$ be a C^∞ map. Let $f \in C^\infty(M, N)$, and let $\mathcal{N} = \mathcal{N}^r(h \circ f, B, (U, \varphi), (W, \omega), \varepsilon)$, $1 \le r < \infty$, be an elementary neighborhood of $h_*(f) = h \circ f \in C^\infty(M, P)$. Then there exist finitely many elementary neighborhoods $\mathcal{M}_j = \mathcal{N}^r(f, B_j, (U, \varphi), (V_j, \psi_j), \varepsilon_j)$ of f, $1 \le j \le t$, such that*

$$\bigcup_{j=1}^{t} B_j = B,$$

and

$$h_*(\bigcap_{j=1}^{t} \mathcal{M}_j) \subset \mathcal{N}.$$

Proof. Let $y \in f(B)$. We can choose two open, relatively compact neighborhoods V_y^* and V_y of y and a chart (V'_y, ψ'_y) at y in N such that

$$y \in V_y^* \subset \overline{V}_y^* \subset V_y \subset \overline{V}_y \subset V'_y \subset h^{-1}(W).$$

Then the family $\{V_y^*\}_{y \in f(B)}$ is an open covering of $f(B)$, and since $f(B)$ is compact in N there exist $V_1^*, ..., V_t^*$ such that

$$f(B) \subset V_1^* \cup ... \cup V_t^*.$$

For $1 \leq j \leq t$ denote

$$D_j = \overline{V}_j^*, \quad \text{and} \quad B_j = B \cap f^{-1}(D_j).$$

Then we have:

$$B = \bigcup_{j=1}^{t} B_j,$$

and

$$f(B_j) \subset D_j \subset V_j \subset \overline{V}_j \subset V'_j \subset h^{-1}(W), \quad 1 \leq j \leq t.$$

Denote $\psi_j = \psi'_j | V_j$, and note that, for each $j = 1, ..., t$, (V_j, ψ_j) is also a chart at $y_j \in V_j$ in N. Thus $0 \in \psi_j(V_j)$, $1 \leq j \leq t$, and we can find for each $j = 1, ..., t$ a positive number ε_j such that

$$E^n(\varepsilon_j) \subset \psi_j(V_j), \quad 1 \leq j \leq t$$

(where $E^n(\varepsilon_j) = \{x \in \mathbb{R}^n \mid \max_{1 \leq i \leq n} |x_i| < \varepsilon_j\}$, $1 \leq j \leq t$). For $1 \leq j \leq t$, define the elementary neighborhoods of f

$$\mathcal{M}_j = \mathcal{N}^r(f; B_j, (U, \varphi), (V_j, \psi_j), \varepsilon).$$

Fix $j \in \{1, ..., t\}$, and assume that $\tilde{f} \in \mathcal{M}_j$: this means that $\tilde{f}(B_j) \subset V_j$ and

$$||\psi_j \circ \tilde{f} \circ \varphi^{-1} - \psi_j \circ f \circ \varphi^{-1}||^r_{\varphi(B_j)} < \varepsilon_j.$$

Then, in particular,

$$\max_{a \in \varphi(B_j)} \{|(\psi_j \circ \tilde{f} \circ \varphi^{-1} - \psi_j \circ f \circ \varphi^{-1})(a)_i|\} < \varepsilon_j, \quad 1 \leq j \leq t.$$

Hence for each $j = 1, ..., t$ we have

$$(\psi_j \circ \tilde{f} \circ \varphi^{-1} - \psi_j \circ f \circ \varphi^{-1})(a) \in E^n(\varepsilon_j) \quad \text{for every } a \in \varphi(B_j),$$

that is,

$$(\psi_j \circ \tilde{f} \circ \varphi^{-1} - \psi_j \circ f \circ \varphi^{-1})(\varphi(B_j)) \subset \psi_j(V_j). \tag{5}$$

Now, since $\psi_j(V_j) \subset \psi_j(\overline{V}_j) \subset \psi'_j(V_j)$, and $\psi_j(\overline{V}_j)$ is compact, we have that

$$||\omega \circ h \circ \psi_j^{-1}||^r_{\psi_j(V_j)} \leq ||\omega \circ h \circ \psi_j^{-1}||^r_{\psi_j(\overline{V}_j)} < \infty. \tag{6}$$

Thus by 3.3.7 and 6 follows that we can choose ε_j to be so small that $\tilde{f} \in \mathcal{M}_j$ implies

$$||(\omega \circ h \circ \psi_j^{-1})(\psi_j \circ \tilde{f} \circ \varphi^{-1}) - (\omega \circ h \circ \psi_j^{-1})(\psi_j \circ f \circ \varphi^{-1})||^r_{\varphi(B_j)} =$$

$$||\omega \circ h \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ f \circ \varphi^{-1}||^r_{\varphi(B_j)} < \varepsilon.$$

Since $\bigcup_{j=1}^t \varphi(B_j) = B$, we have shown that, with a suitable choice for the ε_j, $1 \leq j \leq t$, it follows from $\tilde{f} \in \bigcap_{j=1}^t \mathcal{M}_j$ that

$$||\omega \circ h \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ f \circ \varphi^{-1}||^r_{\varphi(B)} < \varepsilon.$$

Furthermore, $\tilde{f} \in \bigcap_{j=1}^{t} \mathcal{M}_j$ implies that

$$\tilde{f}(B) = \tilde{f}(\bigcup_{j=1}^{t} B_j) \subset \bigcup_{j=1}^{t} V_j \subset h^{-1}(W),$$

that is, $(h \circ \tilde{f})(B) \subset W$. This shows that if $\tilde{f} \in \bigcap_{j=1}^{t} \mathcal{M}_j$, then $h_*(\tilde{f}) \in \mathcal{N}$, i.e. that

$$h_*(\bigcap_{j=1}^{t} \mathcal{M}_j) \subset \mathcal{N},$$

and the claim is proved. □

Corollary 2.2.2. *Let $1 \leq r < \infty$, and let $\mathcal{N} = \mathcal{N}^r(f; B, (U, \varphi), (V, \psi), \varepsilon)$ be an elementary C^r neighborhood of $f \in C^\infty(M, N)$. For $t \in \mathbb{N}$, let $\{B_j\}_{1 \leq j \leq t}$ be a family of compact subsets of M, and $\{(V_j, \psi_j)\}_{1 \leq j \leq t}$ a family of charts of N, such that the following conditions are satisfied:*

(a) $B = \cup_{j=1}^{t} B_j$

(b) $f(B_j) \subset V_j \subset \overline{V_j} \subset V'_j \subset V$, *where* (V'_j, ψ'_j) *are charts in* N, $\psi_j = \psi'_j \mid_{V_j}$ *and* $\overline{V}_j$ *are compact,* $1 \leq j \leq t$.

Then there exists $\varepsilon_j > 0$, $1 \leq j \leq t$, such that if we set $\mathcal{N}_j = \mathcal{N}^r(f; B_j, (U, \varphi), (V_j, \psi_j), \varepsilon_j)$, then $\cap_{j=1}^{t} \mathcal{N}_j \subset \mathcal{N}$.

Proof. It is enough to take $N = P$ and $h = id_N$ in Lemma 2.2.1 above. □

Lemma 2.2.3. *Let M, N and P be C^∞-manifolds, and let $(f, h) \in C^\infty(M, N) \times C^\infty(N, P)$. Let $\mathcal{N} = \mathcal{N}^r(h \circ f, B, (U, \varphi), (W, \omega), \varepsilon)$, $1 \leq r < \infty$, be an elementary neighborhood of $\Gamma(f, h) = h \circ f \in C^\infty(M, P)$. Then there exist finitely many elementary neighborhoods $\mathcal{M}_j = \mathcal{N}^r(f, B_j, (U, \varphi), (V_j, \psi_j), \varepsilon_j)$ of f, and $\mathcal{M}'_j = \mathcal{N}^r(h; K_j, (V'_j, \psi'_j), (W, \omega), \delta_j)$ of h, $1 \leq j \leq t$, such that*

$$\Gamma(\bigcap_{j=1}^{t} (\mathcal{M}_j \times \mathcal{M}'_j)) \subset \mathcal{N}.$$

Proof. Like in the proof of Lemma 2.2.1 we can consider for $1 \leq j \leq t$ the following subspaces of N:

$$f(B_j) \subset D_j \subset V_j \subset \overline{V}_j \subset V'_j \subset h^{-1}(W),$$

where D_j and $\overline{V}_j$ are compact, and (V'_j, ψ'_j) and $(V_j, \psi_j = \psi'_j | V_j)$ are charts of N. Recall also that $B_j \subset M$ is compact, and $B = \cup_{j=1}^{t} B_j$. In particular, we can choose for $1 \leq j \leq t$ positive numbers ε_j, so that if we set $\mathcal{M}_j = \mathcal{N}^r(f; B_j, (U, \varphi), (V_j, \psi_j), \varepsilon_j)$, $1 \leq j \leq t$, then from

$$\tilde{f} \in \bigcap_{j=1}^{t} \mathcal{M}_j$$

follows

$$(7) \qquad ||\omega \circ h \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ f \circ \varphi^{-1}||^r_{\varphi(B)} < \frac{\varepsilon}{2}.$$

Now, fix $j \in \{1, ..., t\}$. Since $\varphi(B_j)$ is compact we have that

$$||\psi_j \circ f \circ \varphi^{-1}||^r_{\varphi(B_j)} < \infty.$$

Then we can find a finite number C_j such that

$$||\psi_j \circ \tilde{f} \circ \varphi^{-1}||^r_{\varphi(B_j)} < C_j, \quad \text{for every } \tilde{f} \in \mathcal{M}_j.$$

Denote

$$\mathcal{M}'_j = \mathcal{N}^r(h; \overline{V}_j, (V'_j, \psi'_j), (W, \omega), \delta_j), \quad 1 \leq j \leq t.$$

Then for each $j \in \{1, ..., t\}$ we can choose δ_j to be so small that for each $\tilde{h} \in \mathcal{M}'_j$, that is, for each $\tilde{h} : N \to P$ such that $\tilde{h}(\overline{V}_j) \subset W$ and

$$||\omega \circ \tilde{h} \circ (\psi'_j)^{-1} - \omega \circ h \circ (\psi'_j)^{-1}||^r_{\psi'_j(\overline{V}'_j)} < \delta_j,$$

we have

$$||[\omega \circ \tilde{h} \circ (\psi'_j)^{-1} - \omega \circ h \circ (\psi'_j)^{-1}](\psi'_j \circ \tilde{f} \circ \varphi^{-1})||^r_{\varphi(B_j)} < \frac{\varepsilon}{2},$$

that is,

$$||\omega \circ \tilde{h} \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ \tilde{f} \circ \varphi^{-1}||^r_{\varphi(B_j)} < \frac{\varepsilon}{2}.$$

Since $\bigcup_{j=1}^t \varphi(B_j) = B$, we have shown that, with a suitable choice for the δ_j, $1 \leq j \leq t$, it follows from $\tilde{h} \in \bigcap_{j=1}^t \mathcal{M}'_j$ that

$$(\tilde{h} \circ f)(B) \subset W,$$

and

$$(8) \qquad ||\omega \circ \tilde{h} \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ \tilde{f} \circ \varphi^{-1}||^r_{\varphi(B)} < \frac{\varepsilon}{2}.$$

Thus, if $(\tilde{f}, \tilde{h}) \in \bigcap_{j=1}^t (\mathcal{M}_j \times \mathcal{M}'_j)$, then

$$(\tilde{f} \circ \tilde{h})(B) \subset W,$$

and from (7) and (8) follows that

$$||\omega \circ \tilde{h} \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ f \circ \varphi^{-1}||^r_{\varphi(B)} < \varepsilon.$$

Thus we have shown that

$$\tilde{h} \circ \tilde{f} = \Gamma(f, h) \in \mathcal{N},$$

and the claim is proved. □

The following Corollary is Lemma 2.3 in [I-Ka1]:

Corollary 2.2.4. *Under the same assumptions and notation of Lemma 2.2.3, let* $\mathcal{N} = \mathcal{N}^r(h \circ f, B, (U, \varphi), (W, \omega), \varepsilon)$*,* $1 \leq r < \infty$*, be an elementary neighborhood of* $f^*(h) = h \circ f \in C^\infty(M, P)$*. Then there exist finitely many elementary neighborhoods*

$$\mathcal{M}'_j = \mathcal{N}^r(h; K_j, (V'_j, \psi'_j), (W, \omega), \delta_j)$$

of h*,* $1 \leq j \leq t$*, such that*

$$f^*(\bigcap_{j=1}^{t} \mathcal{M}'_j) \subset \mathcal{N}. \quad \square$$

Now, let M and N be two real analytic manifolds of dimension m and n, respectively, let g be a Riemannian metric on N and d the induced distance function. We will end this section by showing how we can conveniently express a basis for the C^0-strong topology on $C^\infty(M, N)$ in terms of the Riemannian structure on N (see [H], Section 2.1). In fact, let $f \in C^\infty(M, N)$, and let $\varepsilon > 0$ be a real number (or $\varepsilon = \infty$). Let (V, ψ) be a chart in N, and let B be a compact subset of M such that $f(B) \subset V$. Then an elementary C^0 neighborhood of f in $C^\infty(M, N)$ is a set of the form

$$\mathcal{N}^0(f; B, (V, \psi), \varepsilon) = \{h \in C^\infty(M, N) \mid h(B) \subset V, \quad \|\psi \circ f(x) - \psi \circ h(x)\|^0_B < \varepsilon\},$$

where $\|\cdot\|^0$ is the norm in $\mathbb{R}^n$ defined as follows: if $z = (z_1, ..., z_{n_2}) \in \mathbb{R}^n$, then $\|z\|^0 = \max\{|z_1|, ..., |z_n|\}$. Note that $\|\cdot\|^0$ is equivalent to the standard norm in $\mathbb{R}^n$. A basis for the strong C^0 topology on $C^\infty(M, N)$ is given by all the basic C^0-strong neighborhoods of f, i.e. all sets of the form

$$\mathcal{S}^0 = \bigcap_{i \in \Lambda} \mathcal{N}^0(f; B_i, (V_i, \psi_i), \varepsilon_i),$$

where $f \in C^\infty(M, N)$ and the family $\{B_i\}_{i \in \Lambda}$ is locally finite.

Now, for every $f \in C^\infty(M, N)$, and for every positive continuous function $\delta : M \to \mathbb{R}_+$ denote

$$\mathcal{M}(f; \delta) = \{h \in C^\infty(M, N) \mid d(f(x), h(x)) < \delta(x), \quad \forall\, x \in M\}.$$

Theorem 2.2.5. *The family* $\mathcal{B} = \{\mathcal{M}(f; \delta) \mid f \in C^\infty(M, N), \quad \delta \in C^0(M, \mathbb{R}_+)\}$ *is a basis for the strong* C^0 *topology on* $C^\infty(M, N)$*.*

Proof. Let $\delta : M \to \mathbb{R}_+$ be a positive continuous function, and let $f \in C^\infty(M, N)$. First we will prove that it is possible to find a basic C^0-strong neighborhood (of f) which is contained in $\mathcal{M}(f, \delta)$: let $\{B_i\}_{i \in \Lambda}$ be a locally finite family of compact subsets of M such that:

- $\bigcup_{i \in \Lambda} B_i = M$, and
- for every $i \in \Lambda$ there is a chart (V'_i, ψ'_i) in N such that $f(B_i) \subset V'_i$.

For the existence of the family $\{B_i\}$ see for example [K-N], Theorem IV.3.7. For every $i \in \Lambda$ consider an open subset $V_i \subset N$ such that its closure is compact, and the following holds:

$$f(B_i) \subset V_i \subset \overline{V}_i \subset V'_i.$$

Then (V_i, ψ_i), where $\psi_i = \psi' |_{V_i}$, is a chart in N. Define, for every $i \in \Lambda$,

$$\delta_i := \min\{\delta(x) \mid x \in B_i\} > 0.$$

Now consider, for each $i \in \Lambda$, the homeomorphism

$$(\psi_i')^{-1} : \psi_i'(V_i') \to V_i' \subset N.$$

Since $\psi_i'(\overline{V}_i)$ is compact, the restriction

$$(\psi_i')^{-1}| : \psi_i'(\overline{V}_i) \to V_i'$$

is uniformly continuous, and thus for every $i \in \Lambda$ there exists $\overline{\varepsilon}_i > 0$ such that

$$\forall\, z,\, z' \in \psi_i'(\overline{V}_i), \quad \|z - z'\|^0 < \overline{\varepsilon}_i \quad \Rightarrow \quad d((\psi_i')^{-1}(z), (\psi_i')^{-1}(z')) < \delta_i.$$

In particular we have that

$$(*) \qquad \forall\, z,\, z' \in \psi_i(V_i), \quad \|z - z'\|^0 < \overline{\varepsilon}_i \quad \Rightarrow \quad d(\psi_i^{-1}(z), \psi_i^{-1}(z')) < \delta_i.$$

For every $i \in \Lambda$ consider the following elementary C^0-neighborhood of f:

$$\mathcal{N}_i := \mathcal{N}^0(f; B_i, (V_i, \psi_i), \overline{\varepsilon}_i).$$

We now claim that

$$\mathcal{S} := \bigcap_{i \in \Lambda} \mathcal{N}_i \subset \mathcal{M}(f, \delta).$$

In fact, let $h \in \mathcal{S}$. We have to show that $d(f(x), h(x)) < \delta(x)$, for every $x \in M$. Now, if $x_0 \in M$ there exists $i_0 \in \Lambda$ such that $x_0 \in B_{i_0}$. Since in particular $h \in \mathcal{N}_{i_0} = \mathcal{N}^0(f; B_{i_0}, (V_{i_0}, \psi_{i_0}), \overline{\varepsilon}_{i_0})$, we have that

$$h(B_{i_0}) \subset V_{i_0} \subset \overline{V}_{i_0} \subset V_{i_0}',$$

and thus:

$$\psi_{i_0} \circ f(x_{i_0}),\, \psi_{i_0} \circ h(x_{i_0}) \in \psi_{i_0}(V_{i_0}) \quad \text{and} \quad \|\psi_{i_0} \circ f(x_{i_0}) - \psi_{i_0} \circ h(x_{i_0})\|^0 < \overline{\varepsilon}_{i_0}.$$

By $(*)$ this implies that

$$d((\psi_{i_0})^{-1} \circ \psi_{i_0} \circ f(x_{i_0}), (\psi_{i_0})^{-1} \circ \psi_{i_0} \circ h(x_{i_0})) = d(f(x_{i_0}), h(x_{i_0})) < \delta_{i_0} \le \delta(x_{i_0}),$$

and so our claim is proved.

In order to complete the proof of the theorem it will be enough to show the converse statement to the above, i.e. that given a basic C^0-strong neighborhood $\mathcal{S}$ it is possible to find $h \in C^\infty(M, N)$ and a continuous positive function $\delta : M \to \mathbb{R}_+$ such that $\mathcal{M}(h; \delta) \subset \mathcal{S}$. Let then $\mathcal{S} = \bigcap_{i \in \Lambda} \mathcal{N}^0(f; B_i, (V_i, \psi_i), \varepsilon_i)$ be a basic C^0-strong neighborhood of $f \in C^\infty(M, N)$. For each $i \in \Lambda$ choose an open subset U_i of N with compact closure, and such that

$$f(B_i) \subset U_i \subset \overline{U}_i \subset V_i.$$

Now, for every $i \in \Lambda$ the restriction

$$\psi_i \,|: \overline{U}_i \to \psi_i(\overline{U}_i)$$

is uniformly continuous (again since $\overline{U}_i$ is compact), and thus for every $i \in \Lambda$ there exists $\mu_i > 0$ such that

$$\forall\, y, y' \in \overline{U}_i, \quad d(y, y') < \mu_i \quad \Rightarrow \quad \|\psi_i(y) - \psi_i(y')\|^0 < \varepsilon_i.$$

Choose, for each $i \in \Lambda$, a real number $0 < \delta_i \leq \mu_i$ such that

$$B_{\delta_i}(f(x)) = \{y \in N \mid d(f(x), y) < \delta_i\} \subset \overline{U}_i,$$

for every $x \in B_i$. This is always possible since $f(B_i)$ and $\overline{U}_i$ are compact, and hence $d(f(B_i), \overline{U}_i) > 0$. Thus if $x \in B_i$, and $y \in N$ is such that $d(f(x), y) < \delta_i$, this will imply that $y \in \overline{U}_i \subset V_i$ and hence that $\|\psi_i \circ f(x) - \psi_i(y)\|^0 < \varepsilon_i$. Suppose that we can find a continuous positive function $\delta : M \to \mathbb{R}_+$ which satisfies the following property:

$$(**) \qquad \delta(x) \leq \delta_i, \quad \forall\, x \in B_i.$$

Then we claim that $\mathcal{M}(f; \delta) \subset \mathcal{S}$. In fact, let $h \in \mathcal{M}(f; \delta)$. Then, for every $x \in B_i$ we have that $d(f(x), h(x)) < \delta(x) \leq \delta_i$, hence $h(x) \in \overline{U}_i \subset V_i$, and $\|\psi_i \circ f(x) - \psi_i \circ h(x)\|^0 < \varepsilon_i$. In other words we have that, for every $i \in \Lambda$,

$$h(B_i) \subset V_i,$$

and

$$\|\psi_i \circ f(x) - \psi_i \circ h(x)\|^0 < \varepsilon_i, \quad \forall\, x \in B_i,$$

i.e. $h \in \mathcal{S}$. So, it remains to show that given any locally finite family $\{B_i\}_{i \in \Lambda}$ of compact subsets of M and given a family of positive real numbers $\{\delta_i\}_{i \in \Lambda}$ it is always possible to construct a positive continuous function $\delta : M \to \mathbb{R}_+$ that satisfies the property $(**)$. Let then $\{B_i\}_{i \in \Lambda}$ be such a family, and suppose first that $M = \cup_{i \in \Lambda} B_i$. Now, each B_i has a neighborhood A_i such that the family $\{A_i\}_{i \in \Lambda}$ is a locally finite open cover of M (see for example [I-Ka1], Lemma 1.1). For every $x \in M$ let W_x be a neighborhood of x meeting only finitely many A_i, and define for every $x \in M$:

$$\eta_x = \min\{\delta_i \mid W_x \cap A_i \neq \emptyset\} > 0.$$

Now define for every $x \in M$ constant maps

$$\delta_x : W_x \to \mathbb{R}_+\, ; \quad z \mapsto \eta_x.$$

After relabeling the families $\{W_x\}$ and $\{\delta_x\}$ by $\{W_\alpha\}_{\alpha \in \Gamma}$ and $\{\delta_\alpha\}_{\alpha \in \Gamma}$, respectively, we now have that:

$$z \in W_\alpha \cap A_i \quad \Rightarrow \quad \delta_\alpha(z) = \eta_\alpha \leq \delta_i.$$

Let $\{\varphi_\alpha\}_{\alpha \in \Gamma}$ be a (smooth) partition of unity subordinate to the cover $\{W_\alpha\}$, and define a map

$$\delta : M \to \mathbb{R}_+\, ; \quad \delta(z) = \sum_{\alpha \in \Gamma} \varphi_\alpha(z) \delta_\alpha(z).$$

Then the map δ is C^∞, and for every $x \in B_i \subset A_i$ we have:

$$|\delta(x)| = |\sum_{\alpha\in\Gamma} \varphi_\alpha(x)\delta_\alpha(x)| \leq \sum_{\alpha\in\Gamma} |\varphi_\alpha(x)||\delta_\alpha(x)| \leq \sum_{\alpha\in\Gamma} \varphi_\alpha(x)\delta_i = \delta_i.$$

Thus the map δ has the properties required. Now assume that the family $\{B_i\}$ does not cover M, and denote $W = M \setminus \cup_\imath B_i$. Then we can associate to the open set W an arbitrary positive real number, say $\overline{\delta}$, and defining W_x and η_x, for every $x \in \cup_{i\in\Lambda}$, like before, we can repeat the above construction for the cover $\{W, W_x\}$ of M and the family $\{\overline{\delta}, \delta_x\}$. In fact, in this case, we do not care about the behaviour of δ outside $\cup_{i\in\Lambda} B_i$. Hence the theorem is completely proved. □

Remark 2.2.6. Note that in Theorem 2.2.5 d can be assumed to be any distance function on M. Note also that although the definition of the basis $\mathcal{B}$ depends on the metric d, the topology that $\mathcal{B}$ generates is independent of the choice of the metric.

2.3. Topologies on $C^{\infty,G}(M,N)$

Throughout this section let G be a Lie group, and let M and N be two smooth G-manifolds. In the previous section we saw how the very-strong topology can be considered as the most appropriate topology to be used on $C^\infty(M,N)$. What happens when we consider on the subset $C^{\infty,G}(M,N)$ of G-equivariant map the relative topology from $C^\infty_{vS}(M,N)$? In the case of the C^∞ strong topology we have the following result by Illman and Kankaanrinta:

Theorem 2.3.1. *Let G be a non-compact Lie group which acts properly on two C^r manifolds M and N, $1 \leq r \leq \omega$. Then the strong C^r topology on $C^{r,G}(M,N)$ is the discrete topology.*

Proof. See [I-Ka1], Proposition 4.7. □

Since the very-strong topology is more fine than the C^∞ strong topology on $C^\infty(M,N)$, we obtain as a consequence of Theorem 2.3.1 the following

Corollary 2.3.2. *Let G be a non-compact Lie group which acts properly on two C^∞ manifolds M and N. Then the very-strong topology on $C^{\infty,G}(M,N)$ is the discrete topology.*

Thus, for a very significant class of Lie group actions the very strong topology on $C^{\infty,G}(M,N)$ turns out to be completely "useless". In [I-Ka1] a new topology is defined on the set $C^{r,G}(M,N)$, $1 \leq r \leq \infty$. This topology is called the *strong-weak topology*, since the idea behind it is to "mix" the strong and weak topologies on $C^{r,G}(M,N)$. Recall that the weak topology on $C^{r,G}(M,N)$ has as a basis the family of all sets of the form

$$\mathcal{W}^r = \bigcap_{i=1}^{s} \mathcal{N}^r(f; B_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i),$$

where s is finite. The strong-weak topology allows one to avoid the situation described in Theorem 2.3.1, but in the case $r = \infty$ it has the same limitations that the strong C^∞ topology has (see Section 2.1). Then, in 2002, Illman defined a topology that should be considered as the right one to be used on $C^{\infty,G}(M,N)$ (see [I3]). This topology was named the *very-strong-weak topology*, as it comes up as a mixture between the very-strong and the weak topology on $C^{\infty,G}(M,N)$. Before giving the definition of very-strong-weak topology on $C^{\infty,G}(M,N)$, we need the following:

Definition 2.3.3. Let G be a Lie group, and let M and N be two smooth G-manifolds. Let $p : M \to M/G$ be the projection onto the orbit space. The *very-strong-weak topology on* $C^\infty(M,N)$ *with respect to* $p : M \to M/G$ is the topology which has as a basis the family of all sets of the form

$$\mathcal{N}_{vSW[p]} = \bigcap_{i\in\Lambda} \mathcal{N}^{r_i}(f; B_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_i),$$

where $f \in C^\infty(M,N)$, $1 \le r_i < \infty$ for $i \in \Lambda$, and the family $\{p(B_i)\}_{i\in\Lambda}$ is locally finite in M/G. A set like $\mathcal{N}_{vSW[p]}$ is called a basic neighborhood with respect to p of f. We denote the set $C^\infty(M,N)$ with this topology by $C^\infty_{vSW[p]}(M,N)$.

Remark 2.3.4. Note that the definition of very-strong-weak topology with respect to p seen in Definition 2.3.3 above could be generalized to very-strong-weak topology with respect to any "phase map" $p : M \to \Omega$, where Ω is a topological space (see Definition 1.6 in [I-Ka1] for the strong-weak topology case).

Lemma 2.3.5 below shows that the family of all basic neighborhoods with respect to p forms in fact a basis for a topology on $C^\infty(M,N)$.

Lemma 2.3.5. *Under the assumptions of Definition* 2.3.3 *let* f, $f' \in C^\infty(M,N)$, *and let* $\mathcal{U}$ *and* $\mathcal{U}'$ *be two basic neighborhoods with respect to* p *of* f *and* f', *respectively. Then, if* $f_0 \in \mathcal{U} \cap \mathcal{U}'$, *there exists a basic neighborhood with respect to* p *of* f_0, *say* $\mathcal{U}_0$, *such that* $\mathcal{U}_0 \subset \mathcal{U} \cap \mathcal{U}'$.

Proof. Assume that $\mathcal{U} = \bigcap_{i\in\Lambda} \mathcal{N}_i$, where

$$\mathcal{N}_i = \mathcal{N}^{r_i}(f; B_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_i), \quad i \in \Lambda,$$

is an elementary neighborhood of f, $1 \le r_i < \infty$, and the family $\{p(B_i)\}_{i\in\Lambda}$ is locally finite in M/G. Now, for each $i \in \Lambda$ we can choose a positive real number $\varepsilon_{i,0}$ such that the elementary neighborhood

$$\mathcal{N}_{i,0} = \mathcal{N}^{r_i}(f_0, B_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_{i,0})$$

is such that $\mathcal{N}_{i,0} \subset \mathcal{N}_i$. In fact, it is enough to choose $\varepsilon_{i,0} < \varepsilon_i - d_i$, where $d_i = ||\psi_i \circ f_0 \circ \varphi_i^{-1} - \psi_i \circ f \circ \varphi_i^{-1}||^{r_i}_{\varphi_i(B_i)}$. Similarly, suppose that $\mathcal{U}' = \bigcap_{j\in\Gamma} \mathcal{N}'_j$, where

$$\mathcal{N}'_j = \mathcal{N}^{s_j}(f'; B'_j, (U'_j, \varphi'_j), (V'_j, \psi'_j), \varepsilon'_j), \quad j \in \Gamma,$$

is an elementary neighborhood of f', $1 \leq s_j < \infty$, and the family $\{p(B'_j)\}_{j\in\Gamma}$ is locally finite in M/G. Like before, we can choose for each $j \in \Gamma$ an $\varepsilon'_{j,0}$ such that the elementary neighborhood

$$\mathcal{N}'_{j,0} = \mathcal{N}^{s_j}(f_0, B'_j, (U'_j, \varphi'_j), (V'_j, \psi'_j), \varepsilon'_{j,0})$$

is contained in $\mathcal{N}'_j$. Thus, since the family $\{p(B_i), p(B'_j) \mid i \in \Lambda,\ j \in \Gamma\}$ is locally finite in M/G, we have that the set

$$\mathcal{U}_0 = \bigcap_{i\in\Lambda} \mathcal{N}_{i,0} \cap \bigcap_{j\in\Gamma} \mathcal{N}'_{j,0}$$

is a basic neighborhood of f_0, and $\mathcal{U}_0 \subset \mathcal{U} \cap \mathcal{U}'$. □

Let now $f \in C^{\infty,G}(M, N)$, and let $1 \leq r < \infty$; we denote

$$\mathcal{N}^{r,G}(f; B, (U, \varphi), (V, \psi), \varepsilon) = \mathcal{N}^r(f; B, (U, \varphi), (V, \psi), \varepsilon) \cap C^{\infty,G}(M, N).$$

Definition 2.3.6 below is Definition 2 in [I3].

Definition 2.3.6. Let G be a Lie group, and let M and N be two smooth G-manifolds. Let $p : M \to M/G$ be the projection onto the orbit space. The *very-strong-weak topology* on $C^{\infty,G}(M, N)$ is the relative topology of $C^{\infty,G}(M, N)$ as a subset of $C^{\infty}_{vSW[p]}(M, N)$. A basis for this topology is then given by the family of all sets of the form

$$\mathcal{N}_{vSW} = \bigcap_{i\in\Lambda} \mathcal{N}^{r_i,G}(f; B_i, (U_i, \varphi_i), (V_i, \psi_i), \varepsilon_i), \tag{9}$$

where $f \in C^{\infty,G}(M, N)$, $1 \leq r_i < \infty$ for $i \in \Lambda$, and the family $\{p(B_i)\}_{i\in\Lambda}$ is locally finite in M/G. A set of the form (9) is called a basic vsw-neighborhood of f. We denote by $C^{\infty,G}_{vSW}(M, N)$ the set $C^{\infty,G}(M, N)$ endowed with the very-strong-weak topology.

From Definition 2.3.6 follows, that the very-strong-weak topology is always at least as fine as the weak C^∞ topology, and at most as fine as the very-strong topology. Thus, the identity map

$$id : C^{\infty,G}_{vS}(M, N) \to C^{\infty,G}_{vSW}(M, N) \to C^{\infty,G}_{W}(M, N)$$

is continuous. The idea behind the very-strong-weak topology is that when we consider a family of compact sets in M, and we "move" along an orbit, the very-strong-weak topology behaves like the weak topology; on the other hand, if we "move" in the direction which is perpendicular to the orbit, the very-strong-weak topology behaves like the very-strong topology. Roughly speaking, the very-strong-weak topology can range between the weak C^∞ topology and the very-strong one in correspondence to the fact that the projection map can "range" from the constant map to the identity. If for example the action is transitive (i.e. M is homogeneus), then M/G is a one-point space, the projection $p : M \to M/G$ is the constant map, and the very strong topology on $C^{\infty,G}(M, N)$ coincides with the weak C^∞ topology.

If instead the action of G on M is trivial, then $M/G = M$, and the projection map is the identity on M; in this case, the very-strong-weak topology clearly coincides with the very-strong one. More generally we have the two lemmas below:

Lemma 2.3.7. *Let G be a compact Lie group, and let M and N be two C^∞ G-manifolds. Then*

$$C^{\infty,G}_{vSW}(M,N) = C^{\infty,G}_{vS}(M,N).$$

Proof. Let $f \in C^{\infty,G}(M,N)$, and let $\mathcal{N} = \bigcap_{i\in\Lambda} \mathcal{N}^{r_i,G}(f; B_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i)$ be a basic neighborhood of f in the very-strong topology. It is enough to prove that $\mathcal{N}$ is also a basic neighborhood of f in the very-strong-weak topology. Since G is compact, the projection $p : M \to M/G$ is a proper map, and the orbit space M/G is locally compact. Hence the map p is of finite type, and local finiteness of the family $\{B_i\}_{i\in\Lambda}$ in M implies local fineteness of the family $\{p(B_i)\}_{i\in\Lambda}$ in M/G. □

Lemma 2.3.8. *Let G be a Lie group, let M and N be C^∞ G-manifolds, and assume that the orbit space M/G is compact. Then*

$$C^{\infty,G}_{vSW}(M,N) = C^{\infty,G}_{W}(M,N).$$

Proof. Let $f \in C^{\infty,G}(M,N)$, and let $\mathcal{N} = \bigcap_{i\in\Lambda} \mathcal{N}^{r_i,G}(f; B_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i)$ be a basic neighborhood of f in the very-strong-weak topology. Then the family $\{p(B_i)\}_{i\in\Lambda}$ is locally finite in M/G, and since M/G is compact this means that $|\Lambda| < \infty$. Thus we can write

$$\mathcal{N} = \bigcap_{i=1}^{s} \mathcal{N}^{r_i,G}(f; B_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i),$$

where $s < \infty$. Then $\mathcal{N}$ is open in the weak C^∞ topology, and this is enough to prove the claim. □

2.4. Properties of the very-strong-weak topology

In this section we establish some basic properties of the very-strong-weak topology. For the corresponding results in the case of the strong-weak topology see Section 4 in [I-Ka1].

Like for the other topologies mentioned in Section 2.1, the composition map is not, in general, continuous in the very-strong-weak topology. By "composition map" we mean in the non-equivariant case the map

$$\Gamma : C^\infty(M,N) \times C^\infty(N,P) \to C^\infty(M,P), \quad (f,h) \mapsto h \circ f,$$

where M, N and P are smooth manifolds. For a counterexample in the case of the strong C^∞ topology see [Ma], Remark 2 on page 259.

Nevertheless, also in the case of the very-strong-weak topology it is possible to prove a series of very useful results, using the results in Section 2.2.

Let then G and H be two Lie groups, and let $\theta : G \to H$ be a continuous homomorphism. If M is a G-manifold and N is an H-manifold we will say that a map $f : M \to N$ is θ-equivariant if

$$f(gx) = \theta(g)f(x), \quad \text{for all } g \in G,\ x \in M.$$

We will denote the set of all smooth, θ-equivariant maps from M to N by $C^{\infty,\theta}(M,N)$. Furthermore, we will refer to the "induced map $\bar{f} : M/G \to N/H$" as the (continuous) map which makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M/G & \xrightarrow[\ \bar{f}\ ]{} & N/H \end{array}$$

commute (here p and q are the projections). Now we can prove the following:

Proposition 2.4.1. *Let G and H be two Lie groups, and let $\theta : G \to H$ be a continuous homomorphism. Let M and N be smooth G-manifolds, and let P be a smooth H-manifold. If $h : N \to P$ is a C^∞, θ-equivariant map, then the induced map*

$$h_* : C^{\infty,G}_{vSW}(M,N) \to C^{\infty,\theta}_{vSW}(M,P), \quad f \mapsto h \circ f,$$

is continuous.

Proof. First, let $p : M \to M/G$ be the projection onto the orbit space of M, and consider the map

$$\hat{h}_* : C^{\infty}_{vSW[p]}(M,N) \to C^{\infty}_{vSW[p]}(M,P), \quad f \mapsto h \circ f.$$

Let then $f \in C^{\infty}_{vSW[p]}(M,N)$, and let $\mathcal{U} = \bigcap_{i\in\Lambda} \mathcal{N}_i$ be a basic neighborhood with respect to p of $h \circ f$, where

$$\mathcal{N}_i = \mathcal{N}^{r_i}(h \circ f; B_i, (U_i,\varphi_i), (W_i,\omega_i), \varepsilon_i),$$

and the family $\{p(B_i)\}_{i\in\Lambda}$ is locally finite in M/G. By Lemma 2.2.1 there exist for each $i \in \Lambda$ finitely many elementary neighborhoods of f, say

$$\mathcal{M}_{ij} = \mathcal{N}^{r_i}(f, B_{ij}, (U_i,\varphi_i), (V_{ij},\psi_{ij}), \varepsilon_{ij}),$$

$1 \le j \le t(i)$, such that

$$\hat{h}_*(\bigcap_{j=1}^{t(i)} \mathcal{M}_{ij}) \subset \mathcal{N}_i.$$

Now, the family $\{p(B_{ij}) \mid 1 \le j \le t(i),\ i \in \Lambda\}$ is locally finite in M/G, and so we have that

$$\mathcal{M} = \bigcap_{i\in\Lambda} \bigcap_{j=1}^{t(i)} \mathcal{M}_{ij}$$

is a basic neighborhood of f in $C^{\infty}_{vSW[p]}(M,N)$, and

$$\hat{h}_*(\mathcal{M}) \subset \bigcap_{i\in\Lambda} \mathcal{N}_i = \mathcal{U}.$$

Hence we have shown that $\hat{h}_*$ is continuous. Now, since for every $f \in C^{\infty,G}_{vSW}(M,N)$ the composition map $h \circ f = h_*(f) : M \to P$ is a θ-equivariant map, we have that the continuity of

$$h_* = \hat{h}_*| : C^{\infty,G}_{vSW}(M,N) \to C^{\infty,\theta}_{vSW}(M,P)$$

follows from the continuity of $\hat{h}_*$. □

Proposition 2.4.2. *Let G and H be two Lie groups, and let $\theta : G \to H$ be a continuous homomorphism. Let M be a smooth G-manifold, and let N and P be smooth H-manifolds. Assume $f \in C^{\infty,\theta}_{vSW}(M,N)$ is such that the induced map $\bar{f} : M/G \to N/H$ is of finite type. Then the induced map*

$$f^* : C^{\infty,H}_{vSW}(N,P) \to C^{\infty,\theta}_{vSW}(M,P), \quad h \mapsto h \circ f,$$

is continuous.

Proof. Let $p : M \to M/G$ and $p' : N \to N/H$ be the projection onto the orbit space of M and N, respectively. We first want to show that the map

$$\hat{f}^* : C^{\infty}_{vSW[p']}(N,P) \to C^{\infty}_{vSW[p]}(M,P), \quad f \mapsto h \circ f,$$

is continuous. Let then $h \in C^{\infty}_{vSW[p']}(N,P)$, and let $\mathcal{U} = \bigcap_{i\in\Lambda} \mathcal{N}_i$ be a basic neighborhood with respect to p of $h \circ f$, where

$$\mathcal{N}_i = \mathcal{N}^{r_i}(h \circ f; B_i, (U_i, \varphi_i), (W_i, \omega_i), \varepsilon_i),$$

and the family $\{p(B_i)\}_{i\in\Lambda}$ is locally finite in M/G. By Corollary 2.2.4, there exist finitely many elementary neighborhoods of h, say

$$\mathcal{M}'_{ij} = \mathcal{N}^{r_i}(h; L_{ij}, (V_{ij}, \psi_{ij}), (W_i, \omega_i), \delta_{ij}),$$

$1 \le j \le t(i)$, such that

$$\hat{f}^*(\bigcap_{j=1}^{t(i)} \mathcal{M}'_{ij}) \subset \mathcal{N}_i.$$

Note that here we can choose the sets L_{ij} to be such that

$$L_i = \bigcap_{j=1}^{t(i)} L_{ij} = f(B_i), \quad \text{for all } i \in \Lambda.$$

Now $\bar{f}$ is of finite type, and the family $\{p(B_i)\}_{i\in\Lambda}$ is locally finite in M/G, hence the family

$$\{\bar{f}(p(B_i))\}_{i\in\Lambda} = \{p'(f(B_i))\}_{i\in\Lambda} = \{p'(L_i)\}_{i\in\Lambda}$$

is locally finite in N/H. Thus, the family

$$\{p'(L_{ij}) \mid i \in \Lambda,\ 1 \leq j \leq t(i)\}$$

is also locally finite in N/H, and

$$\mathcal{M}' = \bigcap_{i\in\Lambda} \bigcap_{j=1}^{t(i)} \mathcal{M}'_{ij}$$

is a basic neighborhood of h in $C^{\infty}_{vSW[p']}(N,P)$. Furthermore, we have that

$$\hat{f}^*(\mathcal{M}') \subset \bigcap_{i\in\Lambda} \mathcal{N}_i = \mathcal{U},$$

which shows that $\hat{f}^*$ is continuous. Now, since $f \in C^{\infty,\theta}_{vSW}(M,N)$ we have that for each smooth, H-equivariant map $h : N \to P$ the map $h \circ f = f^*(h) : M \to P$ is smooth and θ-equivariant. Thus the continuity of the restriction

$$f^* = \hat{f}^*| : C^{\infty,H}_{vSW}(N,P) \to C^{\infty,\theta}_{vSW}(M,P)$$

follows from the continuity of $\hat{f}^*$. □

Using Proposition 2.4.1 and Corollary 2.2.2 it is possible to prove the so-called "product theorem" (Theorem 2.4.3 below).

Theorem 2.4.3. *Let G be a Lie group, and let M, N_1 and N_2 be C^∞ G-manifolds. Let G act on $N_1 \times N_2$ by the diagonal action, and let $q_i : N_1 \times N_2 \to N_j$, $j = 1, 2$, denote the projection maps. Then the natural bijection*

$$\iota : C^{\infty,G}_{vSW}(M, N_1 \times N_2) \to C^{\infty,G}_{vSW}(M,N_1) \times C^{\infty,G}_{vSW}(M,N_2), \quad f \mapsto (q_1 \circ f, q_2 \circ f),$$

is a homeomorphism.

Proof. A map $f : M \to N_1 \times N_2$ is G-equivariant if and only if both its components are G-equivariant maps. Hence it will be enough to show that if $p : M \to M/G$ denotes the projection onto the orbit space of M, then the map

$$\hat{\iota} : C^{\infty}_{vSW[p]}(M, N_1 \times N_2) \to C^{\infty}_{vSW[p]}(M,N_1) \times C^{\infty}_{vSW[p]}(M,N_2), \quad f \mapsto (q_1 \circ f, q_2 \circ f),$$

is a homeomorphism. The map $\hat{\iota}$ is clearly bijective, and continuous by Proposition 2.4.1, hence it remains to show that $\hat{\iota}^{-1}$ is continuous. Let $(f_1, f_2) \in C^{\infty}_{vSW[p]}(M,N_1) \times C^{\infty}_{vSW[p]}(M,N_2)$, and set $f = \hat{\iota}^{-1}(f_1,f_2) \in C^{\infty}_{vSW[p]}(M, N_1 \times N_2)$. Let $\mathcal{V} = \bigcap_{i\in\Lambda} \mathcal{N}_i$ be a basic neighborhood of f in $C^{\infty}_{vSW[p]}(M, N_1 \times N_2)$. By Corollary 2.2.2 we can assume that, for each $i \in \Lambda$, $\mathcal{N}_i$ is of the form

$$\mathcal{N}_i = \mathcal{N}^{r_i}(f; B_i, (U_i, \varphi_i), (V_i^1 \times V_i^2, \psi_i^1 \times \psi_i^2), \varepsilon_i),$$

where (V_i^j, ψ_i^j) is a chart of N_j, for $j = 1, 2$. Denote

$$\mathcal{N}_i^j = \mathcal{N}^{r_i}(f_j; B_i, (U_i, \varphi_i), (V_i^j, \psi_i^j), \varepsilon_i), \quad j = 1, 2.$$

Then, clearly, $\hat{\imath}^{-1}(\mathcal{N}_i^1 \times \mathcal{N}_i^2) \subset \mathcal{N}_i$. Thus, for $j = 1, 2$, the set $\mathcal{U}^j = \bigcap_{i\in\Lambda} \mathcal{N}_i^j$ is a basic neighborhood of $f_j = q_j \circ f$ in $C^\infty_{vSW[p]}(M, N_j)$, and

$$\hat{\imath}^{-1}(\mathcal{U}^1 \times \mathcal{U}^2) \subset \bigcap_{i\in\Lambda} \mathcal{N}_i = \mathcal{V},$$

and this proves the claim. □

We end this section with two results regarding the very-strong-weak topology and proper G-manifolds:

Proposition 2.4.4. *Let G be a Lie group, and let M and N be two proper C^∞ G-manifolds. Let P be a G-manifold, and let $f : M \to N$ be a G-equivariant, proper C^∞ map. Then the induced map*

$$f^* : C^{\infty,G}_{vSW}(N, P) \to C^{\infty,G}_{vSW}(M, P), \quad h \mapsto h \circ f,$$

is continuous.

Proof. If f is proper, we have that the induced map $\bar{f} : M/G \to N/G$ is also proper (see [I-Ka1], Lemma 3.7). Since N/G is locally compact this implies that $\bar{f}$ is of finite type (see [I-Ka1], Lemma 1.8), and hence the claim follows from Proposition 2.4.2. □

Lemma 2.4.5. *Let G be a Lie group, and let M and N be two smooth G-manifolds. Assume that the action of G on N is proper, and let $V \subset N$ be open and G-invariant. Then the set $C^{\infty,G}(M, V)$ is open in $C^{\infty,G}_{vSW}(M, N)$.*

Proof. Follows by Lemma 4.8 in [I-Ka1], since the identity map

$$id : C^{\infty,G}_{vSW}(M, N) \to C^{\infty,G}_{SW}(M, N)$$

is continuous (here $C^{\infty,G}_{SW}(M, N)$ denotes $C^{\infty,G}(M, N)$ with the strong-weak topology). □

Chapter 3. The compact case: a non-linear average

In this Chapter we prove that if K is a compact Lie group, then each smooth, K-equivariant map between two K-manifolds can be approximated arbitrarily well in the very-strong topology with a real analytic, K-equivariant map. An analogous result was previously proven by S. Illman under the additional assumption that the number of K-isotropy types in N is finite. We generalize Illman's result following the work of F. Kutzschebauch in the case of the strong C^∞ topology (see [Ku2]).

3.1. Center of mass

The Riemannian "center of mass" is a generalization of the notion of "center of gravity" (see [K-N], Theorem 9.1). It was introduced in the paper [Gr-Kar] (see also [Gr]), and it will be crucial for the construction of a "non-linear average" in the following sections. For a description of other interesting applications of the center of mass the reader is referred to section 2 in [Kar2].

Let then (N, g) be a Riemannian manifold. We can give the following:

Definition 3.1.1. Let $B_\rho(p)$ be a Riemannian ball of radius $\rho > 0$ around $p \in N$. We will say that the ball $B_\rho(p)$ is *strongly convex* if it is geodesically convex and one of the following two conditions is satisfied:

(i) The sectional curvatures of N in $B_\rho(p)$ are at most 0.
(ii) The sectional curvatures of N in $B_\rho(p)$ are at most $\Delta > 0$, and $\rho < \frac{\pi}{4} \cdot \frac{1}{\sqrt{\Delta}}$.

The following theorem was proven by Karcher:

Theorem 3.1.2. *Let K be a compact Lie group, and let (N, g) be a complete Riemannian manifold. Assume that $\eta : K \to N$ is a smooth map whose image $\eta(K)$ is contained in a strongly convex ball $B_\rho(p)$. Consider the function*

$$\Phi_\eta : \overline{B}_\rho(p) \to \mathbb{R}, \quad \Phi_\eta(y) = \frac{1}{2} \int_K d^2(y, \eta(k))\, dk.$$

Then Φ_η has only interior minima on the compact ball $\overline{B}_\rho(p)$. Furthermore, Φ_η is a strictly convex function on $B_\rho(p)$. Thus, the function Φ_η obtains its minimum at exactly one point C_η in $\overline{B_\rho(p)}$, and $C_\eta \in B_\rho(p)$.

Proof. See [Kar2], Theorem 1.2. See also [Gr-Kar]. □

Remark 3.1.3. Under the same assumptions and notation of Theorem 3.1.2, let $\eta : K \to B_{\rho'}(p') \subset B_\rho(p)$ be a smooth map, where $B_{\rho'}(p')$ and $B_\rho(p)$ are two strongly convex balls in N. Denote

$$\Phi_\eta : \overline{B_\rho(p)} \to \mathbb{R}, \quad y \mapsto \frac{1}{2} \int_K d^2(y, \eta(k))\, dk,$$

and

$$\Phi_{\eta'} : \overline{B_{\rho'}(p')} \to \mathbb{R}, \quad y \mapsto \frac{1}{2} \int_K d^2(y, \eta(k))\, dk.$$

Then clearly $C_\eta = C_{\eta'}$.

We can now give the following

Definition 3.1.4. Let K be a compact Lie group, and let (N, g) be a complete Riemannian manifold. We will call a smooth map $\eta : K \to N$ *almost constant* if there exists a strongly convex ball $B_\rho(p)$ in N such that

$$\eta(K) \subset B_{\frac{1}{3}\rho}(p) \subset B_\rho(p).$$

Example Let N be a complete, simply connected Riemannian manifold of non-positive curvature: in this case, the exponential map $exp_p : T_pN \to N$ is a diffeomorphism for every $p \in N$ (see [K-N], Vol. II, Theorem VIII.8.1). Thus, every ball in N is strongly convex, and hence every map $\eta : K \to N$ is almost constant.

If a map $\eta : K \to B_{\frac{1}{3}\rho}(p) \subset B_\rho(p)$ is almost constant, it follows from Theorem 3.1.2 and Corollary 3.1.3 that Φ_η takes its minimum value at only one point C_η in $\overline{B_\rho(p)}$, and $C_\eta \in B_{\frac{1}{3}\rho}(p)$. In the following lemma we prove that C_η *does not depend on the choice of the ball* $B_\rho(p)$.

Lemma 3.1.5. *Let* $\eta : K \to N$ *be an almost constant map such that*

$$\eta(K) \subset B_{\frac{1}{3}\rho_1}(p_1) \subset B_{\rho_1}(p_1),$$

and

$$\eta(K) \subset B_{\frac{1}{3}\rho_2}(p_2) \subset B_{\rho_2}(p_2),$$

where $B_{\rho_1}(p_1)$ *and* $B_{\rho_2}(p_2)$ *are two strongly convex balls in* N. *Denote*

$$\eta_1 : K \to B_{\rho_1}(p_1), \quad k \mapsto \eta(k),$$

and

$$\eta_2 : K \to B_{\rho_2}(p_2), \quad k \mapsto \eta(k).$$

Then $C_{\eta_1} = C_{\eta_2}$.

Proof. Without loss of generality, we may assume that $\rho_1 \leq \rho_2$. Since $\eta(K) \subset B_{\frac{1}{3}\rho_1}(p_1) \cap B_{\frac{1}{3}\rho_2}(p_2)$, we can write by the triangle inequality:

$$d(p_1, p_2) \leq d(p_1, \eta(k)) + d(\eta(k), p_2) < \frac{1}{3}\rho_1 + \frac{1}{3}\rho_2 \leq \frac{2}{3}\rho_2$$

(here $k \in K$). Thus

$$B_{\frac{1}{3}\rho_1}(p_1) \subset B_{d(p_1,p_2)+\frac{1}{3}\rho_1}(p_2) \subset B_{\frac{2}{3}\rho_2+\frac{1}{3}\rho_1}(p_2) \subset B_{\rho_2}(p_2),$$

and hence, by Corollary 3.1.3, we have that $C_{\eta_1} = C_{\eta_2}$. □

The following definition follows naturally from Lemma 3.1.5:

Definition 3.1.6. Let $\eta : K \to N$ be an almost constant map. By Lemma 3.1.5 the point $C_\eta \in N$ is uniquely determined by η. We will call C_η the *center of mass* of the almost constant map η.

Lemma 3.1.7. *Let* $\eta : K \to N$ *be an almost constant map. Then the following properties are easy consequences of the construction of* C_η*:*

1. *If* η *is a constant map, i.e.,* $\eta(k) = y_0$ *for every* $k \in K$, *then* $C_\eta = y_0$.
2. *If* $T : K \to K$ *is the left or right translation with respect to an element of* K, *then* $\eta \circ T : K \to N$ *is an almost constant map, and*

$$\Phi_{\eta\circ T}(y) = \frac{1}{2}\int_K d^2(y, \eta(T(k)))\, dk = \frac{1}{2}\int_K d^2(y, \eta(k))\, dk = \Phi_\eta(y).$$

Thus, $C_{\eta\circ T} = C_\eta$.

3. *If* $f : N \to N$ *is an isometry, the map* $f \circ \eta : K \to N$ *is an almost constant map (see Remark* 1.3.5*). Moreover,*
$$\Phi_{f\circ\eta}(f(y)) = \frac{1}{2}\int_K d^2(f(y), f(\eta(k)))\, dk = \Phi_\eta(y),$$
hence $C_{f\circ\eta} = f(C_\eta)$. □

Later on we will need the following estimate:

Proposition 3.1.8. *Let* $\eta_1, \eta_2 : K \to N$ *be two almost constant maps, and assume that there exists a strongly convex ball* $B_\rho(p)$ *in* N *such that* $\eta_1, \eta_2 : K \to B_{\frac{1}{3}\rho}(p) \subset B_\rho(p)$*. If* $\delta \leq \mathcal{K} \leq \Delta$ *are lower and upper curvature bounds in* $B_{\frac{1}{3}\rho}(p)$*, then*

$$d(C_{\eta_1}, C_{\eta_2}) \leq (1 + c(\delta, \Delta) \cdot (2\rho)^2) \cdot \int_K d(\eta_1(k), \eta_2(k))\, dk, \tag{10}$$

where $c(\delta, \Delta)$ *is a positive constant which depends on* δ *and* Δ*.*

Proof. See [Kar2], Corollary 1.6. □

From now on let K be a compact Lie group, and let M and N be smooth Riemannian manifolds. Furthermore, assume that N is complete.

Definition 3.1.9. We will call a continuous map $\theta : K \times M \to N$ a *mass distribution* if, for each $x \in M$, the map
$$\theta_x = \theta(\cdot, x) : K \to N, \quad k \mapsto \theta_x(k) = \theta(k, x),$$
is almost constant.

Definition 3.1.10. Let $\theta : K \times M \to N$ be a mass distribution. We define the *center* of θ to be the map
$$\hat{\mathcal{C}}(\theta) : M \to N, \quad x \mapsto C_{\theta_x},$$
where, for every $x \in M$, C_{θ_x} is the center of mass of the map θ_x defined in 3.1.6.

Using Proposition 3.1.8, we are going to prove that the center of a mass distribution is a continuous map. We first need Lemma 3.1.11 below:

Lemma 3.1.11. *Let* A *be a compact topological space,* X *a topological space and* $Y = (Y, d)$ *a metric space. Let* $f : A \times X \to Y$ *be a continuous map, and fix* $x' \in X$*. Then for every* $\varepsilon > 0$ *there exists a neighborhood* V *of* x' *such that*
$$d(f(a, x), f(a, x')) < \varepsilon, \quad \forall\, a \in A, \quad \forall\, x \in V.$$

Proof. Consider the map
$$\chi : A \times X \to \mathbb{R}, \quad (a, x) \mapsto d(f(a, x), f(a, x')).$$
The map χ is continuous since $\chi = d \circ (f \times f) \circ (id_{A\times X} \times (id_A \times c_{x'})) \circ \phi$, where ϕ is the natural homeomorphism from $A \times X$ onto the diagonal of $(A \times X) \times (A \times X)$,

and $c_{x'} : X \to X$ is the constant map with value x'. Thus, for every $a' \in A$ we can find a neighborhood $U_{a'} \times V_{x'}$ of (a', x') in $A \times X$ such that

$$|\chi(a,x) - \chi(a',x')| = d(f(a,x), f(a,x')) < \varepsilon, \quad \forall\ (a,x) \in U_{a'} \times V_{x'}.$$

Now, the family $\{U_{a'}\}_{a' \in A}$ is an open covering of A, hence, since A is compact, it is possible to find a finite subfamily $\{U_{a_i} \mid i = 1, ..., n\}$ such that $\cup_{i=1}^n U_{a_i} = A$. Furthermore, in correspondence to each U_{a_i} we can consider the neighborhood V_i of x' so that, for $i = 1, ..., n$, we have:

$$d(f(a,x), f(a,x')) < \varepsilon, \quad \forall\ (a,x) \in U_{a_i} \times V_i.$$

Let $V := \cap_{i=1}^n V_i$; then we can write:

$$d(f(a,x), f(a,x')) < \varepsilon, \quad \forall\, a \in A, \quad \forall\, x \in V,$$

and the claim is proved. □

Proposition 3.1.12. *Let K be a compact Lie group, let M and N be smooth Riemannian manifold, and assume that N is complete. Let $\theta : K \times M \to N$ be a mass distribution, and let $\hat{\mathcal{C}}(\theta) : M \to N$ be its center. Then $\hat{\mathcal{C}}(\theta)$ is a continuous map.*

Proof. Let $x' \in M$. By Lemma 3.1.11, for every $\varepsilon > 0$ there exists a neighborhood V of x' such that, for all $k \in K$, and for all $x \in V$,

$$d(\theta(k,x), \theta(k,x')) < \varepsilon. \tag{11}$$

Since $\theta(\cdot, x')(K) \subset N$ is compact, we can choose $\varepsilon > 0$ to be so small that the inequality (11) implies that, for every $x \in V$, the images of the almost constant maps θ_x and $\theta_{x'}$ are contained in the same ball $B_{\frac{1}{3}\rho}(p)$ (where $B_\rho(p)$ is a strongly convex ball in N). Let δ and Δ be lower and upper curvature bounds in $B_{\frac{1}{3}\rho}(p)$, respectively: by Proposition 3.1.8 and (11) it now follows that, for every $x \in V$,

$$d(C_{\theta_x}, C_{\theta_{x'}}) \leq (1 + c(\delta, \Delta) \cdot (2\rho)^2) \cdot \int_K d(\theta(k,x), \theta(k,x'))\, dk <$$
$$< (1 + c(\delta, \Delta) \cdot (2\rho)^2) \cdot \varepsilon,$$

and this shows that the map $\hat{\mathcal{C}}(\theta)$ is continuous. □

3.2. Non-linear average

In this section K will be a compact Lie group. We saw in Chapter 2 how the very-strong-weak topology should be considered as the most appropriate topology for the set $C^{\infty,G}(M,N)$: nevertheless, by Lemma 2.3.7 we can reformulate and discuss the approximation problem in the compact case in terms of very-strong topology. The following result was proven by Illman:

Theorem 3.2.1. *Let M and N be real analytic K-manifolds, and assume that the number of K-isotropy types in N is finite. Then $C^{\omega,K}(M,N)$ is dense in $C^{\infty,K}_{vS}(M,N)$.*

Proof. See [I2], Theorem 7.2. □

We are going to generalize Theorem 3.2.1 by dropping out the assumption on the K-isotropy types. Let us first consider the case when $N = \mathbb{R}^n(\rho)$ is a representation space for K. We saw in Section 1.1 that, under the assumptions of Theorem 1.1.5, we have an "averaging" map

$$A : C^\infty(M, \mathbb{R}^n(\rho)) \to C^{\infty,K}(M, \mathbb{R}^n(\rho)), \quad f \mapsto A(f) = \int_K kf(k^{-1})dk\,,$$

which is a retraction and preserves real-analyticity. Now, the map A is continuous in the very-strong topology (see [I2], Theorem 6.4), thus if $f \in C^{\infty,K}(M, \mathbb{R}^n(\rho))$ it is possible to first approximate f with a real analytic map using the non-equivariant result by Illman (Theorem 2.1.6), and then average continuously with A. Let now N be again any real analytic K-manifold. In this case one needs to have an *equivariant*, real analytic embedding of N into some linear K-space, because by means of the map A we can only average maps which take their values in a linear space. Since the existence of such an embedding is only granted under the additional assumption that the number of K-isotropy types in N is finite (see [M-S], theorem 1.1), one gets the result stated in Theorem 3.2.1.

Thus, in order to avoid embeddings we need to be able to (continuously) associate to each map $f \in C^\infty_{vS}(M, N)$ a map $\tilde{f} \in C^{\infty,K}_{vS}(M, N)$, in such a way that $C^{\infty,K}_{vS}(M, N)$ is fixed, and real analyticity is preserved. We will show that such a "non-linear average" exists for those maps in $C^\infty_{vS}(M, N)$ which are suitably C^0-close to K-equivariant maps.

In the following let M and N be real analytic K-manifolds, and assume that on N a real analytic, complete, K-invariant Riemannian metric is fixed.

Definition 3.2.2. Let $h \in C^\infty(M, N)$. We will say that h is an *almost K-equivariant* map if the map

$$\theta_h : K \times M \to N, \quad (k, x) \mapsto kh(k^{-1}x),$$

is a mass distribution, or, equivalently, if for every $x \in M$ the map

$$\theta_{h,x} : K \to N, \quad x \mapsto kh(k^{-1}x),$$

is almost constant.

Definition 3.2.3. Let $h \in C^\infty(M, N)$ be an almost K-equivariant map. Then by Definition 3.1.10 the associated mass distribution θ_h has a well defined center $\hat{\mathcal{C}}(\theta_h)$: in this case we will call it the *center* of the almost K-equivariant map h, and we will denote it by

$$\mathcal{C}(h) : M \to N, \quad x \mapsto \mathcal{C}(h)(x) = \hat{\mathcal{C}}(\theta_h)(x) = C_{\theta_{h,x}}.$$

Proposition 3.2.4. *The center $\mathcal{C}(h) : M \to N$ of an almost K-equivariant map $h \in C^\infty(M, N)$ is a continuous, K-equivariant map. Furthermore, if h is K-equivariant, then $\mathcal{C}(h) = h$.*

Proof. If $h \in C^\infty(M,N)$ is almost K-equivariant, then $\mathcal{C}(h) : M \to N$ is continuous by Proposition 3.1.12. Furthermore, $\mathcal{C}(h)$ is K-equivariant: in fact, let $g \in K$, and consider the isometry $\bar{g} : N \to N$, $y \mapsto gy$. Then it is easy to see that

$$\theta_{h,gx} = \bar{g} \circ \theta_{h,x} \circ L_{g^{-1}},$$

where $L_{g^{-1}} : K \to K$ denotes the left translation with respect to $g^{-1} \in K$. Thus, by Properties 2 and 3 in Lemma 3.1.7, we have for every $g \in K$:

$$\mathcal{C}(h)(gx) = C_{\theta_{h,gx}} = C_{\bar{g}\circ\theta_{h,x}\circ L_{g^{-1}}} = C_{\bar{g}\circ\theta_{h,x}} = gC_{\theta_{h,x}} = g\mathcal{C}(h)(x).$$

Note that if h is K-equivariant, then h is also almost K-equivariant. In fact, in this case the map

$$\theta_{h,x} : K \to N, \quad k \mapsto h(x),$$

is constant for every $x \in M$. By Property 1 in Lemma 3.1.7 we have that $\mathcal{C}(h)(x) = C_{\theta_{h,x}} = h(x)$ for every $x \in M$, i.e. $\mathcal{C}(h) \equiv h$. □

Now we construct in Lemma 3.2.5 below a special convexity function (see Proposition 1.3.1):

Lemma 3.2.5. *Let N be a real analytic K-manifold, where K is a compact Lie group. Assume that g is a complete, K-invariant, real analytic Riemannian metric on N, and let d be the induced Riemannian distance. Then there exists a continuous, K-invariant "strong-convexity function" on N, that is, a function $r : N \to \mathbb{R}_+$ such that for every $y \in N$ the ball $B_{r(y)}(y) = \{z \in N \mid d(y,z) < r(y)\}$ is strongly convex.*

Proof. By Proposition 1.3.6 we can fix on N a K-invariant convexity function, that is, a continuous, K-invariant function $r' : N \to \mathbb{R}_+$ such that, for every $y \in N$, the ball $B_{r'(y)}(y)$ is geodesically convex. Moreover, we can assume that $r'(y) < \frac{\pi}{4}$, for all $y \in N$ (see Remark 1.3.2). For each $y \in N$ let Δ_y denote the maximum of the sectional curvatures of N in $\overline{B_{r'(y)}(y)}$, and consider the function

$$\Delta : N \to \mathbb{R}, \quad y \mapsto \Delta_y.$$

By Proposition A.3 (eventually replacing r' with a smaller function), Δ is continuous and K-invariant. We define $r : N \to \mathbb{R}_+$ in the following way:

$$r(y) = \begin{cases} r'(y) & \text{if} \quad \Delta_y \leq 1, \\ \bar{r}(y) & \text{if} \quad \Delta_y \geq 1, \end{cases} \tag{12}$$

where

$$\bar{r} : \{y \in N \mid \Delta_y > 0\} \to \mathbb{R}, \quad y \mapsto \bar{r}(y) := \frac{1}{\sqrt{\Delta_y}} \cdot r'(y).$$

Now, if $y \in N$ is such that $\Delta_y = 1$, we clearly have that $r'(y) = \bar{r}(y)$. Thus, since both r' and $\bar{r}$ are continuous and K-invariant, the map r is also continuous and K-invariant. Let $y \in N$: by Definition 3.1.1, if $-\infty < \Delta_y \leq 0$ the ball $B_{r(y)}(y)$

is strongly convex. Similarly, if $0 < \Delta_y \leq 1$, the ball $B_{r(y)}(y)$ is strongly convex because

$$r(y) = r'(y) < \frac{\pi}{4} \leq \frac{\pi}{4} \cdot \frac{1}{\sqrt{\Delta_y}}.$$

Assume now that $\Delta_y > 1$. We have that the ball $B_{r(y)}(y)$ is geodesically convex since $\bar{r}(y) < r'(y)$. Furthermore, since we chose $r' < \frac{\pi}{4}$, we have that $\bar{r}(y) < \frac{\pi}{4}\frac{1}{\sqrt{\Delta_y}}$, and hence $B_{r(y)}(y)$ is strongly convex. □

Thus, let M, N and K be as before, and fix on N a K-invariant strong-convexity function $r : N \to \mathbb{R}_+$ like in Lemma 3.2.5. For every $f \in C^{\infty,K}(M, N)$ we define

$$\mathcal{M}(f) = \{h \in C^\infty(M, N) \mid d(f(x), h(x)) < \varepsilon r(f(x)), \quad \text{for every}\, x \in M\},$$

where ε is a positive real number such that $\varepsilon < \frac{1}{3}$. Furthermore, we will use the following notation:

$$\mathcal{M}^\omega(f) = \mathcal{M}(f) \cap C^\omega(M, N),$$

and

$$\mathcal{M}^K(f) = \mathcal{M}(f) \cap C^{\infty,K}(M, N).$$

Proposition 3.2.6. *Let $f \in C^{\infty,K}_{vS}(M, N)$, and let $\mathcal{M}(f)$ be as above. Then every $h \in \mathcal{M}(f)$ is almost K-equivariant.*

Proof. Let $h \in \mathcal{M}(f)$, fix $x \in M$ and consider the map

$$\theta_{h,x} : K \to N, \quad k \mapsto kh(k^{-1}x).$$

Then for every $k \in K$ we have:

$$\begin{aligned} d(f(x), kh(k^{-1}x)) &= d(f(kk^{-1}x), kh(k^{-1}x)) = d(kf(k^{-1}x), kh(k^{-1}x)) = \\ &= d(f(k^{-1}x), h(k^{-1}x)) < \varepsilon r(f(k^{-1}x)) = \varepsilon r(k^{-1}f(x)) = \varepsilon r(f(x)), \end{aligned}$$

thus

$$\theta_{h,x}(K) \subset B_{\varepsilon r(f(x))}(f(x)) \subset B_{r(f(x))}(f(x)).$$

Since $0 < \varepsilon < \frac{1}{3}$, and $B_{r(f(x))}(f(x))$ is strongly convex, the map $\theta_{h,x}$ is almost constant. □

We saw in 3.2.3 that each almost K-equivariant map h has a well defined *center*, that is we can associate to h the continuous, K-equivariant map

$$\mathcal{C}(h) : M \to N, \quad x \mapsto \mathcal{C}(h)(x) = C_{\theta_{h,x}}.$$

Thus, by Proposition 3.2.6 we can define a map

$$\mathcal{C}_f : \mathcal{M}(f) \to C^{0,K}(M, N), \quad h \mapsto \mathcal{C}_f(h) = \mathcal{C}(h),$$

such that

$$\mathcal{C}_f \mid \mathcal{M}^K(f) = id.$$

Furthermore, we will show in Proposition 3.2.8 that for every $h \in \mathcal{M}(f)$ the map $\mathcal{C}_f(h)$ is C^∞, and moreover it is real analytic if $h \in \mathcal{M}^\omega(f)$. First we need the following lemma:

Lemma 3.2.7. *For all fixed $x_0 \in M$ there exists a neighborhood $A(x_0) \subset M$ of x_0 such that*

$$B_{\varepsilon r(f(x))}(f(x)) \subset B_{r(f(x_0))}(f(x_0)) \quad \forall\, x \in A(x_0). \tag{13}$$

Proof. Denote $B_0 := B_{r(f(x_0))}(f(x_0))$, and let $y \in B_0$. Since r is continuous, there exists a neighborhood V of $f(x_0)$ such that for every $y \in V$ we have:

$$d(y, f(x_0)) + \varepsilon r(y) < r(f(x_0)),$$

that is, $B_{\varepsilon r(y)}(y) \subset B_0$. By continuity of f, there exists a neighborhood $A(x_0)$ of x_0 in M such that $f(x) \in V$, for every $x \in A(x_0)$, and the claim is proved. □

Proposition 3.2.8. *Let $f \in C^{\infty,K}(M,N)$. If $h \in \mathcal{M}(f)$, then $\mathcal{C}(h) \in C^{\infty,K}(M,N)$. In particular, if $h \in \mathcal{M}^\omega(f)$ then $\mathcal{C}(h)$ is real analytic.*

Proof. Fix $x_0 \in M$, and let $A(x_0) \subset M$ be a neighborhood of x_0 for which the property (13) in Lemma 3.2.7 holds. Consider for every $h \in \mathcal{M}(f)$ the map

$$K \times A(x_0) \times B_{r(f(x_0))}(f(x_0)) \to \mathbb{R}_{\geq 0}$$

$$(k, x, y) \mapsto d^2(y, kh(k^{-1}x)).$$

By Proposition 1.3.3, and since the action of K on N is real analytic, the map above is smooth (real analytic) if h is smooth (real analytic). Integration over K preserves smoothness (real analyticity), hence for every $h \in \mathcal{M}(f)$ the map

$$\Phi_h \colon A(x_0) \times B_{r(f(x_0))}(f(x_0)) \to \mathbb{R}$$

$$(x, y) \mapsto \Phi_h(x, y) = \Phi_{h,x}(y) = \frac{1}{2}\int_K d^2(y, kh(k^{-1}x))dk$$

is smooth (real analytic) if h is smooth (real analytic). Now take charts (U, φ) in M and (V, ψ) in N such that $A(x_0) \subseteq U$ and $B_{r(f(x_0))}(f(x_0)) \subseteq V$ (for example we can take (V, ψ) to be the normal chart given by $\exp_{f(x_0)} : T_{f(x_0)}N \to N$), and consider the corresponding local representation of Φ_h, i.e. the smooth (or real analytic, if h is real analytic) map

$$\tilde{\Phi}_h := \Phi_h \circ (\varphi^{-1} \times \psi^{-1}) : \tilde{U} \times \tilde{V} \to \mathbb{R}$$

(here $\tilde{U} = \varphi(A(x_0)) \subset \mathbb{R}^m$ and $\tilde{V} = \psi(B_{r(f(x_0))}(f(x_0))) \subset \mathbb{R}^n$, with $m = \dim M$ and $n = \dim N$). From now on, by abuse of notation, we will identify $x \in A(x_0)$ and $y \in B_{r(f(x_0))}(f(x_0))$ with their coordinates $\varphi(x) \in \tilde{U}$ and $\psi(y) \in \tilde{V}$, respectively. By (13) we have that for every $x \in A(x_0)$ the value

$$\mathcal{C}_f(h)(x) = \Phi_h(x, \cdot)^{-1}\big(\min_{y \in B_{r(f(x_0))}(f(x_0))} \Phi_h(x, \cdot)\big)$$

is the unique solution of

$$\alpha_h(x, y) = d_2\tilde{\Phi}_h(x, y) = 0, \tag{14}$$

where $\alpha_h = d_2\tilde{\Phi}_h$ is the smooth (or real analytic, if h is real analytic) map $\tilde{U} \times \tilde{V} \to \mathbb{R}^n$ that associates to every pair $(x, y) \in \tilde{U} \times \tilde{V}$ the linear map

$$d_2\tilde{\Phi}_h(x, y) : \mathbb{R}^n \to \mathbb{R},$$

that is

$$d_2\tilde{\Phi}_h(x, y) = \left(\frac{\partial \tilde{\Phi}_{h,x}}{\partial y_1}(y), ..., \frac{\partial \tilde{\Phi}_{h,x}}{\partial y_n}(y) \right) \in M_{1,n}(\mathbb{R}).$$

Now consider the partial differential

$$d_2\alpha_h(x, y) = d\alpha_{h,x}(y) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$$

(where $\alpha_{h,x} = \alpha_h(x, \cdot) = d\tilde{\Phi}_{h,x}(\,\cdot\,) : \mathbb{R}^n \to \mathbb{R}^n$, for all $x \in \tilde{U}$). By Theorem 3.1.2 $\tilde{\Phi}_{h,x}$ is strictly convex in $B_{r(f(x))}(f(x))$, for every $x \in \tilde{U}$ (see also [Kar2], Theorem 1.2). Hence we have that

$$\frac{\partial^2 \tilde{\Phi}_h}{\partial y^2}(x, y) := d_2\alpha_h(x, y) = \left(\frac{\partial^2 \tilde{\Phi}_{h,x}}{\partial y_i \partial y_j}(y) \right)_{i,j=1,...,n} \in M_n(\mathbb{R})$$

is positive definite in $y = \mathcal{C}_f(h)(x)$. In other words, by what we have seen so far we have a smooth (or real analytic) map $\alpha_h : \tilde{U} \times \tilde{V} \to \mathbb{R}^n$ such that for every $x \in \tilde{U}$:

$$\alpha_h(x, \mathcal{C}_f(h)(x)) = 0,$$

and

$$\mathrm{rank}(d_2\alpha_h)(x, \mathcal{C}_f(h)(x)) = n.$$

Hence, by the implicit function theorem (See [Na], Theorem 1.3.5, Corollary 1.3.9 and Remark 1.3.10), we have that the unique map

$$\tilde{U} \to \tilde{V}\,, \quad x \mapsto \mathcal{C}_f(h)(x),$$

is smooth (real analytic) if h is smooth (real analytic). □

Remark 3.2.9. Let

$$d\mathcal{C}_f(h)(x) = \left(\frac{\partial \mathcal{C}_f(h)_i}{\partial x_j}(x) \right)_{\substack{i=1,...,n \\ j=1,...,m}} \in M_{n,m}(\mathbb{R}).$$

Using the local description of $\mathcal{C}_f(h)$ described in the proof of Proposition 3.2.8, the implicit function theorem gives us the following expression for the differential of $\mathcal{C}_f(h)$ at $x \in M$ (actually at $x \in \tilde{U}$), which we will use later (see [Na], Lemma 1.3.7):

$$d\mathcal{C}_f(h)(x) = -[(d_2\alpha_h)(x, \mathcal{C}_f(h)(x))]^{-1} \cdot ((d_1\alpha_h)(x, \mathcal{C}_f(h)(x))) =$$
$$= -\left(\tfrac{\partial^2 \tilde{\Phi}_h}{\partial y^2}(x, \mathcal{C}_f(h)(x)) \right)^{-1} \cdot \left(\tfrac{\partial^2 \tilde{\Phi}_h}{\partial x \partial y}(x, \mathcal{C}_f(h)(x)) \right),$$

where

$$\frac{\partial^2 \tilde{\Phi}_h}{\partial x \partial y}(x, y) := \left(\frac{\partial^2 \tilde{\Phi}_{h,x}}{\partial y_i \partial x_j}(y) \right)_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \in M_{n,m}(\mathbb{R}).$$

Thus we have shown that for every $f \in C^{\infty,K}(M, N)$ there exist a set $\mathcal{M}(f)$ and a map

$$\mathcal{C}_f : \mathcal{M}(f) \to C^{\infty,K}(M, N),$$

such that

$$\mathcal{C}_f \mid \mathcal{M}^K(f) = id, \tag{15}$$

and

$$\mathcal{C}_f \mid \mathcal{M}^\omega(f) \subset C^{\omega,K}(M, N). \tag{16}$$

Example (linear case) Let $N = \mathbb{R}^n(\rho)$ be a linear representation space for K, and consider on N the usual euclidean distance. Since $\mathcal{K}(N) = 0$, and N is simply connected, every C^∞ map $\eta : K \to N$ is almost constant. Thus, every $h \in C^\infty(M, \mathbb{R}^n(\rho))$ is almost K-equivariant. In particular, if $\eta : K \to N$ we have that

$$\Phi_\eta(y) = \frac{1}{2} \int_K \sum_{i=1}^n (y_i - \eta(k)_i)^2 \, dk.$$

Using the properties of the Haar integral (see [Kaw], Theorem 2.34) we obtain:

$$d\Phi_\eta(y) = \frac{1}{2} \left(\dots, \int_K \frac{d}{dy_j} \sum_{i=1}^n (y_i - \eta(k)_i)^2 \, dk, \dots \right) = \frac{1}{2} \left(\dots, \int_K 2(y_i - \eta(k)_i) \, dk, \dots \right) =$$

$$= y - \int_K \eta(k) \, dk,$$

hence

$$C_\eta = \int_K \eta(k) \, dk.$$

Thus, if $h \in C^\infty(M, \mathbb{R}^n(\rho))$, and θ_h is the mass distribution associated to it, we have that

$$\mathcal{C}(h) : M \to \mathbb{R}^n, \quad x \mapsto \int_K \theta_h(k)(x) \, dk = \int_K k h(k^{-1}x) \, dk,$$

that is, in the linear case the map $\mathcal{C}(h)$ coincides with the map $A(h)$ defined in Section 1.1, for all $h \in C^\infty(M, \mathbb{R}^n(\rho))$.

3.3. Approximation result in the compact case

In Section 3.2 we saw how classic results on approximation of smooth maps between two manifolds M and N are obtained by using embeddings. We also mentioned the fact that in the equivariant case the use of embeddings requires an additional assumption on the number of isotropy types in N. Instead, the use of the "non-linear" average constructed using the center of mass gives us the possibility to approximate equivariant maps directly, without turning to embeddings. In fact, we have Theorem 3.3.1 below:

Theorem 3.3.1. *Let K be a compact Lie group, and let M and N be two real analytic K-manifolds. There exists in $C^\infty_{vS}(M,N)$ an open neighborhood $\mathcal{M} \supset C^{\infty,K}_{vS}(M,N)$ of almost K-equivariant maps and a continuous map*

$$\mathcal{C} : \mathcal{M} \to C^{\infty,K}_{vS}(M,N),$$

which is a retraction and preserves real analyticity, i.e.

$$\mathcal{C} \mid_{C^{\infty,K}_{vS}(M,N)} = id_{C^{\infty,K}_{vS}(M,N)}, \tag{17}$$

and

$$\mathcal{C}(\mathcal{M}^\omega) = C^{\omega,K}_{vS}(M,N). \tag{18}$$

(where $\mathcal{M}^\omega = \mathcal{M} \cap C^\omega_{vS}(M,N)$).

Corollary 3.3.2. *Let K be a compact Lie group, and let M and N be two real analytic K-manifolds. Then $C^{\omega,K}_{vS}(M,N)$ is dense in $C^{\infty,K}_{vS}(M,N)$.*

Proof of Corollary 3.3.2. Assume that the map $\mathcal{C}$ in Theorem 3.3.1 exists, and let $\mathcal{U}$ be a non-empty open subset of $C^{\infty,K}_{vS}(M,N)$. We want to show that $\mathcal{U} \cap C^{\omega,K}_{vS}(M,N) \neq \emptyset$. Since $\mathcal{C}$ is continuous and, by (17), surjective, it follows that $\mathcal{C}^{-1}(\mathcal{U})$ is non-empty and open in $\mathcal{M}$, and hence open in $C^\infty_{vS}(M,N)$. Then, by Theorem 2.1.6, we have that $\mathcal{C}^{-1}(\mathcal{U}) \cap C^\omega_{vS}(M,N) \neq \emptyset$. Hence

$$\begin{aligned}\emptyset \neq \mathcal{C}(\mathcal{C}^{-1}(\mathcal{U}) \cap C^\omega_{vS}(M,N)) = \mathcal{C}(\mathcal{C}^{-1}(\mathcal{U}) \cap \mathcal{M} \cap C^\omega_{vS}(M,N)) = \\ = \mathcal{C}\mathcal{C}^{-1}(\mathcal{U}) \cap \mathcal{C}(\mathcal{M} \cap C^\omega_{vS}(M,N)) = \mathcal{U} \cap C^{\omega,K}_{vS}(M,N). \quad \square\end{aligned}$$

The rest of this section will be devoted to the proof of Theorem 3.3.1. In the following, K will denote a compact Lie group, and M and N will be two real analytic K-manifolds. Furthermore, we will assume that on N a real analytic, complete, K-invariant metric is fixed (see Theorem 1.4.5), and that $r : N \to \mathbb{R}_+$ is a K-invariant strong-convexity function on N. For every $f \in C^{\infty,K}_{vS}(M,N)$, let then

$$\mathcal{M}(f) = \{h \in C^\infty(M,N) \mid d(f(x),h(x)) < \varepsilon r(f(x)), \quad \text{for every } x \in M\},$$

where $0 < \varepsilon < 1/3$, and let

$$\mathcal{C}_f : \mathcal{M}(f) \to C^{\infty,K}_{vS}(M,N)$$

be the map defined in Section 3.2. By Theorem 2.2.5, the set $\mathcal{M}(f)$ is open in the strong C^0 topology, and hence also in the very-strong topology. Thus the set

$$\mathcal{M} := \bigcup_{f \in C^{\infty,K}_{vS}(M,N)} \mathcal{M}(f) \subset C^{\infty}_{vS}(M,N)$$

is an open neighborhood of $C^{\infty,K}_{vS}(M,N)$. Furthermore, the map

$$\mathcal{C} : \mathcal{M} \to C^{\infty,K}_{vS}(M,N), \quad \mathcal{C}\mid_{\mathcal{M}(f)} = \mathcal{C}_f,$$

is well-defined, and it satisfies properties (17) and (18) (by (15) and (16), respectively). Thus, to complete the proof of Theorem 3.3.1, it will be enough to prove that $\mathcal{C}_f$ is continuous in the very-strong topology, for all $f \in C^{\infty,K}_{vS}(M,N)$.

Lemma 3.3.3. *The map $\mathcal{C}_f : \mathcal{M}(f) \to C^{\infty,K}_{vS}(M,N)$ is continuous in the strong C^0 topology.*

Proof. By Theorem 2.2.5 a basis for the strong C^0 topology on $\mathcal{M}(f)$ is given by all the sets of the form

$$\mathcal{M}(g,\delta) := \{h \in \mathcal{M}(f) \mid d(g(x),h(x)) < \delta(x),\ \forall\ x \in M\},$$

where $g \in \mathcal{M}(f)$ and $\delta \in C^0(M,\mathbb{R}_+)$ are arbitrary. So, let $h \in \mathcal{M}(f)$. For every $\epsilon \in C^0(M,\mathbb{R}_+)$ we would like to find a (without loss of generality) K-invariant function $\delta : M \to \mathbb{R}_+$ such that:

$$h_1 \in \mathcal{M}(h,\delta) \quad \Rightarrow \quad \mathcal{C}_f(h_1) \in \mathcal{M}(\mathcal{C}_f(h),\epsilon).$$

Suppose then $h_1 \in \mathcal{M}(h,\delta)$, with $\delta \in C^{0,K}(M,\mathbb{R}_+)$, and consider for every $x \in M$ the almost constant maps $\theta_{h,x}, \theta_{h_1,x} : K \to N$. Then for every $k \in K$ we have that:

$$d(\theta_{h,x}(k),\theta_{h_1,x}(k)) = d(kh(k^{-1}x),kh_1(k^{-1}x)) = d(h(k^{-1}x),h_1(k^{-1}x)) < \delta(x).$$

Now, for each $x \in M$ the images $\theta_{h,x}(K)$ and $\theta_{h_1,x}(K)$ are contained in the same strongly convex ball $B_{\varepsilon r(f(x))}(f(x)) \subset B_{r(f(x))}(f(x))$. Let then Δ_x and δ_x be respectively the maximum and the minimum of the sectional curvatures of N in the closure of $B_{\varepsilon r(f(x))}(f(x))$, for every $x \in M$. By Proposition A.3 and Proposition 3.1.8, we get for every $x \in M$:

$$\begin{aligned} d(\mathcal{C}_f(h)(x),\mathcal{C}_f(h_1)(x)) &\leq (1+a(x)(2r(f(x)))^2) \cdot \int_K d(\theta_{h,x}(k),\theta_{h_1,x}(k))\,dk < \\ &< (1+a(x)(2r(f(x)))^2) \cdot \int_K \delta(x)\,dk = (1+a(x)(2r(f(x)))^2) \cdot \delta(x) =: \gamma(x), \end{aligned}$$

where $a(x) \in \mathbb{R}$ is a positive constant that depends continuously on x. Then clearly $\gamma : M \to \mathbb{R}_+$ is continuous, and δ can be chosen so that $\gamma(x) \leq \epsilon(x)$, for all $x \in M$. Thus $\mathcal{C}_f$ is continuous in the strong C^0 topology. □

For every $h \in C^{\infty}_{vS}(M,N)$ we defined:

$$\theta_h : K \times M \to N, \quad (k,x) \mapsto kh(k^{-1}x).$$

We have the following Lemma:

Lemma 3.3.4. *The map*

$$\Theta : C^\infty_{vS}(M,N) \to C^\infty_{vS}(K \times M, N), \quad h \mapsto \theta_h,$$

is continuous.

Proof. See Lemma 6.3 in [I2] for the case $N = \mathbb{R}^n$. If N is any smooth K-manifold the proof goes exactly in the same way. □

Consider the map

$$\chi : C^\infty_{vS}(M,N) \to C^\infty_{vS}(M \times K, N \times K), \quad g \mapsto g \times id.$$

Then χ is continuous (see Corollary 3.2 in [I2]); the proof uses the fact that by the compactness of K, the natural projection $p_1 : M \times K \to M$ is proper. Now, in our situation we would like the map

$$\chi : C^\infty_{vS}(K \times M, N) \to C^\infty_{vS}(K \times M \times N, N \times N), \quad g \mapsto g \times id,$$

to be continuous; unfortunately, we cannot apply Corollary 3.2 in [I2] as long as N is not compact. Nevertheless, due to the fact that we are only interested in a special class of maps, it will be enough to prove Proposition 3.3.5 below.

First we notice that the *graph* of our $f \in C^{\infty,K}_{vS}(M,N)$, i.e., the set

$$\mathcal{G}_f = \{(x, f(x)) \mid x \in M\} \subset M \times N,$$

is a closed, K-invariant submanifold of $M \times N$. In fact, $\mathcal{G}_f$ is the image of the smooth embedding

$$\phi : M \to M \times N, \quad m \mapsto \phi(m) = (m, f(m)). \tag{19}$$

Thus, by Theorem 1.3.7 there exists a tubular neighborhood $(p', E, \mathcal{G}_f)$ of $\mathcal{G}_f$ in $M \times N$, where $E \subset M \times N$ and $p' : E \to \mathcal{G}_f$ is a retraction. Fix on N two K-invariant convexity functions R and $\overline{R}$ such that

$$\overline{R}(y) > R(y) > r(y), \quad \text{for all} \quad y \in N$$

(in fact, we can always replace r with a smaller function). Consider then the restriction T of E to the disc bundle whose radius is the continuous function

$$r' : \mathcal{G}_f \to \mathbb{R}_+, \quad r'(x, f(x)) = R(f(x)). \tag{20}$$

By the properties of R, this means

$$\begin{aligned} T = \{v \in E \mid v \in E_{(x,f(x))},\ \|v\| \leq r'(x, f(x)),\ (x, f(x)) \in \mathcal{G}_f\} = \\ = \{(x,y) \in E \mid d(y, f(x)) \leq R(f(x)),\ x \in M\}. \end{aligned}$$

Denote

$$p := p' \mid T \to \mathcal{G}_f, \quad (x,y) \mapsto (x, f(x)). \tag{21}$$

Note that the projection p is proper. Now T is a manifold with boundary; hence we are allowed to consider the space

$$C^\infty_{vS}(K \times T, N \times N)$$

(see Remark 2.1.5), and we can define the map

$$\chi' : \Theta(\mathcal{M}(f)) \to C^\infty_{vS}(K \times T, N \times N),$$
$$\theta_h \mapsto \theta_h \times id.$$

Thus we can prove:

Proposition 3.3.5. *Let* $\Theta : \mathcal{M}(f) \to C^\infty_{vS}(K \times M, N)$ *be as in Lemma* 3.3.4. *Then the map*

$$\chi' : \Theta(\mathcal{M}(f)) \to C^\infty_{vS}(K \times T, N \times N),$$
$$\theta_h \mapsto \theta_h \times id,$$

is continuous.

Proof. By Proposition 3.1 in [I2], there exists a homeomorphism

$$\iota : C^\infty_{vS}(K \times T, N \times N) \to C^\infty_{vS}(K \times T, N) \times C^\infty(K \times T, N)$$
$$g \mapsto (q_1 \circ g, q_2 \circ g),$$

where q_1 and q_2 are the natural projections of $N \times N$ onto the first and the second factor, respectively. Hence it will be enough to show that the maps

(1) $\Theta(\mathcal{M}(f)) \to C^\infty_{vS}(K \times T, N), \quad \theta_h \mapsto q_1 \circ (\theta_h \times id)$
(2) $\Theta(\mathcal{M}(f)) \to C^\infty_{vS}(K \times T, N), \quad \theta_h \mapsto q_2 \circ (\theta_h \times id)$

are continuous. Now, let

$$p_1 : K \times T \to K \times M, \quad (k, x, y) \mapsto (k, x)$$

and

$$p_2 : K \times T \to N, \quad (k, x, y) \mapsto y$$

be projection maps. Then the map in (2) is the constant map from $\Theta(\mathcal{M}(f))$ to the element $p_2 \in C^\infty_{vS}(K \times T, N)$, and thus it is continuous. Furthermore, it is easy to see that for every $\theta_h \in \Theta(\mathcal{M}(f))$ we have that $q_1 \circ (\theta_h \times id) = \theta_h \circ p_1$. Hence the map in (1) equals the map

$$p_1^* : \Theta(\mathcal{M}(f)) \to C^\infty_{vS}(K \times T, N)$$
$$\theta_h \mapsto \theta_h \circ p_1.$$

By Proposition 2.5 in [I2], it will be enough to show that p_1 is a proper map. We have:

$$p_1 = id_K \times (\phi^{-1} \circ p) : K \times T \to M,$$

where ϕ is the embedding of M into $M \times N$ defined in 19 and $p : T \to \mathcal{G}_f$ is the bundle projection defined in (21). Since ϕ is a homeomorphism onto its image and p is proper, the claim is proved. □

Note that for every $h \in \mathcal{M}(f)$ and for every $(k, x, y) \in K \times T$, we have that

$$(\theta_h \times id)(k, x, y) = (kh(k^{-1}x), y) \in B_{\varepsilon r(f(x))}(f(x)) \times \overline{B_{R(x)}(f(x))}.$$

Thus, if $\overline{R}$ is the function defined in (20) we have:

$$\chi' : \Theta(\mathcal{M}(f)) \to C^\infty_{vS}(K \times T, \cup_{x\in M}(B_{\overline{R}(x)}(f(x)) \times B_{\overline{R}(x)}(f(x))).$$

For every $h \in \mathcal{M}(f)$ consider the map:

$$\Phi_h : T \to \mathbb{R}, \quad (x, y) \mapsto \frac{1}{2}\int_K d^2(kh(k^{-1}x), y)\, dk.$$

Proposition 3.3.6. *The map*

$$\Phi : \mathcal{M}(f) \to C^\infty_{vS}(T, \mathbb{R}),$$
$$h \mapsto \Phi_h,$$

is continuous.

Proof. Let Θ and χ' be definied as in Lemma 3.3.4 and Proposition 3.3.5, respectively. Let also

$$\hat{A} : C^\infty_{vS}(K \times T, \mathbb{R}) \to C^\infty_{vS}(T, \mathbb{R}),$$

where

$$\hat{A}(f) : T \to \mathbb{R}, \quad x \mapsto \hat{A}(f)(x) = \int_K f(k, x)\, dk.$$

We have that

$$\Phi = \hat{A} \circ d^2_* \circ \chi' \circ \Theta,$$

where

$$\mathcal{M}(f) \xrightarrow{\Theta} \Theta(\mathcal{M}(f)) \xrightarrow{\chi'} C^\infty_{vS}(K \times T, \cup_{x\in M}(B_{\overline{R}(x)}(f(x)) \times B_{\overline{R}(x)}(f(x)))) \longrightarrow$$
$$\xrightarrow{d^2_*} C^\infty_{vS}(K \times T, \mathbb{R}) \xrightarrow{\hat{A}} C^\infty_{vS}(T, \mathbb{R}).$$

Note that the map d^2 is C^∞ (actually real analytic) on each product of the type

$$B_{\overline{R}(x)}(f(x)) \times B_{\overline{R}(x)}(f(x))$$

(see 1.3.3), hence

$$d^2 \in C^\infty_{vS}(\cup_{x\in M}(B_{\overline{R}(x)}(f(x)) \times B_{\overline{R}(x)}(f(x))), \mathbb{R}),$$

and d^2_* is well defined. Now Θ and χ' are continuous by Lemma 3.3.4 and Proposition 3.3.5, respectively. Furthemore, $(d^2)_*$ is continuous by Proposition 2.6 in [I2], and $\hat{A}$ is continuous by Proposition 6.2 in [I2], hence the claim follows. □

Remark 3.3.7. For every $x \in M$ let $A(x)$ be a neighborhood of x like in Lemma 3.2.7. Since the neighborhood $A(x)$ can be chosen as small as we like, we can also assume that $A(x)$ is contained in some chart (U, φ) of M, and that the following property is satisfied:

$$(*) \quad \tilde{x} \in A(x) \Rightarrow B_{\varepsilon r(f(\tilde{x}))}(f(\tilde{x})) \subset B_{r(f(x))}(f(x)) \subset B_{R(\tilde{x})}(f(\tilde{x})),$$

where $R : N \to \mathbb{R}_+$ is the convexity function that we fixed as radius of our closed tubular neighborhood $T \supset \mathcal{G}_f$ (the proof is similar to the proof of Lemma 3.2.7). In the following it will be useful to consider locally finite subfamilies of

$\{A(x)\}_{x\in M}$, say $\{A(x_i)\}_{i\in\Lambda}$: we will refer to them as families "with the $*$-property". To each family $\{A(x_i)\}_{i\in\Lambda}$ of M with the $*$-property we can associate the family $\{B_{r(f(x_i))}(f(x_i))\}_{i\in\Lambda}$; note that if $\{A(x_i)\}_{i\in\Lambda}$ is a cover of M then

$$f(M) \subseteq \cup_{x\in M} B_{\varepsilon r(f(x))}(f(x)) \subseteq \cup_{i\in\Lambda} B_{r(f(x_i))}(f(x_i)) \subseteq N.$$

Note also that the second part of the inequality $(*)$ implies that, for every $i \in \Lambda$, we have

$$A(x_i) \times B_{r(f(x_i))}(f(x_i)) \subset T.$$

Proposition 3.3.8. *Let $f \in C^{\infty,K}_{vS}(M,N)$ and let $h \in \mathcal{M}(f)$. Let*

$$\mathcal{N} = \mathcal{N}^r(h; D, (U,\varphi), (V,\psi), \delta)$$

be an elementary C^r-neighborhood of h, $1 \le r < \infty$. Then there exist finitely many elementary C^r-neighborhoods $\mathcal{N}_j = \mathcal{N}^r(h; D_j, (A(x_j),\varphi), (B_{r(f(x_j))}(f(x_j)),\psi_j), \delta_j)$ of h, $1 \le j \le t$, such that the following conditions are satisfied:

(i) *The family $\{A(x_j)\}_{j=1,\dots,t}$ has the $*$-property.*
(ii) $\bigcap_{j=1}^t \mathcal{N}_j \subset \mathcal{N}$.

Proof. For every $x \in D$ let $A(x) \subset U$ be an open neighborhood of x with the property (13). Then the family $\{A(x)\}_{x\in D}$ is an open cover of D, and since D is compact we can find a finite subcover $\{A(x_j)\}_{j=1,\dots,t}$ of D and compact subsets $D_j \subset A(x_j)$, $1 \le j \le t$, such that $\cup_{j=1}^t D_j = D$. Then for every $x \in D_j \subset A(x_j)$ we have that $h(x) \in B_{\varepsilon r(f(x))}(f(x)) \subset B_{r(f(x_j))}(f(x_j))$, hence $h(D_j) \subset B_{r(f(x_j))}(f(x_j))$, $1 \le j \le t$. Thus we can denote $V_j := B_{r(f(x_j))}(f(x_j))$ and apply Corollary 2.2.2, and the claim is proved. □

Corollary 3.3.9. *Let $f \in C^{\infty,K}_{vS}(M,N)$, and let $h \in \mathcal{M}(f)$. Let*

$$\mathcal{S} = \bigcap_{i\in\Lambda} \mathcal{N}^{r_i}(h; F_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i)$$

be any basic very-strong neighborhood of h. Then there exists a basic very-strong neighborhood $\mathcal{U}$ of h such that $\mathcal{U} \subset \mathcal{S}$, and

$$\mathcal{U} = \bigcap_{s\in\Omega} \mathcal{N}^{r_s}(h; D_s, (A(x_s),\tilde\varphi_s), (B_{r(f(x_s))}(f(x_s)),\tilde\psi_s), \delta_s),$$

where the family $\{A(x_s)\}_{s\in\Omega}$ has the $*$-property. We will call a neighborhood like $\mathcal{U}$ a $*$-basic very-strong neighborhood of h. □

Theorem 3.3.10. *The map*

$$\mathcal{C}_f : \mathcal{M}(f) \to C^{\infty,K}_{vS}(M,N), \quad h \mapsto \mathcal{C}_f(h)$$

is continuous in the very-strong topology.

Proof. Let $h \in \mathcal{M}(f)$, and let $\mathcal{S}' = \bigcap_{i\in\Lambda} \mathcal{N}^{r'_i}(\mathcal{C}_f(h); D'_i, (U'_i, \varphi'_i), (V'_i, \psi'_i), \varepsilon'_i)$ be a basic very-strong neighborhood of $\mathcal{C}_f(h)$. We want to find a basic very-strong neighborhood $\mathcal{U}'$ of h such that

$$\tilde{h} \in \mathcal{U}' \quad \Rightarrow \quad \mathcal{C}_f(\tilde{h}) \in \mathcal{S}'.$$

Since $\mathcal{C}_f(h) \in \mathcal{M}(f)$, there exists by Corollary 3.3.9 a $*$-basic very-strong neighborhood

$$\mathcal{S} = \bigcap_{s\in\Omega} \mathcal{N}^{r_s}(\mathcal{C}_f(h); D_s, (A(x_s), \varphi_s), (B_{r(f(x_s))}(f(x_s)), \psi_s), \varepsilon_s)$$

of $\mathcal{C}_f(h)$ such that $\mathcal{S} \subset \mathcal{S}'$. We will use the notation $B_s := B_{r(f(x_s))}(f(x_s))$, for every $s \in \Omega$. Now, fix $s \in \Omega$, and for every $z \in \varphi_s(D_s)$, consider the first derivatives of the local representation of $\mathcal{C}_f(h)$ with respect to the charts $A(x_s)$ and B_s in the point z, i.e.:

$$\frac{d\mathcal{C}^s_f(h)_l}{dx_q}(z)\,, \quad l = 1, ..., n\,; \;\; q = 1, ..., m. \tag{22}$$

By the formula given in (3.2.9), each of the terms in 22 depends continuously on the second-order derivatives of the local representation Φ^s_h of the function Φ_h with respect to the chart $A(x_s) \times B_s$, calculated in the point $(z, \mathcal{C}^s_f(h)(z))$. Hence there exists a positive real number $\mu_{s,1}$ such that if $\Phi_{\tilde{h}} \in \Phi(\mathcal{M}(f))$ and if

$$\left| \frac{\partial^2 \Phi^s_h}{\partial y_l \partial y_p}(z, \mathcal{C}^s_f(h)(z)) - \frac{\partial^2 \Phi^s_{\tilde{h}}}{\partial y_l \partial y_p}(z, \mathcal{C}^s_f(\tilde{h})(z)) \right| < \mu_{s,1}, \quad l, p = 1, ..., n,$$

and

$$\left| \frac{\partial^2 \Phi^s_h}{\partial y_l \partial x_q}(z, \mathcal{C}^s_f(h)(z)) - \frac{\partial^2 \Phi^s_{\tilde{h}}}{\partial y_l \partial x_q}(z, \mathcal{C}^s_f(\tilde{h})(z)) \right| < \mu_{s,1}, \quad l = 1, ..., n;\ q = 1, ..., m,$$

for every $z \in \varphi_s(D_s)$, then the following inequalities are satisfied:

$$\left| \frac{d\mathcal{C}^s_f(h)_l}{dx_q}(z) - \frac{d\mathcal{C}^s_f(\tilde{h})_l}{dx_q}(z) \right| < \varepsilon_s\,, \quad l = 1, ..., n\,; \;\; q = 1, ..., m;\ z \in \varphi_s(D_s).$$

Similarly, from the same formula we can see that the values of the derivatives of order $t > 1$ of $\mathcal{C}^s_f(h)$, calculated in $z \in \varphi_s(D_s)$, depend continuously on the values of the derivatives up to the order $t+1$ of Φ^s_h calculated in $(z, \mathcal{C}^s_f(h)(z))$. Like before, it is possible to find positive real numbers $\mu_{s,\rho}$, $\rho = 1, ..., r_s$, such that if $\Phi_{\tilde{h}} \in \Phi(\mathcal{M}(f))$, and if

$$|D^\alpha \Phi^s_h(z, \mathcal{C}^s_f(h)(z)) - D^\alpha \Phi^s_{\tilde{h}}(z, \mathcal{C}^s_f(\tilde{h})(z))| < \mu_{s,\rho}, \quad \forall\, \alpha \;|\; 2 \le |\alpha| \le \rho + 1, \tag{23}$$

for every $z \in \varphi_s(D_s)$, then

$$|D^\beta \mathcal{C}^s_f(h)(z) - D^\beta \mathcal{C}^s_f(\tilde{h})(z)| < \varepsilon_s, \quad \forall\, \beta \;|\; |\beta| = \rho, \forall\, z \in \varphi_s(D_s). \tag{24}$$

Denote $\mu_s = \min\{\mu_{s,\rho} \mid 1 \le \rho \le r_s\}$, for every $s \in \Omega$. Now, consider the basic very-strong neighborhood of Φ_h:

$$\mathcal{N} = \bigcap_{s\in\Omega} \mathcal{N}^{r_s+1}(\Phi_h; D_s \times \overline{B_{\varepsilon r(f(x_s))}(f(x_s))}, (A(x_s) \times B_s), \varphi_s \times \psi_s), \mu_s) \subset$$

$$\subset C^\infty_{vS}(T, \mathbb{R}).$$

By Proposition 3.3.6, there exists a basic very-strong neighborhood $\mathcal{U}$ of h in $\mathcal{M}(f) \subset C^\infty_{vS}(M, N)$ such that

$$\tilde{h} \in \mathcal{U} \quad \Rightarrow \quad \Phi_{\tilde{h}} \in \mathcal{N}.$$

In particular, we have for every $\tilde{h} \in \mathcal{U}$, $z \in \varphi_s(D_s)$ and $s \in \Omega$:

$$|D^\alpha \Phi^s_h(z, \mathcal{C}^s_f(h)(z)) - D^\alpha \Phi^s_{\tilde{h}}(z, \mathcal{C}^s_f(h)(z))| < \mu_s, \quad 0 \le |\alpha| \le r_s.$$

Now, since each term $D^\alpha \Phi^s_{\tilde{h}} \mid_{A(x_s)\times B_s}$ is continuous, it is always possible to find a positive real number ξ_s such that if $\tilde{h} \in \mathcal{U}$, $z \in \varphi_s(D_s)$, and if

$$|\mathcal{C}^s_f(h)(z) - \mathcal{C}^s_f(\tilde{h})(z)| < \xi_s\,,$$

then the inequalities (23) (and hence also (24)) are satisfied. Denote by h^s and $\tilde{h}^s$ the local representations with respect to the charts $A(x_s)$ and B_s of h and $\tilde{h}$, respectively. By Lemma 3.3.3, $\mathcal{C}_f$ is continuous in the strong C^0 topology; hence, for each $s \in \Omega$ there exists a $\delta'_s > 0$ such that

$$|h^s(z) - \tilde{h}^s(z)| < \delta'_s \Rightarrow |\mathcal{C}^s_f(h)(z) - \mathcal{C}^s_f(\tilde{h})(z)| < \xi_s, \quad \forall\, z \in \varphi_s(D_s).$$

Similarly, for every $s \in \Omega$ we can find a positive δ''_s such that

$$|h^s(z) - \tilde{h}^s(z)| < \delta''_s \Rightarrow |\mathcal{C}^s_f(h)(z) - \mathcal{C}^s_f(\tilde{h})(z)| < \varepsilon_s, \quad \forall\, z \in \varphi_s(D_s).$$

If $\delta_s := \min\{\delta'_s, \delta''_s\}$ for all $s \in \Omega$ we define

$$\mathcal{U}_0 = \bigcap_{s\in\Omega} \mathcal{N}^1(h; D_s, (A(x_s), \varphi_s), (B_{r(f(x_s))}(f(x_s)), \psi_s), \delta_s).$$

Then it is clear that if $\mathcal{U}' := \mathcal{U} \cap \mathcal{U}_0$, then

$$\tilde{h} \in \mathcal{U}' \Rightarrow \Phi_{\tilde{h}} \in \mathcal{S} \subset \mathcal{S}',$$

and this proves the claim. □

Chapter 4. Approximation of smooth, G-equivariant maps

In this Chapter we show that the space $C^{\infty,G}_{vSW}(G \times_H M, N)$ is homeomorphic to $C^{\infty,H}_{vSW}(M, N)$. Using this result and the approximation result for the compact case established in Chapter 3, we then prove our main result: if G is a good Lie group which acts properly on the manifolds M and N, then every smooth, G-equivariant map from M to N can be approximated in the very-strong-weak topology with a real analytic, G-equivariant map.

4.1. Induced G-maps in the very-strong-weak topology

The results proved in this section are the very-strong-weak versions of the results proved in Section 5 of [I-Ka1] for the strong-weak topology.

Let H be a closed subgroup of the Lie group G, and let M and N be two smooth H-manifolds. Let $G \times H$ act on G by

$$(G \times H) \times G \to G, \quad ((\bar{g}, h), g) \mapsto \bar{g}gh^{-1}, \tag{25}$$

and on M and N by

$$(G \times H) \times M \to M, \quad ((\bar{g}, h), x) \mapsto hx, \tag{26}$$

and

$$(G \times H) \times N \to N, \quad ((\bar{g}, h), y) \mapsto hy, \tag{27}$$

respectively. Then we can consider the corresponding diagonal actions of $G \times H$ on $G \times M$ and $G \times N$, that is

$$(G \times H) \times (G \times M) \to G \times M, \quad ((\bar{g}, h), (g, x)) \mapsto (\bar{g}gh^{-1}, hx),$$

and

$$(G \times H) \times (G \times N) \to G \times N, \quad ((\bar{g}, h), (g, y)) \mapsto (\bar{g}gh^{-1}, hy),$$

respectively, and the set $C^{\infty, G\times H}_{vSW}(G \times M, G \times N)$. It is then easy to see that the map

$$\chi : C^{\infty,H}_{vSW}(M, N) \to C^{\infty, G\times H}_{vSW}(G \times M, G \times N), \quad f \mapsto id \times f,$$

is well defined. We have the following:

Proposition 4.1.1. *Under the assumptions above, the map*

$$\chi : C^{\infty,H}_{vSW}(M, N) \to C^{\infty, G\times H}_{vSW}(G \times M, G \times N), \quad f \mapsto id \times f,$$

is continuous.

Proof. Let $q_1 : G \times N \to G$ and $q_2 : G \times N \to N$ be the projections. By Theorem 2.4.3 it is enough for us to prove the continuity of the maps

$$\chi_1 : C^{\infty,H}_{vSW}(M, N) \to C^{\infty, G\times H}_{vSW}(G \times M, G), \quad f \mapsto q_1 \circ (id \times f),$$

and

$$\chi_2 : C^{\infty,H}_{vSW}(M, N) \to C^{\infty, G\times H}_{vSW}(G \times M, N), \quad f \mapsto q_2 \circ (id \times f),$$

where the group $G \times H$ acts on G and N by the actions (25) and (27), respectively. Now, the map χ_1 associates to each $f \in C^{\infty,H}_{vSW}(M, N)$ the following map:

$$q_1 \circ (id \times f) : G \times M \to G, \quad (g, x) \mapsto q_1(g, f(x)) = g,$$

that is,

$$\chi_1(f) = r_1, \quad \text{for every } f \in C^{\infty,H}_{vSW}(M, N),$$

where $r_1 : G \times M \to G$ is the projection onto the first factor. Thus χ_1 is a constant map, and hence continuous. Denote by $r_2 : G \times M \to M$ the projection onto the second factor, and note that for each $f \in C^{\infty,H}_{vSW}(M, N)$ we have

$$q_2 \circ (id \times f) = f \circ r_2.$$

Thus the map χ_2 equals the map

$$r_2^* : C^{\infty,H}_{vSW}(M, N) \to C^{\infty,G\times H}_{vSW}(G \times M, N), \quad f \mapsto f \circ r_2.$$

Now, let $\pi_2 : G \times H \to H$ be the projection, and let $z \in C^{\infty,G\times H}_{vSW}(G \times M, N)$. Then for every $(\bar{g}, h) \in G \times H$ and $(g, x) \in M$ we have:

$$z((\bar{g}, h)(g, x)) = hz(g, x) = \pi_2(\bar{g}, h)z(g, x),$$

hence

$$C^{\infty,G\times H}_{vSW}(G \times M, N) = C^{\infty,\pi_2}_{vSW}(G \times M, N).$$

Clearly, the map $r_2 : G \times M \to M$ is π_2-equivariant. Furthermore, the induced map

$$\overline{r}_2 : (G \times M)/(G \times H) \to M/H$$

is a homeomorphism, and hence of finite type. Thus, we can apply Lemma 2.4.2, and the claim is proved. □

Let again H be a closed subgroup of a Lie group G, and let M be a smooth H-manifold. We saw that the twisted product $G \times_H M$ is defined as the orbit space of the action

$$H \times (G \times M) \to (G \times M), \quad (h, (g, x)) \mapsto (gh^{-1}, hx). \tag{28}$$

We denote by

$$p : G \times M \to G \times_H M, \quad (g, x) \mapsto [g, x],$$

the usual projection. Recall also that there exists a canonical bijection

$$\mu : C^{\infty,H}_{vSW}(M, N) \to C^{\infty,G}_{vSW}(G \times_H M, N), \quad f \mapsto \mu(f),$$

where

$$\mu(f) : G \times_H M \to N, \quad [g, x] \mapsto gf(x).$$

Note that μ preserves real analyticity. Our aim in this Section is to prove that μ is, in fact, a homeomorphism. Let then $\pi_1 : G \times H \to G$ be the projection; consider the map

$$p^* : C^{\infty,G}_{vSW}(G \times_H M, N) \to C^{\infty,\pi_1}_{vSW}(G \times M, N), \quad z \mapsto z \circ p,$$

and let

$$\chi : C^{\infty,H}_{vSW}(M, N) \to C^{\infty,G\times H}_{vSW}(G \times M, G \times N), \quad f \mapsto id \times f,$$

be like in Proposition 4.1.1. Furthermore, let $\Phi : G \times N \to N$, $(g, y) \mapsto gy$, denote the action of G on N: then Φ is π_1-equivariant, hence we can consider the map

$$\Phi_* : C^{\infty,G\times H}_{vSW}(G \times M, G \times N) \to C^{\infty,\pi_1}_{vSW}(G \times M, N), \quad q \mapsto \Phi \circ q.$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc} C^{\infty,H}_{vSW}(M,N) & \xrightarrow{\chi} & C^{\infty,G\times H}_{vSW}(G\times M, G\times N) & \xrightarrow{\Phi_*} & C^{\infty,\pi_1}_{vSW}(G\times M,N) \\ \downarrow{\scriptstyle \mu} & & & \nearrow{\scriptstyle p^*} & \\ C^{\infty,G}_{vSW}(G\times_H M,N) & & & & \end{array}$$

We will need the following two results:

Lemma 4.1.2. *Let H be a Lie group, let M be a proper, free, C^∞ H-manifold and let $p : M \to M/H$ be the projection. Assume that $\rho' : M/H \to \Omega$ is a continuous map, where Ω is a topological space, and take ρ' and $\rho = \rho' \circ p : M \to \Omega$ as phase maps for M/H and M, respectively (see Remark* 2.3.4*). Then, if P is a smooth manifold, the map*

$$p^* : C^\infty_{vSW[\rho']}(M/H, P) \to C^{\infty,H}_{vSW[\rho]}(M,P), \quad k \mapsto k \circ p,$$

is a homeomorphism.

Proof. Note that, since the action of H on M is proper and free, M/H is a smooth manifold. It is clear that p^* is a bijective map. Furthermore, the continuity of p^* follows from the continuity of $\hat{f}^*$ in the proof of Proposition 2.4.2, when we take generic phase maps instead of the projections. Note that, in this case, the induced map $\bar{p} : \Omega \to \Omega$ is the identity map. Denote by

$$p^{**} : C^{\infty,H}_{vSW[\rho]}(M,P) \to C^\infty_{vSW[\rho']}(M/H,P), \quad z \mapsto \bar{z},$$

the inverse of p^*: it remains to show that p^{**} is continuous. Let then $f \in C^{\infty,H}_{vSW[\rho]}(M,P)$, and let $\mathcal{U} = \bigcap_{i\in\Lambda} \mathcal{N}_i$ be a basic neighborhood of $p^{**}(f) = \bar{f}$ in $C^\infty_{vSW[\rho']}(M/H,P)$, where

$$\mathcal{N}_i = \mathcal{N}^{r_i}(\bar{f}; L_i, (V_i,\psi_i), (W_i,\omega_i), \varepsilon_i), \quad i \in \Lambda,$$

and the family $\{\rho'(L_i)\}_{i\in\Lambda}$ is locally finite in Ω. Since the action of H on M is proper and free, the projection $p : M \to M/H$ is a smooth principal H-bundle over M/H. By Corollary 2.2.2 we can assume that for each $i \in \Lambda$ there exists a bundle chart $\tilde{\varphi}$ over $V_i \subset M/H$, i.e. a C^∞ diffeomorphism $\tilde{\varphi} : p^{-1}(V_i) \to \mathbb{R}^q \times V_i$ such that $p = pr_2 \circ \tilde{\varphi}$ (here $q = \dim H$, and $pr_2 : \mathbb{R}^q \times V_i \to V_i$ is the projection). Thus, if we set $U_i = p^{-1}(V_i)$, $i \in \Lambda$, we can find smooth maps $\varphi_i : U_i \to \mathbb{R}^q \times \psi_i(V_i)$ such that, for every $i \in \Lambda$, (U_i, φ_i) is a chart in M and the diagram

$$\begin{array}{ccc} \mathbb{R}^q \times \psi_i(V_i) & \xleftarrow{\varphi_i} & U_i \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p|} \\ \psi_i(V_i) & \xleftarrow{\psi_i} & V_i \end{array}$$

commutes. We set

$$K_i = \varphi^{-1}(\{0\} \times \psi_i(L_i)) \subset U_i\,, \quad i \in \Lambda.$$

Then, since $p(K_i) = L_i$, we have that

$$\rho(K_i) = (\rho' \circ p)(K_i) = \rho'(L_i)$$

for $i \in \Lambda$, and so the family $\{\rho(K_i)\}_{i\in\Lambda}$ is locally finite in Ω. Thus, if we set

$$\mathcal{M}_i^H = \mathcal{N}^{r_i}(f; K_i, (U_i, \varphi_i), (W_i, \omega_i), \varepsilon_i) \cap C^{\infty,H}(M, P), \quad i \in \Lambda,$$

we can define the following basic neighborhood of f in $C^{\infty,H}_{vSW[\rho]}(M, P)$:

$$\mathcal{M}^H = \bigcap_{i\in\Lambda} \mathcal{M}_i^H.$$

Now, for any smooth, H-equivariant map $k : M \to P$ such that $k(K_i) \subset W_i$ we have that

$$k \circ \varphi_i^{-1} = \bar{k} \circ \psi_i \circ pr_2 : \mathbb{R}^q \times \psi_i(V_i) \to P,$$

and hence, since $\varphi_i(K_i) = \{0\} \times \psi_i(L_i)$, we have for every $i \in \Lambda$:

$$||\omega_i \circ k \circ \varphi_i^{-1} - \omega_i \circ f \circ \varphi_i^{-1}||^{r_i}_{\varphi_i(K_i)} = ||\omega_i \circ \bar{k} \circ \psi_i^{-1} - \omega_i \circ \bar{f} \circ \psi_i^{-1}||^{r_i}_{\psi_i(L_i)}.$$

It follows from the above equation that if $k \in \mathcal{M}_i^H$, then $\bar{k} \in \mathcal{N}_i$. Thus $p^{**}(\mathcal{M}^H) \subset \mathcal{U}$, and the claim is proved. □

Proposition 4.1.3. *Let H be a closed, normal subgroup of a Lie group G, and let $\pi : G \to G/H$ be the projection. Let M be a smooth G-manifold, such that the action of H on M is proper and free. Let $p : M \to M/H$ denote the projection, and let P be a smooth, G/H-manifold. Then the map*

$$p^* : C^{\infty,G/H}_{vSW}(M/H, P) \to C^{\infty,\pi}_{vSW}(M, P), \quad k \mapsto k \circ p,$$

is a homeomorphism.

Proof. Note that, since $H < G$ is closed and normal, the action of G on M induces an action of G/H on M/H, and $(M/H)(G/H) = M/G$ (see [Kaw], Proposition 1.59). Take the projection maps $q' : M/H \to (M/H)(G/H)$ and $q : M \to M/G$ as phase maps for M/H and M, respectively. Since $q = q' \circ p$ we can apply Lemma 4.1.2, and thus the map

$$p^* : C^{\infty}_{vSW[q']}(M/H, P) \to C^{\infty,H}_{vSW[q]}(M, P), \quad k \mapsto k \circ p,$$

is a homeomorphism. Now consider the (continuous) restriction of p^* to the space

$$C^{\infty}_{vSW[q']}(M/H, P) \cap C^{\infty,G/H}(M/H, P) = C^{\infty,G/H}_{vSW}(M/H, P).$$

Now, if $k : M/H \to P$ is a G/H-equivariant map, we have for every $g \in G$ and $x \in M$:

$$k \circ p(gx) = k(gxH) = k(gH \cdot xH) = gHk(xH) = \pi(g)k \circ p(x).$$

Thus $p^*(k) = k \circ p : M \to P$ is π-equivariant, and the converse is also true, hence

$$p^*(C^{\infty,G/H}_{vSW}(M/H, P)) = C^{\infty,\pi}_{vSW}(M, P),$$

and the claim is proved. □

We are now ready to prove a crucial result:

Theorem 4.1.4. *Let G be a Lie group, and let $H < G$ be a closed subgroup. Assume M is a smooth H-manifold, and N a smooth G-manifold. Then the bijection*

$$\mu : C^{\infty,H}_{vSW}(M,N) \to C^{\infty,G}_{vSW}(G \times_H M, N), \quad f \mapsto \mu(f),$$

is a homeomorphism.

Proof. We saw before that

$$p^* \circ \mu = \Phi_* \circ \chi : C^{\infty,H}_{vSW}(M,N) \to C^{\infty,\pi_1}_{vSW}(G \times M, N),$$

where

$$\chi : C^{\infty,H}_{vSW}(M,N) \to C^{\infty,G\times H}_{vSW}(G \times M, G \times N), \quad f \mapsto id \times f,$$
$$\Phi_* : C^{\infty,G\times H}_{vSW}(G \times M, G \times N) \to C^{\infty,\pi_1}_{vSW}(G \times M, N), \quad q \mapsto \Phi \circ q,$$

and

$$p^* : C^{\infty,G}_{vSW}(G \times_H M, N) \to C^{\infty,\pi_1}_{vSW}(G \times M, N), \quad z \mapsto z \circ p$$

(here $\pi_1 : G \times H \to G$ is the projection, and Φ is the action of G on N). Now, the map χ is continuous by Proposition 4.1.1, and the map Φ_* is continuous by Proposition 2.4.1. Furthermore, $\{e\} \times H = H$ is a closed, normal subgroup of $G \times H$, which acts on $G \times M$ by

$$H \times (G \times M) \to G \times M, \quad (h,(g,x)) \mapsto (gh^{-1}, hx). \tag{29}$$

The action (29) is clearly free, and it is proper. Thus, we can apply Proposition 4.1.3, and so the map p^* is a homeomorphism. This shows that the map μ is continuous. Consider now the inverse map μ^{-1}. If $i : M \to G \times_H M$, $x \mapsto [e,x]$, denotes the natural inclusion, we have that

$$\mu^{-1} = i_* : C^{\infty,G}_{vSW}(G \times_H M, N) \to C^{\infty,H}_{vSW}(M,N).$$

Since the induced map

$$\bar{i} : M/H \to (G \times_H M)/G$$

is a homeomorphism (see Lemma 1.1.6), and hence of finite type, i_* is continuous by Proposition 2.4.1, and the claim follows. □

We end this section by remarking an important consequence of Lemma 4.1.2 above (see also Corollary 5.3 in [I-Ka1] for the "strong-weak" case):

Theorem 4.1.5. *Let H be a Lie group, and let M be a proper, free, smooth H-manifold. Let $p : M \to M/H$ denote the projection, and let P be a smooth manifold. Then the map*

$$p^* : C^{\infty}_{vS}(M/H, P) \to C^{\infty,H}_{vSW}(M,P), \quad f \mapsto f \circ p,$$

is a homeomorphism.

Proof. It is enough to take $\rho' = id : M/H \to M/H$ in Lemma 4.1.2. □

4.2. Approximation result

Before proving our main result, Theorem 4.2.2, we need to prove the approximation result in the case when G is a Lie group with only finitely many connected components. The strong-weak topology form of Theorem 4.2.1 below is Proposition 2.1 of [I-Ka2]: in that proof use is made of the Corollary on page 19 in [Ku2]. that is, the equivariant approximation result for a compact Lie group in the case of the strong C^∞ topology. In the case of the very-strong-weak topology, we instead have to rely on our result, Corollary 3.3.2.

Theorem 4.2.1. *Let G be a Lie group with only finitely many connected components, and let M and N be real analytic G-manifolds. If the action of G on M is proper, then $C^{\omega,G}(M,N)$ is dense in $C^{\infty,G}_{vSW}(M,N)$.*

Proof. Let K be a maximal compact subgroup of G. By the real analytic version of Abels' theorem, Theorem 1.2.4, there exists in M a global K-slice, that is, a K-invariant, real analytic submanifold S of M such that M can be written in the form $G \times_K S$. Then by Theorem 4.1.4 we have a homeomorphism

$$\mu : C^{\infty,K}_{vSW}(S,N) \to C^{\infty,G}_{vSW}(G \times_K S, N), \quad f \mapsto \mu(f),$$

and

$$\mu(C^{\omega,K}_{vSW}(S,N)) = C^{\omega,G}_{vSW}(G \times_K S, N).$$

Now, since K is compact we have by Lemma 2.3.7 that

$$C^{\infty,K}_{vS}(S,N) = C^{\infty,K}_{vSW}(S,N) \cong C^{\infty,G}_{vSW}(G \times_K S, N),$$

and

$$C^{\omega,K}_{vS}(S,N) = C^{\omega,K}_{vSW}(S,N) \cong C^{\omega,G}_{vSW}(G \times_K S, N).$$

Since $C^{\omega,K}_{vS}(S,N)$ is dense in $C^{\infty,K}_{vS}(S,N)$ by Corollary 3.3.2, the claim is proved. □

We now come to our main theorem. It establishes the very-strong-weak topology version of Theorem II in [I-Ka2], where the corresponding result is proved for the strong-weak topology.

Theorem 4.2.2. *Let G be a good Lie group, and let M and N be real analytic, proper G-manifolds. Then $C^{\omega,G}(M,N)$ is dense in $C^{\infty,G}_{vSW}(M,N)$.*

Proof. Let $f \in C^{\infty,G}_{vSW}(M,N)$, and let $\mathcal{U}$ be a basic neighborhood of f in the very-strong-weak topology. We can assume that the good Lie group G is a closed subgroup of a Lie goup J with only finitely many connected components. Thus we can construct the induced J-manifolds $J \times_G M$ and $J \times_G N$, which are both real analytic and proper. We denote by

$$i_1 : M \to J \times_G M, \quad x \mapsto [e,x]$$

and

$$i_2 : N \to J \times_G N, \quad y \mapsto [e,y]$$

the (G-equivariant, real analytic, closed) canonical embeddings of M and N into $J \times_G M$ and $J \times_G N$, respectively. We define a map $\tilde{f}$ in the following way:

$$\hat{f} : J \times_G M \to J \times_G N, \quad [j, x] \mapsto [j, f(x)].$$

Then $f = i_2^{-1} \circ \hat{f} \circ i_1$, and

$$\hat{f} \in C^{\infty,J}_{vSW}(J \times_G M, J \times_G N).$$

Now, we can consider on $J \times_G N$ the induced action of G, that is

$$G \times J \times_G N \to J \times_G N, \quad (g, [j, y]) \mapsto [gj, y].$$

Then $J \times_G N$ is a real analytic, proper G-manifold, and we can apply Theorem I in [I-Ka] to its G-invariant, real analytic, closed submanifold N: thus there exist a G-invariant, open neighborhood W of N, and a G-equivariant, real analytic retraction $q : W \to N$ in $J \times_G N$. Thus we can write

$$f = q \circ \hat{f} \circ i_1.$$

Now, the map

$$q_* : C^{\infty,G}_{vSW}(M, W) \to C^{\infty,G}_{vSW}(M, N), \quad \alpha \mapsto q \circ \alpha,$$

is continuous by Proposition 2.4.1. Moreover, by Lemma 2.4.5 the space $C^{\infty,G}_{vSW}(M, W)$ is open in $C^{\infty,G}_{vSW}(M, J \times_G N)$, thus the set $q_*^{-1}(\mathcal{U})$ is open in $C^{\infty,G}_{vSW}(M, J \times_G N)$. Furthermore, the map

$$i_1^* : C^{\infty,J}_{vSW}(J \times_G M, J \times_G N) \to C^{\infty,G}_{vSW}(M, J \times_G N), \quad \beta \mapsto \beta \circ i_1,$$

is continuous by Lemma 1.1.6 and Proposition 2.4.2. Thus we have that the set

$$\mathcal{W} = (i_1^*)^{-1}(q_*^{-1}(\mathcal{U}))$$

is an open neighborhood of $\hat{f}$ in $C^{\infty,J}_{vSW}(J \times_G M, J \times_G N)$. Since J is a Lie group with only finitely many connected components, we can now apply Theorem 4.2.1, hence we can find a map

$$\hat{h} \in \mathcal{W} \cap C^{\omega,J}_{vSW}(J \times_G M, J \times_G N).$$

Now define

$$h = q_*(i_1^*(\hat{h})),$$

that is, $h = q \circ \hat{h} \circ i_1$. Then

$$h \in \mathcal{U} \cap C^{\omega,G}_{vSW}(M, N),$$

and the claim is proved. $\square$

Appendix A.

We start by proving a technical result, which will be needed in the proof of Proposition A.3.

Lemma A.1. *Let (N, g) be a smooth Riemannian manifold, and let $f : N \to \mathbb{R}$ be a continuous function. Let $y \in N$, and assume that the ball $B_R(y)$, $R > 0$, is geodesically convex. Furthermore, let $r < \frac{1}{2}R$ be a positive real number, and let*

$$\Delta = \max_{z \in \overline{B_r(y)}} \{f(z)\}.$$

For every $\delta > 0$, denote

$$\Delta_\delta^+ = \max_{z \in \overline{B_{r+\delta}(y)}} \{f(z)\}.$$

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Delta_\delta^+ - \Delta < \varepsilon. \tag{30}$$

Proof. Clearly, $\Delta_\delta^+ \geq \Delta$ for each $\delta > 0$. Since the function f is uniformly continuous on the compact ball $\overline{B_R(y)}$, we can find $\bar{\delta} > 0$ such that for every z_1, $z_2 \in \overline{B_R(y)}$ with $d(z_1, z_2) < \bar{\delta}$ we have:

$$|f(z_1) - f(z_2)| < \varepsilon.$$

Thus, fix $\delta > 0$ such that $\delta < \min\{\bar{\delta}, \frac{1}{2}R\}$. If $\Delta_\delta^+ = \Delta$, the inequality (30) is satisfied. Assume then $\Delta_\delta^+ > \Delta$, and let

$$z_0 \in \overline{B_{r+\delta}(y)} \setminus \overline{B_r(y)}$$

be such that

$$f(z_0) = \Delta_\delta^+.$$

Consider the radial geodesic $\gamma : I \to N$ from y to z_0, and denote

$$\bar{z}_0 = \gamma(I) \cap \partial B_r(y).$$

Clearly $d(z_0, \bar{z}_0) \leq \delta < \bar{\delta}$, hence we have

$$|f(z_0) - f(\bar{z}_0)| = |\Delta_\delta^+ - f(\bar{z}_0)| < \varepsilon.$$

Now $\Delta \geq f(\bar{z}_0)$, that is, we can write $\Delta = f(\bar{z}_0) + \eta$, where $\eta \geq 0$. Thus we get:

$$\Delta_\delta^+ - \Delta = \Delta_\delta^+ - (f(\bar{z}_0) + \eta) \leq \Delta_\delta^+ - f(\bar{z}_0) < \varepsilon,$$

and this proves the claim. □

Remark A.2. With the same assumptions and notation of Lemma A.1, let $\delta > 0$ be the real number constructed in the proof of A.1 for which the inequality (30) holds. If $\delta < r$ denote

$$\Delta_\delta^- = \max_{z \in \overline{B_{r-\delta}(y)}} \{f(z)\}.$$

Then it is easy to prove that also the inequality

$$\Delta - \Delta_\delta^- < \varepsilon$$

is satisfied.

Proposition A.3. *Let (N, g) be a smooth, Riemannian manifold on which a compact Lie group K acts by isometries, and assume that $f : N \to \mathbb{R}$ is a continuous, K-invariant function. Let $R : N \to \mathbb{R}_+$ be a K-invariant convexity function on N, and define $r : N \to \mathbb{R}_+$ by $r(y) = \frac{1}{2}R(y)$, $y \in N$. Then the function*

$$\Delta : N \to \mathbb{R}, \quad y \mapsto \Delta_y = \max_{z \in \overline{B_{r(y)}(y)}} \{f(z)\},$$

is continuous and K-invariant.

Proof. First we prove the invariance of Δ: let then $y \in N$, and $k \in K$. Since g and r are K-invariant, we have that

$$B_{r(ky)}(ky) = kB_{r(y)}(y).$$

Thus we can write, using invariance of f:

$$\Delta_{ky} = \max_{z \in \overline{B_{r(ky)}(ky)}} \{f(z)\} = \max_{x \in \overline{B_{r(y)}(y)}} \{f(kx)\} = \max_{x \in \overline{B_{r(y)}(y)}} \{f(x)\} = \Delta_y,$$

and the claim is proved. It remains to show that Δ is continuous. Let $y \in N$, and let $\varepsilon > 0$. By Lemma A.1 and Remark A.2 there exists $0 < \delta < r(y)$ such that, if we denote

$$\Delta^+_{y,\delta} = \max_{z \in \overline{B_{r(y)+\delta}(y)}} \{f(z)\}$$

and

$$\Delta^-_{y,\delta} = \max_{z \in \overline{B_{r(y)-\delta}(y)}} \{f(z)\},$$

then the following inequalities are satisfied:

$$\Delta^+_{y,\delta} - \Delta_y < \frac{\varepsilon}{2}, \tag{31}$$

$$\Delta_y - \Delta^-_{y,\delta} < \frac{\varepsilon}{2} \tag{32}$$

(clearly, $\Delta^-_{y,\delta} \leq \Delta_y \leq \Delta^+_{y,\delta}$). Now, since r is continuous, we can choose $\rho > 0$ such that

$$|r(y) - r(y_1)| + d(y, y_1) \leq \delta, \quad \forall \ y_1 \in B_\rho(y),$$

that is,

$$r(y) - (\delta - d(y, y_1)) \leq r(y_1) \leq r(y) + (\delta - d(y, y_1)).$$

The inequality above implies that for every $y_1 \in B_\rho(y)$ we get

$$z \in B_{r(y_1)}(y_1) \ \Rightarrow \ d(z, y) \leq r(y_1) + d(y, y_1) \leq r(y) + \delta,$$

and

$$w \in B_{r(y)-\delta}(y) \ \Rightarrow \ d(w, y_1) \leq r(y) - \delta + d(y, y_1) \leq r(y_1),$$

that is,

$$\overline{B_{r(y)-\delta}(y)} \subset \overline{B_{r(y_1)}(y_1)} \subset \overline{B_{r(y)+\delta}(y)}. \tag{33}$$

Thus, if $y_1 \in B_\rho(y)$ we get by 33 that

$$\Delta^-_{y,\delta} \le \Delta_{y_1} \le \Delta^+_{y,\delta},$$

and hence, by (31) and (32),

$$|\Delta_y - \Delta_{y_1}| \le |\Delta^+_{y,\delta} - \Delta^-_{y,\delta}| \le |\Delta^+_{y,\delta} - \Delta_y| + |\Delta_y - \Delta^-_{y,\delta}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and the proof is completed. □

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Department of Mathematics and Statistics
P.O. Box 68, FI-00014 University of Helsinki, Finland
E-mail: elena.ravaioli@helsinki.fi

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