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HELSINKI 2010
SUOMALAINEN TIEDEAKATEMIA

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Jyväskylä, September 2010

Tuula Ripatti

Contents

1. Introduction

Different concepts of dimension and their geometry is an active area of interest in geometric measure theory. One line of research is to study the behaviour of these different dimensions, of both measures and sets, under orthogonal projections, plane sections and general intersections. This work continues the study of dimensions of intersection measures, in particular their local dimensions. Our results on local dimensions lead to new results on Hausdorff and packing dimensions of intersection measures. We also consider the case where intersection measures are defined using general linear maps or continuously differentiable functions instead of previously studied cases of isometries and similarities.

The lower and upper local dimensions of a Radon measure μ on $\mathbb{R}^n$ at point $x \in \mathbb{R}^n$, denoted by $\underline{\dim}_{\mathrm{loc}}\mu(x)$ and $\overline{\dim}_{\mathrm{loc}}\mu(x)$ respectively, are defined as the lower and upper limits of the quantity $\log \mu(B(x,h))/\log h$ as h goes to 0. Here $B(x,r)$ is a closed ball with centre at x and with radius r. The local dimensions of μ are related to the Hausdorff and packing dimensions of μ via

$$\dim_{\mathrm{H}} \mu = \mu\text{-}\operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \underline{\dim}_{\mathrm{loc}}\mu(x)$$

and

$$\dim_{\mathrm{p}} \mu = \mu\text{-}\operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \overline{\dim}_{\mathrm{loc}}\mu(x),$$

and, if the essential infimum is replaced by the essential supremum, then we get the upper Hausdorff and packing dimensions of μ, denoted by $\dim^*_{\mathrm{H}}$ and $\dim^*_{\mathrm{p}}$ respectively. It turns out that local dimensions behave like their global counterparts when considering dimensions of slices and intersections.

The relation between dimension of a Borel set A in $\mathbb{R}^n$ and dimension of its intersections with affine planes, called slices of A, is well known for both Hausdorff and packing dimensions. First Marstrand [21] proved in the plane and later Mattila [22] generalized to $\mathbb{R}^n$ the following result. Let m and n be integers and let $m \leq s \leq n$. Denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure and by $\gamma_{n,n-m}$ the natural measure on the space of $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$. If $A \subset \mathbb{R}^n$ is a

Borel set with $0 < \mathcal{H}^s(A) < \infty$, then we have for $\gamma_{n,n-m}$-almost all $(n-m)$-planes V that

$$\mathcal{H}^m(\{a \in V^\perp : \dim_H(A \cap V_a) = s - m\}) > 0. \tag{1.1}$$

Here $V^\perp$ is the orthogonal complement of V and V_a is the affine $(n-m)$-plane which is parallel to V and goes through a.

The behaviour of packing dimension is more irregular than that of Hausdorff dimension when considering slices of sets, and as we see later, the same phenomenon happens with slices of measures and with general intersections. In other words, equation (1.1) does not hold for packing dimension, as was shown by Falconer, Järvenpää and Mattila [5], see also Csörnyei [2]. However, Järvenpää, Järvenpää and Llorente obtained in [15] a sufficient condition for the stability of packing dimensions of slices of sets, that is, under this condition it holds that for $\gamma_{n,n-m}$-almost all $(n-m)$ -planes V

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a\in V^\perp} \dim_p(A \cap V_a) = d - m.$$

Here d is a constant independent of the plane V, and it may be strictly less than $\dim_p A$.

Slices of a Radon measure μ on $\mathbb{R}^n$ by affine $(n-m)$-planes V_x through a point $x \in \mathbb{R}^n$, denoted by $\mu_{V,x}$, were introduced by Mattila in [23]. For the definition, see Section 2.1. These measures, which are supported on $\operatorname{spt}\mu \cap V_x$, were originally used to study capacities of slices of sets. Here $\operatorname{spt}\mu$ is the support of μ. Dimensional properties of sliced measures are also well known, see [7], [6], [19] and [15]. Järvenpää and Mattila proved in [19] that if μ is a Radon measure on $\mathbb{R}^n$ with compact support, then for $\gamma_{n,n-m}$-almost all $(n-m)$-planes V

$$\mathcal{H}^m\text{-}\operatorname{ess\,inf}\{\dim_H \mu_{V,a} : a \in V^\perp \text{ with } \mu_{V,a}(\mathbb{R}^n) > 0\} = \dim_H \mu - m \tag{1.2}$$

provided that $\dim_H \mu > m$, and

$$\mathcal{H}^m\text{-}\operatorname{ess\,inf}\{\dim_p \mu_{V,a} : a \in V^\perp \text{ with } \mu_{V,a}(\mathbb{R}^n) > 0\} = \mu\text{-}\operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \overline{d}^m_\mu(x) - m \tag{1.3}$$

provided that $I_{m+\varepsilon}(\mu) < \infty$ for some $\varepsilon > 0$. Here $I_{m+\varepsilon}(\mu)$ is the $(m+\varepsilon)$-energy of μ and $\overline{d}^m_\mu(x)$ is a modified local dimension which is obtained as a convolution of μ with a certain kernel (see definition in Section 2.2.) They also got following relations for upper Hausdorff and packing dimensions of sliced measures. For $\gamma_{n,n-m}$-almost all $(n-m)$-planes V

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a\in V^\perp} \dim^*_H \mu_{V,a} = \dim^*_H \mu - m, \tag{1.4}$$

provided that $\dim_H \mu > m$, and

$$\mathcal{H}^m\text{-}\operatorname*{ess\,sup}_{a\in V^\perp} \dim^*_p \mu_{V,a} = \mu\text{-}\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \overline{d}^m_\mu(x) - m, \tag{1.5}$$

provided that $I_{m+\varepsilon}(\mu) < \infty$ for some $\varepsilon > 0$.

Properties of local dimensions of sliced measures are similar to related global dimensions. Local dimensions of sliced measures were first studied by Falconer and O'Neil in [7]. Extending their result Järvenpää, Järvenpää and Llorente showed in [15] that for $\gamma_{n,n-m}$-almost all $(n-m)$-planes V and for μ-almost all $x \in \mathbb{R}^n$

$$\underline{\dim}_{\mathrm{loc}}\mu_{V,x}(x) = \underline{\dim}_{\mathrm{loc}}\mu(x) - m$$

if $\dim_{\mathrm{H}} \mu > m$, and

$$\overline{\dim}_{\mathrm{loc}}\mu_{V,x}(x) = \overline{d}^{m}_{\mu}(x) - m$$

if $I_{m+\varepsilon}(\mu) < \infty$ for some $\varepsilon > 0$. Using the methods of the proof of Theorem 3.13 it is possible to show that the latter equality also holds under the assumption $\dim_{\mathrm{H}} \mu > m$ by approximating μ by measures with finite $(m+\varepsilon)$-energy.

Another local concept of dimension is the average dimension $\dim_{\mathrm{A}}$, which is defined in a similar way as local dimensions, see Section 2.2. The average dimension of a measure always lies between corresponding lower and upper local dimensions. Llorente [20] showed that the average dimension of sliced measures behaves like the lower local dimension. In other words, she proved that assuming $\dim_{\mathrm{H}} \mu > m$ we have for $\gamma_{n,n-m}$-almost all $(n-m)$ -planes V and for μ-almost all $x \in \mathbb{R}^n$

$$\dim_{\mathrm{A}} \mu_{V,x}(x) = \dim_{\mathrm{A}} \mu(x) - m.$$

Intersection measures $\mu \cap f_\sharp\nu$, where μ and ν are Radon measures on $\mathbb{R}^n$, f is a mapping $\mathbb{R}^n \to \mathbb{R}^n$ and $f_\sharp\nu$ is the image of ν under f, can be considered as natural measures on $\operatorname{spt}\mu \cap f(\operatorname{spt}\nu)$. They are defined by slicing the product measure $\mu \times f_\sharp\nu$ by affine n-planes, which are parallel to the diagonal $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$, see definition in Section 2.1. Thus dimensional properties of these measures are closely related to the dimensions of sliced measures.

Mattila introduced intersection measures in [24] in order to study dimensions of intersections of two Borel sets in $\mathbb{R}^n$. He considered the cases where $f : \mathbb{R}^{\to}\mathbb{R}^n$ is an isometry or a similarity map, i.e. a map for which there is $r > 0$ such that $|f(x) - f(y)| = r|x-y|$ for all $x, y \in \mathbb{R}^n$. Every similarity map has a unique decomposition as

$$f = \tau_z \circ g \circ \delta_r,$$

where $\tau_z : \mathbb{R}^n \to \mathbb{R}^n$, $\tau_z(x) = x+z$, $z \in \mathbb{R}^n$, $g \in \mathcal{O}(n)$ and $\delta_r : \mathbb{R}^n \to \mathbb{R}^n$, $\delta_r(x) = rx$, $r > 0$. Here $\mathcal{O}(n)$ is the orthogonal group of linear isometries $\mathbb{R}^n \to \mathbb{R}^n$. Mattila proved that if $0 < s < n$, $0 < t < n$, $s + t > n$ and A and B are Borel sets in $\mathbb{R}^n$ such that $\mathcal{H}^s(A) < \infty$ and $\mathcal{H}^t(B) < \infty$, then

$$\dim_{\mathrm{H}}(A \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})(B)) \geq s + t - n \tag{1.6}$$

for $\mathcal{H}^s \times \mathcal{H}^t \times \theta_n \times \mathcal{L}^1$-almost all $(x, y, g, r) \in A \times B \times \mathcal{O}(n) \times (0, \infty)$. Here θ_n is the normalized Haar measure on $\mathcal{O}(n)$ and $\mathcal{L}^1$ is the 1-dimensional Lebesgue measure.

In [24] Mattila showed that the same result holds for isometries under the additional assumption $t > \frac{1}{2}(n+1)$. However, it is not known if this assumption is necessary. In general, the opposite inequality in (1.6) is false for both similarities and isometries. For any $0 < s \leq n$ there are classes of sets in $\mathbb{R}^n$ with Hausdorff dimension s, such that even their countable intersection has Hausdorff dimension s, see Falconer [3]. However, Mattila proved that equality holds in (1.6) if we additionally assume that the Hausdorff dimensions of A and B satisfy the equality $\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B = \dim_{\mathrm{H}}(A \times B)$.

Hausdorff and packing dimensions of intersections measures in both isometry and similarity cases have been studied by Järvenpää in [16], [17] and [18]. She showed in [18] that if μ and ν are Radon measures on $\mathbb{R}^n$ with compact supports such that $\dim_{\mathrm{H}}(\mu \times \nu) = \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu > n$ and $I_t(\nu) < \infty$ for all $0 < t < \dim_{\mathrm{H}} \nu < n$, then for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$

(1.7)
$$\begin{aligned}\mathcal{L}^n\text{-}\operatorname{ess\,inf}\{ &\dim_{\mathrm{H}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) : z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbb{R}^n) > 0\} \\ &= \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu - n.\end{aligned}$$

Moreover, if we assume that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < s < n$ and $0 < t < n$ with $s + t > n$, then for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$

(1.8)
$$\begin{aligned}\mathcal{L}^n\text{-}\operatorname{ess\,inf}\{ &\dim_{\mathrm{p}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) : z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbb{R}^n) > 0\} \\ &= \mu \times \nu\text{-} \operatorname*{ess\,inf}_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} \overline{d}^n_{\mu \times \nu}(x, y) - n.\end{aligned}$$

Here $\overline{d}^n_{\mu \times \nu}(x, y)$ is as in (1.3). She also got similar relations for upper Hausdorff and packing dimensions of intersection measures, and these results are analogous to (1.4) and (1.5).

In the cases of the Hausdorff dimension and the upper Hausdorff dimension the same results can be obtained for isometries using the same methods, provided that $\dim_{\mathrm{H}}(\mu \times \nu) = \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu > n$ and $I_t(\nu) < \infty$ for all $(n+1)/2 < t < \dim_{\mathrm{H}} \nu < n$. For packing dimension the methods used to prove (1.8) cannot be applied in the case of isometries, since an integration with respect to r is not involved. However, the following lower bound for the packing dimensions of intersection measures is obtained in [17]. Assuming $I_\alpha(\mu) < \infty$, $I_\beta(\nu) < \infty$, $0 < \alpha < n$, $(n+1)/2 \leq \beta < n$ and $\alpha + \beta > n$, then we have for $\mu \times \nu \times \theta_n$-almost all $(x, y, g) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{O}(n)$

$$\dim_{\mathrm{p}}(\mu \cap (\tau_x \circ g \circ \tau_{-y})_\sharp \nu) \geq \frac{\beta \dim_{\mathrm{p}}(\alpha + \beta - n)}{n\alpha - (n - \beta) \dim_{\mathrm{p}} \mu}.$$

In Section 3 we will use our results concerning local dimensions of intersection measures to improve equalities (1.7) and (1.8) in the case of similarities. We will show that assuming only $\dim_{\mathrm{H}}(\mu \times \nu) > n$, we get instead of (1.7) that

$$\begin{aligned}\mathcal{L}^n\text{-}\operatorname{ess\,inf}\{ &\dim_{\mathrm{H}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) : z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbb{R}^n) > 0\} \\ &= \dim_{\mathrm{H}}(\mu \times \nu) - n\end{aligned}$$

for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$. Moreover, the equality (1.8) holds. We will also make similar improvements for the results on upper Hausdorff and upper packing dimensions of intersection measures. In other words, we will show that assuming $\dim_H(\mu\times\nu)>n$ we have for $\theta_n\times\mathcal{L}^1$-almost all $(g,r)\in\mathcal{O}(n)\times(0,\infty)$

$$\mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z\in\mathbb{R}^n}\dim^*_{\mathrm{H}}(\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu)=\dim^*_{\mathrm{H}}(\mu\times\nu)-n$$

and

$$\mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z\in\mathbb{R}^n}\dim^*_{\mathrm{p}}(\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu)=\mu\times\nu\text{-}\operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n}\overline{d}^n_{\mu\times\nu}(x,y)-n.$$

In this paper we study local dimensions of intersection measures for the first time and it turns out that they behave just like the corresponding global dimensions. We will prove in Section 3 the following result. Assuming $\dim_H(\mu\times\nu)>n$ we have for $\theta_n\times\mathcal{L}^1$-almost all $(g,r)\in\mathcal{O}(n)\times(0,\infty)$ that

$$\underline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_{x-g\circ\delta_r(y)}\circ g\circ\delta_r)_\sharp\nu)(x)=\underline{\dim}_{\mathrm{loc}}(\mu\times\nu)(x,y)-n=\underline{d}^n_{\mu\times\nu}(x,y)-n$$

and

$$\overline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_{x-g\circ\delta_r(y)}\circ g\circ\delta_r)_\sharp\nu)(x)=\overline{d}^n_{\mu\times\nu}(x,y)-n$$

for $\mu\times\nu$-almost all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$.

When proving this we have to slice general Radon measures on $\mathbb{R}^n\times\mathbb{R}^n$, not just the ones which are products of two measures on $\mathbb{R}^n$, and study their properties in a similar way as in [24]. This is done in Section 3.1. This method cannot be used in the case of isometries. Other ingredients needed for the proofs are methods from [15], where local dimensions of sliced measures where studied, combined with those from [18]. We also make use of results concerning Hausdorff and packing dimensions of sliced measures from [19].

Average local dimensions of intersection measures were studied by Llorente in [20]. She proved that if μ and ν are Radon measures on $\mathbb{R}^n$ with compact supports such that $I_s(\mu)<\infty$ and $I_t(\nu)<\infty$ for some $0<s<n$ and $0<t<n$ with $s+t\geq n$, then for $\mu\times\nu\times\theta_n\times\mathcal{L}^1$-almost all $(x,y,g,r)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{O}(n)\times(0,\infty)$

$$\begin{aligned}\dim_{\mathrm{A}}\mu(x)+\underline{\dim}_{\mathrm{loc}}\nu(y)-n&\leq\dim_{\mathrm{A}}(\mu\cap(\tau_{x-g\circ\delta_r(y)}\circ g\circ\delta_r)_\sharp\nu)(x)\\&\leq\dim_{\mathrm{A}}\mu(x)+\overline{\dim}_{\mathrm{loc}}\nu(y)-n.\end{aligned}$$

She also proved the same result in the isometry case. We will show in Section 3.4 that the average dimension of intersection measures behaves like the lower local dimension. That is, assuming $\dim_H(\mu\times\nu)>n$ we have for $\mu\times\nu\times\theta_n\times\mathcal{L}^1$-almost all $(x,y,g,r)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{O}(n)\times(0,\infty)$

$$\dim_{\mathrm{A}}(\mu\cap(\tau_{x-g\circ\delta_r(y)}\circ g\circ\delta_r)_\sharp\nu)(x)=\dim_{\mathrm{A}}(\mu\times\nu)(x,y)-n=\underline{d}^n_{\mu\times\nu}(x,y)-n.$$

Previous results show that dimensions of intersection measures depend on the dimension of the product measure, not just on the dimension of two measures separately. Note that Hausdorff and packing dimensions do not behave nicely under products. For example for the Hausdorff dimension of the product measure the best we can do are the following bounds:

$$\dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu \leq \dim_{\mathrm{H}}(\mu \times \nu) \leq \dim_{\mathrm{H}} \mu + \dim_{\mathrm{p}} \nu.$$

In Section 4 we study intersection measures in the case where similarities are replaced by general (invertible) affine maps. Note that almost all linear maps, with respect to Lebesque measure $\mathcal{L}^{n^2}$, are invertible. Every invertible linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ has a unique representation, the QR-decomposition, as

$$L = g \circ T,$$

where $g \in \mathcal{O}(n)$ and $T \in T(n)_+$, the group of upper triangular matrices with strictly positive diagonal entries. Using this decomposition, we apply the same methods as in the case of similarities to prove results for dimensions of more general intersection measures. We will show for example that if $\dim_{\mathrm{H}}(\mu \times \nu) > n$, then for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g, T) \in \mathcal{O}(n) \times T(n)_+$

$$\underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-g\circ T(y)} \circ g \circ T)_\sharp \nu)(x) = \underline{\dim}_{\mathrm{loc}}(\mu \times \nu)(x, y) - n = \underline{d}^n_{\mu\times\nu}(x, y) - n$$

and

$$\overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-g\circ T(y)} \circ g \circ T)_\sharp \nu)(x) = \overline{d}^n_{\mu\times\nu}(x, y) - n$$

for $\mu \times \nu$-almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. The average dimension behaves as the lower local dimension in this setting. We also use the above equalities to obtain results for Hausdorff and packing dimensions. All these results are analogous with the case of similarities.

In Section 4 we also consider dimensions of intersection measures in the case where similarity maps are replaced by continuously differentiable functions. Note that there is no analogue for Lebesgue measure or Haar measure in the infinite-dimensional space $C^1(\mathbb{R}^n, \mathbb{R}^n)$ of continuously differentiable functions. We use a notion of prevalence by Hunt, Sauer and Yorke in [14], see also [1]. This concept turns out to be a good notion of 'almost every' in infinite dimensional spaces from measure theoretical point of view. In fact, in finite dimensional spaces prevalence is equivalent to 'Lebesgue almost every'. We will show that, if $\dim_{\mathrm{H}}(\mu \times \nu) > n$, then in a prevalent set of functions $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-f(y)} \circ f)_\sharp \nu)(x) \geq \underline{\dim}_{\mathrm{loc}}(\mu \times \nu)(x, y) - n = \underline{d}^n_{\mu\times\nu}(x, y) - n \tag{1.9}$$

and

$$\overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-f(y)} \circ f)_\sharp \nu)(x) \geq \overline{d}^n_{\mu\times\nu}(x, y) - n. \tag{1.10}$$

for $\mu \times \nu$-almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

For the lower local dimension the equality holds in the above theorem at least when the Hausdorff dimension of the product measure behaves nicely. Assuming

$\dim_H(\mu \times \nu) = \dim_H \mu + \dim_H \nu > n$, we have in a prevalent set of functions $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ that

$$\underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-f(y)} \circ f)_\sharp \nu)(x) = \underline{\dim}_{\mathrm{loc}}(\mu \times \nu)(x,y) - n = \underline{d}^n_{\mu\times\nu}(x,y) - n$$

for $\mu \times \nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Again the average dimension behaves like the lower local dimension.

For the upper local dimension we have the following theorem. If $\dim_H(\mu \times \nu) > n$ and $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, then for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$ for which $g \circ T + f$ is injective

$$\overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-(g\circ T+f)(y)} \circ f)_\sharp \nu)(x) = \overline{d}^n_{\mu\times\nu}(x,y) - n$$

for $\mu \times \nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. It remains an open question whether the equality holds in (1.10) in a prevalent set of functions $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

2. Preliminaries

We denote by $B(x,r)$ a closed ball in $\mathbb{R}^n$ with centre at x and with radius r. Further, we denote by $d(x,A) = \inf\{|x-a| : a \in A\}$ the distance between a point $x \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$.

Let m and n be positive integers with $m < n$ and let μ be a Radon measure on $\mathbb{R}^n$ with compact support. We denote by $\mu|_B$ the restriction of a measure μ to a set $B \subset \mathbb{R}^n$, that is,

$$\mu|_B(A) = \mu(A \cap B)$$

for all $A \subset \mathbb{R}^n$. The image of the measure μ under $f : \mathbb{R}^n \to \mathbb{R}^m$ is denoted by $f_\sharp \mu$, that is,

$$f_\sharp \mu(A) = \mu(f^{-1}(A))$$

for all $A \subset \mathbb{R}^m$. Let ν be a Radon measure on $\mathbb{R}^n$ with compact support. The measure μ is absolutely continuous with respect to ν if $\nu(A) = 0$ implies $\mu(A) = 0$ for all $A \subset \mathbb{R}^n$. Then we write $\mu \ll \nu$. We say that measures μ and ν are mutually singular if there exists a set $A \subset \mathbb{R}^n$ such that $\mu(A) = 0 = \nu(\mathbb{R}^n \setminus A)$. In this case we write $\mu \perp \nu$. For $t > 0$ the t-energy of μ is defined by

$$I_t(\mu) = \int\int |x-y|^{-t}\, d\mu(x)\, d\mu(y).$$

The n-dimensional Lebesgue measure is denoted by $\mathcal{L}^n$. For $s \geq 0$ the s-dimensional Hausdorff measure is denoted by $\mathcal{H}^s$ and Hausdorff measures are normalized such that $\mathcal{H}^n(B(x,r)) = (2r)^n$.

2.1. **Slices and intersection measures.** Let $0 < m < n$ be integers and denote by $G(n, n-m)$ the Grassmann manifold of $(n-m)$-dimensional linear subspaces of $\mathbb{R}^n$. The normalized Haar measure on $G(n, n-m)$ is denoted by $\gamma_{n,n-m}$. Let $V \in G(n, n-m)$ and $a \in V^\perp$, where $V^\perp$ is the orthogonal complement of V. Then $V_a = \{v + a : v \in V\}$ is the $(n-m)$-plane parallel to V and going through a. For

$\mathcal{H}^m$-almost all $a \in V^\perp$ there is a Radon measure $\mu_{V,a}$, the slice of μ by V_a, on V_a such that

$$\int \varphi \, d\mu_{V,a} = \lim_{\delta \to 0} (2\delta)^{-m} \int_{V_a(\delta)} \varphi \, d\mu, \tag{2.1}$$

for all non-negative continuous φ on $\mathbb{R}^n$ with compact support. Here we use a notation $V_a(\delta) = \{x \in \mathbb{R}^n : d(x, V_a) \le \delta\}$. For $x \in \mathbb{R}^n$ we define $\mu_{V,x} = \mu_{V,a}$, if $a = P_{V^\perp}(x)$ and $\mu_{V,a}$ is defined. Here $P_{V^\perp} : \mathbb{R}^n \to V^\perp$ is the orthogonal projection. The construction of sliced measures and the proofs of their basic properties can be found in [25].

Sliced measures have the following properties. If φ is a non-negative lower semicontinuous function on $\mathbb{R}^n$, then

$$\int \varphi \, d\mu_{V,a} \le \liminf_{\delta \to 0} (2\delta)^{-m} \int_{V_a(\delta)} \varphi \, d\mu. \tag{2.2}$$

Moreover, if φ is a non-negative Borel function on $\mathbb{R}^n$ such that $\int \varphi \, d\mu < \infty$, then for any Borel set $B \subset V^\perp$

$$\int_B \int \varphi \, d\mu_{V,a} \, d\mathcal{H}^m a \le \int_{P_{V^\perp}^{-1}(B)} \varphi \, d\mu, \tag{2.3}$$

with equality if $P_{V^\perp\sharp}\mu \ll \mathcal{H}^m\big|_{V^\perp}$. Finally, if $B \subset \mathbb{R}^n$ is a Borel set and $P_{V^\perp\sharp}\mu \ll \mathcal{H}^m|_{V^\perp}$, then

$$\mu_{V,a}|_B = (\mu|_B)_{V,a} \tag{2.4}$$

for $\mathcal{H}^m$-almost all $a \in V^\perp$, see [19, Lemma 3.2].

Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports. For $\mathcal{L}^n$-almost all $z \in \mathbb{R}^n$ we can define the intersection of μ and $\tau_{z\sharp}\nu$, where $\tau_z(x) = x + z$, by slicing the product measure $\mu \times \nu$ by the n-plane

$$W_{(z,-z)/2} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x - y = z\},$$

which goes through $(z, -z)/2 \in W^\perp$, and by projecting the sliced measure to $\mathbb{R}^n$. That is, for $\mathcal{L}^n$-almost all $z \in \mathbb{R}^n$ we can define the intersection measure by

$$\mu \cap \tau_{z\sharp}\nu = \pi_\sharp\big[(\mu \times \nu)_{W,(z,-z)/2}\big],$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\pi(x, y) = x$. Then

$$\operatorname{spt}(\mu \cap \tau_{z\sharp}\nu) \subset \operatorname{spt}\mu \cap \operatorname{spt}\tau_{z\sharp}\nu,$$

where spt is the support of a measure.

Later we replace ν by $f_\sharp\nu$, where f is some function, for example a similarity or a linear mapping. We also need to consider the following more general setting. Let $R_f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, $R_f(x, y) = (x, f(y))$ and let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support. We slice measures $R_{f\sharp}\lambda$ by the planes $W_{(z,-z)/2}$. It turns out that for $\mathcal{L}^n$-almost all $z \in \mathbb{R}^n$ there is a Radon measure $(R_{f\sharp}\lambda)_{W,(z,-z)/2}$

such that the following properties hold. For all non-negative lower semicontinuous functions φ on $\mathbb{R}^n \times \mathbb{R}^n$ we have by (2.2)

$$\begin{aligned}\int \varphi \, d(R_{f\sharp}\lambda)_{W,(z,-z)/2} \\ \le \liminf_{\delta\to 0} (\sqrt{2}\delta)^{-n} \int_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : |S_f(x,y)-z|\le\delta\}} \varphi(x,y)\, d\lambda(x,y),\end{aligned} \tag{2.5}$$

where $S_f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $S_f(x,y) = x - f(y)$. Moreover, if φ is a non-negative Borel function on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\int \varphi \, dR_{f\sharp}\lambda < \infty$, then for any Borel set $B \subset W^\perp$

$$\int_B \int \varphi \, d(R_{f\sharp}\lambda)_{W,a} \, d\mathcal{H}^n|_{W^\perp}(a) \le \int_{P_{W^\perp}^{-1}(B)} \varphi \, dR_{f\sharp}\lambda \tag{2.6}$$

with equality if $S_{f\sharp}\lambda \ll \mathcal{L}^n$. This follows from (2.3) since $S_{f\sharp}\lambda \ll \mathcal{L}^n$ if and only if $P_{W^\perp\sharp}(R_{f\sharp}\lambda) \ll \mathcal{H}^n|_{W^\perp}$.

2.2. **Dimensions of measures.** In this section we define some concepts related to dimensions of measures. For more detailed information on this subject, see [4]. Let μ be a Radon measure on $\mathbb{R}^n$ with compact support. The lower and upper local dimensions of μ at a point $x \in \mathbb{R}^n$ are defined by

$$\underline{\dim}_{\text{loc}}\mu(x) = \liminf_{h\to 0} \frac{\log \mu(B(x,h))}{\log h}$$

and

$$\overline{\dim}_{\text{loc}}\mu(x) = \limsup_{h\to 0} \frac{\log \mu(B(x,h))}{\log h}.$$

For $0 \le s < \infty$ let

$$D_\mu^s(x) = \liminf_{\delta\to 0} \frac{1}{|\log\delta|} \int_\delta^1 \frac{\mu(B(x,r))}{r^s} \frac{1}{r} \, d\mathcal{L}^1(r).$$

Then the average dimension of μ at a point $x \in \mathbb{R}^n$ is defined by

$$\dim_{\text{A}} \mu(x) = \sup\{s \ge 0 : D_\mu^s(x) = 0\} = \inf\{s \ge 0 : D_\mu^s(x) = \infty\}.$$

We have the following relations

$$\underline{\dim}_{\text{loc}}\mu(x) \le \dim_{\text{A}} \mu(x) \le \overline{\dim}_{\text{loc}}\mu(x).$$

Hausdorff and packing dimensions of μ are defined as follows

$$\begin{aligned}\dim_{\text{H}} \mu &= \mu\text{-}\operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \underline{\dim}_{\text{loc}}\mu(x) \\ &= \inf\{\dim_{\text{H}} A : A \text{ is a Borel set and } \mu(A) > 0\}\end{aligned}$$

and

$$\begin{aligned}\dim_{\text{p}} \mu &= \mu\text{-}\operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \overline{\dim}_{\text{loc}}\mu(x) \\ &= \inf\{\dim_{\text{p}} A : A \text{ is a Borel set and } \mu(A) > 0\}.\end{aligned}$$

Upper Hausdorff and packing dimensions of μ are defined by

$$\begin{aligned}\dim_{\mathrm{H}}^{*}\mu &= \mu\text{- }\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\underline{\dim}_{\mathrm{loc}}\mu(x)\\ &= \inf\{\dim_{\mathrm{H}} A : A \text{ is a Borel set and } \mu(\mathbb{R}^n\setminus A)=0\}\end{aligned}$$

and

$$\begin{aligned}\dim_{\mathrm{p}}^{*}\mu &= \mu\text{- }\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\overline{\dim}_{\mathrm{loc}}\mu(x)\\ &= \inf\{\dim_{\mathrm{p}} A : A \text{ is a Borel set and } \mu(\mathbb{R}^n\setminus A)=0\}.\end{aligned}$$

Letting ν be a Radon measure on $\mathbb{R}^n$ with compact support, we have

$$\dim_{\mathrm{H}}\mu+\dim_{\mathrm{H}}\nu\le\dim_{\mathrm{H}}(\mu\times\nu)\le\dim_{\mathrm{H}}\mu+\dim_{\mathrm{p}}\nu \tag{2.7}$$

and

$$\dim_{\mathrm{H}}\mu+\dim_{\mathrm{p}}\nu\le\dim_{\mathrm{p}}(\mu\times\nu)\le\dim_{\mathrm{p}}\mu+\dim_{\mathrm{p}}\nu. \tag{2.8}$$

The corresponding inequalities hold if we replace $\dim_{\mathrm{H}}$ by $\dim_{\mathrm{H}}^{*}$ and $\dim_{\mathrm{p}}$ by $\dim_{\mathrm{p}}^{*}$, see [11] and [12].

As in [15], in order to study local dimensions of slices by $(n-m)$-planes, we modify the definitions of local dimensions of measures using the function

$$\psi_h^m(x)=\begin{cases}h^m|x|^{-m}, & \text{if } |x|\le h\\ 0, & \text{if } |x|>h\end{cases},$$

and defining

$$\underline{d}_\mu^m(x)=\liminf_{h\to 0}\frac{\log(\mu*\psi_h^m(x))}{\log h}=\liminf_{h\to 0}\frac{\log(h^m\int_{B(x,h)}|x-a|^{-m}\,d\mu(a))}{\log h}$$

and

$$\overline{d}_\mu^m(x)=\limsup_{h\to 0}\frac{\log(\mu*\psi_h^m(x))}{\log h}=\limsup_{h\to 0}\frac{\log(h^m\int_{B(x,h)}|x-a|^{-m}\,d\mu(a))}{\log h}.$$

Here $\mu*\psi_h^m$ is the convolution of μ and ψ_h^m.

Remark 2.1. (1) It is shown in [7, (4.12)] that, if $\int|x-y|^{-m}\,d\mu(y)<\infty$ for some $x\in\mathbb{R}^n$, then

$$\underline{d}_\mu^m(x)=\underline{\dim}_{\mathrm{loc}}\mu(x)\ge m.$$

(2) If $B\subset\mathbb{R}^n$ is a Borel set, then

$$\underline{\dim}_{\mathrm{loc}}\mu|_B(x)=\underline{\dim}_{\mathrm{loc}}\mu(x) \text{ and } \overline{\dim}_{\mathrm{loc}}\mu|_B(x)=\overline{\dim}_{\mathrm{loc}}\mu(x)$$

for μ-almost all $x\in B$, and moreover,

$$\underline{d}_{\mu|_B}^m(x)=\underline{d}_\mu^m(x) \text{ and } \overline{d}_{\mu|_B}^m(x)=\overline{d}_\mu^m(x)$$

for μ-almost all $x\in B$. These equalities follow from the density point theorem [25, Corollary 2.14].

(3) Since

$$\underline{\dim}_{\rm loc}\mu(x) = \sup\{s \geq 0 : \int |x-y|^{-s}\, d\mu(y) < \infty\},$$

we have $\int |x-a|^{-m}\, d\mu(a) = \infty$ provided that $\underline{\dim}_{\rm loc}\mu(x) < m$. Moreover,

$$\overline{d}^m_\mu(x) = -\infty \text{ if and only if } \int |x-a|^{-m}\, d\mu(a) = \infty.$$

Lemma 2.2. *Let μ be a Radon measure on $\mathbb{R}^n$ and let $s > 0$. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping such that $c_1|x| \leq |Lx| \leq c_2|x|$ for all $x \in \mathbb{R}^n$. Then for all $x \in \mathbb{R}^n$*

$$(1) \qquad \underline{d}^m_{L_\sharp\mu}(Lx) = \underline{d}^m_\mu(x)$$

$$(2) \qquad \overline{d}^m_{L_\sharp\mu}(Lx) = \overline{d}^m_\mu(x).$$

and

$$(3) \qquad \dim_{\rm A} L_\sharp\mu(Lx) = \dim_{\rm A}\mu(x).$$

Proof. First we prove (1) and (2). Since

$$\begin{aligned} c_2^{-m}\int_{B(x,h/c_2)} |x-a|^{-m}\, d\mu(a) &\leq \int_{B(Lx,h)} |Lx-a|^{-m}\, dL_\sharp\mu(a) \\ &\leq c_1^{-m}\int_{B(x,h/c_1)} |x-a|^{-m}\, d\mu(a), \end{aligned} \tag{2.9}$$

we have for all $0 < h < 1$

$$\begin{aligned} &\frac{\log\big((h/c_1)^m \int_{B(x,h/c_1)} |x-a|^{-m}\, d\mu(a)\big)}{\log(h/c_1) + \log c_1} \\ \leq\, &\frac{\log\big(h^m \int_{B(Lx,h)} |Lx-a|^{-m}\, dL_\sharp\mu(a)\big)}{\log h} \\ \leq\, &\frac{\log\big((h/c_2)^m \int_{B(x,h/c_2)} |x-a|^{-m}\, d\mu(a)\big)}{\log(h/c_2) + \log c_2}. \end{aligned}$$

The equalities follow by letting $h \to 0$. The equality (3) follows from the fact

$$\mu(B(x,h/c_2)) \leq L_\sharp\mu(B(Lx,h)) \leq \mu(B(x,h/c_1)).$$

□

3. Similarities

First we consider dimensions of measures $\mu \cap f_\sharp\nu$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a similarity map, that is, there is $r > 0$ such that $|f(x) - f(y)| = r|x-y|$ for all $x, y \in \mathbb{R}^n$. Let $\mathcal{O}(n)$ be the orthogonal group of linear isometries in $\mathbb{R}^n$ and let θ_n be the Haar measure on $\mathcal{O}(n)$ such that $\theta_n(\mathcal{O}(n)) = 1$. Every f has a unique representation as

$$f = \tau_z \circ g \circ \delta_r,$$

where $\tau_z : \mathbb{R}^n \to \mathbb{R}^n$ is the translation $\tau_z(x) = x + z$, $g \in \mathcal{O}(n)$ and $\delta_r : \mathbb{R}^n \to \mathbb{R}^n$ is the dilation $\delta_r(x) = rx$.

3.1. **Properties of intersection measures.** We need to prove analogues of [24, Lemma 6.5, Theorem 6.6 and Theorem 6.7] in a more general setting, where we consider measures $R_{g\circ\delta_r\sharp}\lambda$ instead of the product $\mu \times (g \circ \delta_r)_\sharp \nu$. Here λ is a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support and

$$R_{g\circ\delta_r} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,\ R_{g\circ\delta_r\sharp}(x, y) = (x, g \circ \delta_r(y)).$$

From (2.5) we get that for all non-negative lower semicontinuous functions φ on $\mathbb{R}^n \times \mathbb{R}^n$ we have

$$\begin{aligned} &\int \varphi\, d(R_{g\circ\delta_r\sharp}\lambda)_{W,(z,-z)/2} \\ &\leq \liminf_{\delta\to 0} (\sqrt{2}\delta)^{-n} \int_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : |S_{g\circ\delta_r}(x,y)-z|\leq\delta\}} \varphi(x, y)\, d\lambda(x, y), \end{aligned} \tag{3.1}$$

where

$$S_{g\circ\delta_r} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ S_{g\circ\delta_r}(x, y) = x - g \circ \delta_r(y).$$

Moreover from (2.6) we get that, if φ is a non-negative Borel function on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\int \varphi\, dR_{g\circ\delta_r\sharp}\lambda < \infty$, then for any Borel set $B \subset W^\perp$

$$\int_B \int \varphi\, d(R_{g\circ\delta_r\sharp}\lambda)_{W,a}\, d\mathcal{H}^n|_{W^\perp}(a) \leq \int_{P^{-1}_{W^\perp}(B)} \varphi\, dR_{g\circ\delta_r\sharp}\lambda \tag{3.2}$$

with equality if $S_{g\circ\delta_r\sharp}\lambda \ll \mathcal{L}^n$.

Note that the following lemmas hold for intersection measures since $R_{g\circ\delta_r\sharp}(\mu\times\nu) = \mu \times (g \circ \delta_r)_\sharp \nu$. Moreover [24, Theorem 6.6] follows from Theorem 3.2, since if $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < s < n$ and $0 < t < n$ with $s + t \geq n$, then $I_{s+t}(\mu \times \nu) < \infty$.

Lemma 3.1. *Let α be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support. Assume that $\int |(a, b)|^{-s}\, d\alpha(a, b) < \infty$ for some $s \geq n$. Then for all $0 < r_1 < r_2 < \infty$*

$$\begin{aligned} &\limsup_{\delta\to 0} \delta^{-n} \int\int_{r_1}^{r_2} \theta_n\{g \in \mathcal{O}(n) : |a - g \circ \delta_r(b)| \leq \delta\}\, d\mathcal{L}^1(r)|(a, b)|^{-s+n}\, d\alpha(a, b) \\ &\leq c \int |(a, b)|^{-s}\, d\alpha(a, b), \end{aligned}$$

where c is a constant depending on n, s, r_1 and r_2.

Proof. The proof is a slight modification of that of [24, Lemma 6.5]. We denote by c_1, c_2 and c_3 constants which may depend on n, r_1 and r_2. Define for all $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$

$$I_\delta(a, b) = \int_{r_1}^{r_2} \theta_n\{g \in \mathcal{O}(n) : |a - g \circ \delta_r(b)| \leq \delta\}\, d\mathcal{L}^1(r).$$

If $r \notin \{r \in [r_1, r_2] : \big||a| - r|b|\big| \leq \delta\}$, then

$$|a - g \circ \delta_r(b)| \geq ||a| - |g \circ \delta_r(b)|| = \big||a| - r|b|\big| > \delta,$$

and thus

$$I_\delta(a,b) = \int_{\{r\in[r_1,r_2]:||a|-r|b||\leq\delta\}} \theta_n\{g \in \mathcal{O}(n) : |a - g \circ \delta_r(b)| \leq \delta\}\, d\mathcal{L}^1(r).$$

Define a Borel set

$$A_\delta = \{(a,b) \in \mathbb{R}^n \times \mathbb{R}^n : r_1|b| - \delta \leq |a| \leq r_2|b| + \delta\}.$$

Then for $(a,b) \notin A_\delta$ we have for all $r \in [r_1, r_2]$ that $|a| - r|b| \geq |a| - r_2|b| > \delta$ or $r|b| - |a| \geq r_1|b| - |a| > \delta$, which means that $\{r \in [r_1, r_2] : \big||a| - r|b|\big| \leq \delta\} = \emptyset$, and therefore

$$\int I_\delta(a,b)|(a,b)|^{-s+n}\, d\alpha(a,b) = \int_{A_\delta} I_\delta(a,b)|(a,b)|^{-s+n}\, d\alpha(a,b).$$

Defining

$$\begin{aligned} A_\delta^1 &= \{(a,b) \in A_\delta : |a| \leq 2\delta\} \\ A_\delta^2 &= \{(a,b) \in A_\delta : r_1|b| \leq 2\delta\} \\ A_\delta^3 &= \{(a,b) \in A_\delta : |a| > 2\delta,\ r_1|b| > 2\delta\}, \end{aligned}$$

we have $A_\delta = A_\delta^1 \cup A_\delta^2 \cup A_\delta^3$.

If $(a,b) \in A_\delta^1$, then $|a| \leq 2\delta$ and $r_1|b| \leq 3\delta$ giving

$$|(a,b)| = \sqrt{|a|^2 + |b|^2} \leq c_1\delta.$$

Since $\int |(a,b)|^{-s}\, d\alpha(a,b) < \infty$ we get

$$\begin{aligned} &\limsup_{\delta\to 0} \delta^{-n} \int_{A_\delta^1} I_\delta(a,b)|(a,b)|^{-s+n}\, d\alpha(a,b) \\ &\leq c_1^n(r_2 - r_1) \limsup_{\delta\to 0} \int_{A_\delta^1} |(a,b)|^{-s}\, d\alpha(a,b) = 0. \end{aligned} \tag{3.3}$$

Similarly, if $(a,b) \in A_\delta^2$, then $r_1|b| \leq 2\delta$ and $|a| \leq (1 + \frac{2r_2}{r_1})\delta$, and thus

$$\limsup_{\delta\to 0} \delta^{-n} \int_{A_\delta^2} I_\delta(a,b)|(a,b)|^{-s+n}\, d\alpha(a,b) = 0. \tag{3.4}$$

Finally, let $(a,b) \in A_\delta^3$. Then $\frac{1}{2}r_1|b| \leq |a| \leq 2r_2|b|$, which implies $|a| \geq c_2|(a,b)|$. Moreover, using [25, Lemma 3.8] we have

$$\begin{aligned} I_\delta(a,b) &\leq c_3\delta^{n-1}|a|^{1-n}\mathcal{L}^1\big(\{r \in [r_1, r_2] : \big||a| - r|b|\big| \leq \delta\}\big) \\ &\leq c_3\delta^{n-1}|a|^{1-n}2\delta|b|^{-1} \\ &\leq 4c_3r_2\delta^n|a|^{-n}. \end{aligned}$$

Thus

$$\begin{aligned}
&\limsup_{\delta\to 0} \delta^{-n} \int_{A_\delta^3} I_\delta(a,b)|(a,b)|^{-s+n}\, d\alpha(a,b)\\
&\le 4c_3 r_2 \limsup_{\delta\to 0} \int_{A_\delta^3} |a|^{-n}|(a,b)|^{-s+n}\, d\alpha(a,b)\\
&\le c \limsup_{\delta\to 0} \int_{A_\delta^3} |(a,b)|^{-s}\, d\alpha(a,b)
\end{aligned}$$

and the lemma follows by combining this with (3.3) and (3.4). □

Now we are ready to prove a modification of [24, Theorem 6.6].

Theorem 3.2. *Let λ be Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support such that $I_n(\lambda) < \infty$. Then*

$$S_{g\circ\delta_r\sharp}\lambda \ll \mathcal{L}^n$$

for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$.

Proof. Let $\alpha = S_\sharp(\lambda\times\lambda)$, where $S : (\mathbb{R}^n)^4 \to \mathbb{R}^n \times \mathbb{R}^n$, $S(a,b,x,y) = (a-x, b-y)$. Then

$$\iint |(u,v)|^{-n}\, d\alpha(u,v) = \iint |(x,y)-(a,b)|^{-n}\, d\lambda(a,b)\, d\lambda(x,y) < \infty.$$

Let $0 < r_1 < r_2 < \infty$. Using the methods of [23, Section 2.5] we can show that the function

$$(z,g,r) \mapsto \liminf_{\delta\to 0} \delta^{-n} S_{g\circ\delta_r\sharp}\lambda(B(z,\delta))$$

is Borel measurable. Thus applying Fatou's lemma, Fubini's theorem and Lemma 3.1 for α and $s = n$, we get

$$\begin{aligned}
&\int_{r_1}^{r_2} \iint \liminf_{\delta\to 0} \delta^{-n} S_{g\circ\delta_r\sharp}\lambda(B(z,\delta))\, dS_{g\circ\delta_r\sharp}\lambda(z)\, d\theta_n(g)\, d\mathcal{L}^1(r)\\
&\le \liminf_{\delta\to 0} \delta^{-n} \int \int_{r_1}^{r_2} \theta_n\{g \in \mathcal{O}(n) : |a - x - g\circ\delta_r(b-y)| \le \delta\}\\
&\qquad \times\, d\mathcal{L}^1(r)\, d(\lambda\times\lambda)(a,b,x,y)\\
&= \liminf_{\delta\to 0} \delta^{-n} \int \int_{r_1}^{r_2} \theta_n\{g \in \mathcal{O}(n) : |u - g\circ\delta_r(v)| \le \delta\}\, d\mathcal{L}^1(r)\, d\alpha(u,v)\\
&\le c \int |(u,v)|^{-n}\, d\alpha(u,v) < \infty,
\end{aligned}$$

where c depends on n, r_1 and r_2.

Thus it follows that for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$

$$\liminf_{\delta\to 0} \delta^{-n} S_{g\circ\delta_r\sharp}\lambda(B(z,\delta)) < \infty$$

for $S_{g\circ\delta_r\sharp}\lambda$-almost all $z \in \mathbb{R}^n$. Then [25, Theorem 2.12(3)] implies that for such $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ we have $S_{g\circ\delta_r\sharp}\lambda \ll \mathcal{L}^n$. □

By a simple approximation we only need to assume the following local energy condition.

Corollary 3.3. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support such that $\int |(x,y)-(a,b)|^{-n}\,d\lambda(a,b) < \infty$ for λ-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then*

$$S_{g\circ\delta_r\sharp}\lambda \ll \mathcal{L}^n$$

for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$.

Proof. For $j = 1,2,\dots$ define Borel sets A_j by

$$A_j = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \int |(x,y)-(a,b)|^{-n}\,d\lambda(a,b) < j\}.$$

Now

$$I_n(\lambda|_{A_j}) < \infty$$

and so Theorem 3.2 implies that for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$

$$S_{g\circ\delta_r\sharp}(\lambda|_{A_j}) \ll \mathcal{L}^n.$$

Since

$$\lim_{j\to\infty} \lambda((\mathbb{R}^n \times \mathbb{R}^n)\setminus A_j) = \lambda((\mathbb{R}^n \times \mathbb{R}^n)\setminus \bigcup_j A_j) = 0,$$

we get

$$S_{g\circ\delta_r\sharp}(\lambda)(B) = \lim_{j\to\infty} S_{g\circ\delta_r\sharp}(\lambda|_{A_j})(B)$$

for all $B \subset \mathbb{R}^n$, and the lemma follows. □

We also need the following modification of [24, Theorem 6.7].

Lemma 3.4. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support. Assume $I_s(\lambda) < \infty$ for some $s > n$. Then for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$*

$$I_{s-n}(R_{g\circ\delta_r\sharp}\lambda)_{W,(z,-z)/2}) < \infty$$

for $\mathcal{H}^n$-almost all $(z,-z)/2 \in W^\perp$.

Proof. Let $0 < r_1 < r_2 < \infty$. We denote by c_1 and c_2 constants which may depend on r_1, r_2, s and n. Letting $w = (z,-z)/2$ and using (2.2), Fatou's lemma and

Fubini's theorem we obtain

$$
\begin{aligned}
&\int_{r_1}^{r_2}\iint I_{s-n}(R_{g\circ\delta_r\sharp}\lambda)_{W,w}\,d\mathcal{H}^n|_{W^\perp}(w)\,d\theta_n(g)\,d\mathcal{L}^1(r)\\
&\le \liminf_{\delta\to 0}(2\delta)^{-n}\int_{r_1}^{r_2}\iiint\int_{W_w(\delta)}|(x,y)-(a,b)|^{-s+n}\\
&\quad\times\ dR_{g\circ\delta_r\sharp}\lambda(a,b)\,d(R_{g\circ\delta_r\sharp}\lambda)_{W,w}(x,y)\,d\mathcal{H}^n|_{W^\perp}(w)\,d\theta_n(g)\,d\mathcal{L}^1(r)\\
&= \liminf_{\delta\to 0}(2\delta)^{-n}\int_{r_1}^{r_2}\iiint_{\{(a,b):d((a,b),W_w)\le\delta\}}\int|(x,y)-(a,b)|^{-s+n}\\
&\quad\times\ d(R_{g\circ\delta_r\sharp}\lambda)_{W,w}(x,y)\,dR_{g\circ\delta_r\sharp}\lambda(a,b)\,d\mathcal{H}^n|_{W^\perp}(w)\,d\theta_n(g)\,d\mathcal{L}^1(r)\\
&= \liminf_{\delta\to 0}(2\delta)^{-n}\int_{r_1}^{r_2}\iiint_{\{w\in W^\perp:d((a,b),W_w)\le\delta\}}\int|(x,y)-(a,b)|^{-s+n}\\
&\quad\times\ d(R_{g\circ\delta_r\sharp}\lambda)_{W,w}(x,y)\,d\mathcal{H}^n|_{W^\perp}(w)\,dR_{g\circ\delta_r\sharp}\lambda(a,b)\,d\theta_n(g)\,d\mathcal{L}^1(r).
\end{aligned}
$$

The measurability of a function $(w,g,r)\mapsto I_{s-n}(R_{g\circ\delta_r\sharp}\lambda)_{W,w}$ can be shown in a similar way as in [23, Lemma 4.2].

If $B=\{w\in W^\perp : d((a,b),W_w)\le\delta\}$, then

$$
\begin{aligned}
P_{W^\perp}^{-1}(B) &= \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : |P_{W^\perp}((a,b)-(x,y))|\le\delta\}\\
&= \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : |a-x-(b-y)|\le\sqrt{2}\delta\}.
\end{aligned}
$$

So by Theorem 3.2 and (3.2)

$$
\begin{aligned}
&\int_{r_1}^{r_2}\iint I_{s-n}(R_{g\circ\delta_r\sharp}\lambda)_{W,w}\,d\mathcal{H}^n|_{W^\perp}(w)\,d\theta_n(g)\,d\mathcal{L}^1(r)\\
&\le \liminf_{\delta\to 0}(2\delta)^{-n}\int_{r_1}^{r_2}\iiint_{\{(x,y):|a-x-(b-y)|\le\sqrt{2}\delta\}}|(x,y)-(a,b)|^{-s+n}\\
&\quad\times\ dR_{g\circ\delta_r\sharp}\lambda(x,y)\,dR_{g\circ\delta_r\sharp}\lambda(a,b)\,d\theta_n(g)\,d\mathcal{L}^1(r)\\
&= \liminf_{\delta\to 0}(2\delta)^{-n}\int_{r_1}^{r_2}\iiint_{\{(x,y):|a-x-g\circ\delta_r(b-y)|\le\sqrt{2}\delta\}}|(x,g\circ\delta_r(y))-(a,g\circ\delta_r(b))|^{-s+n}\\
&\quad\times\ d\lambda(x,y)\,d\lambda(a,b)\,d\theta_n(g)\,d\mathcal{L}^1(r).
\end{aligned}
$$

Thus using the fact

$$
|(a,g\circ\delta_r(b))|\ge c_1|(a,b)|,\ \text{ if } r\in[r_1,r_2]
$$

and Fubini's theorem, and then applying Lemma 3.1 for the measure $\alpha = S_\sharp(\lambda \times \lambda)$, where $S : (\mathbb{R}^n)^4 \to \mathbb{R}^n$ is as in the proof of Theorem 3.2, we get

$$\begin{aligned}
&\int_{r_1}^{r_2} \iint I_{s-n}((R_{g\circ\delta_r\sharp}\lambda)_{W,w})\, d\mathcal{H}^n|_{W^\perp}(w)\, d\theta_n(g)\, d\mathcal{L}^1(r) \\
&\le c_1^{-s+n} \liminf_{\delta\to 0} (2\delta)^{-n} \int_{r_1}^{r_2} \iiint_{\{(x,y):|a-x-g\circ\delta_r(b-y)|\le\sqrt{2}\delta\}} |(x,y)-(a,b)|^{-s+n} \\
&\qquad \times\, d\lambda(x,y)\, d\lambda(a,b)\, d\theta_n(g)\, d\mathcal{L}^1(r) \\
&= c_1^{-s+n} (\sqrt{2})^{-n} \liminf_{\delta\to 0} \delta^{-n} \iiint_{r_1}^{r_2} \theta_n\{g \in \mathcal{O}(n) : |a-x-g\circ\delta_r(b-y)| \le \delta\} \\
&\qquad \times |(a-x, b-y)|^{-s+n}\, d\mathcal{L}^1(r)\, d\lambda(x,y)\, d\lambda(a,b) \\
&\le c_2 \int |(a,b)-(x,y)|^{-s}\, d\lambda(x,y)\, d\lambda(a,b) < \infty.
\end{aligned}$$

Thus for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times [r_1, r_2]$

$$I_{s-n}((R_{g\circ\delta_r\sharp}\lambda)_{W,w}) < \infty$$

for $\mathcal{H}^n|_{W^\perp}$-almost all $w \in W^\perp$, and the result follows. □

3.2. **Local dimensions of intersection measures.** In this section we consider local dimensions of intersection measures. We need a lemma whose proof is a modification of the proof of [18, Lemma 5.4].

Lemma 3.5. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support. Fix $0 < r_1 < r_2 < \infty$. If $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ is such that $\int |(x,y)-(a,b)|^{-n}\, d\lambda(a,b) < \infty$, then*

$$\begin{aligned}
&\int_{r_1}^{r_2} \int \pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](B(x,h))\, d\theta_n(g)\, d\mathcal{L}^1(r) \\
&\le c \int_{B((x,y),\tilde{c}h)} |(x,y)-(a,b)|^{-n}\, d\lambda(a,b),
\end{aligned}$$

where c and $\tilde{c}$ are constants depending only on n, r_1 and r_2.

Proof. We denote by $c_1, \dots, c_4$ constants which may depend on n, r_1 and r_2. Let φ be the characteristic function of the open ball with centre x and radius $2h$. Using (3.1) for $\varphi \circ \pi$, Fatou's lemma and Fubini's theorem we have

$$\begin{aligned}
&\int_{r_1}^{r_2} \int \pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](B(x,h))\, d\theta_n(g)\, d\mathcal{L}^1(r) \\
&\le \int_{r_1}^{r_2} \int \liminf_{\delta\to 0} (\sqrt{2}\delta)^{-n} \int_{\{(a,b):|a-x-g\circ\delta_r(b-y)|\le\delta\}} \varphi(a)\, d\lambda(a,b)\, d\theta_n(g)\, d\mathcal{L}^1(r) \\
&\le \liminf_{\delta\to 0} \delta^{-n} \int_{B(x,2h)\times\mathbb{R}^n} I_\delta(a,b)\, d\lambda(a,b),
\end{aligned}$$

where

$$I_\delta(a,b) = \int_{\{r\in[r_1,r_2]:||a-x|-r|b-y||\le\delta\}} \theta_n\{g\in\mathcal{O}(n) : |a-x-g\circ\delta_r(b-y)|\le\delta\}\, d\mathcal{L}^1(r).$$

As in the proof of Lemma 3.1 we get

$$\begin{aligned}&\int_{r_1}^{r_2}\int \pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})](B(x,h))\, d\theta_n(g)\, d\mathcal{L}^1(r)\\ &\le \liminf_{\delta\to 0}\delta^{-n}\int_{(B(x,2h)\times\mathbb{R}^n)\cap B_\delta^3} I_\delta(a,b)\, d\lambda(a,b),\end{aligned}$$

where

$$B_\delta = \{(a,b)\in\mathbb{R}^n\times\mathbb{R}^n : r_1|b-y|-\delta\le|a-x|\le r_2|b-y|+\delta\}$$

and

$$B_\delta^3 = \{(a,b)\in B_\delta : |a-x|>2\delta,\ r_1|b-y|>2\delta\}.$$

Since for $(a,b)\in B_\delta^3$ we have by [25, Lemma 3.8]

$$I_\delta(a,b)\le 4c_1r_2\delta^n|a-x|^{-n}$$

and

$$|a-x|\ge c_2|(a,b)-(x,y)|$$

we get

$$\begin{aligned}&\int_{r_1}^{r_2}\int \pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})](B(x,h))\, d\theta_n(g)\, d\mathcal{L}^1(r)\\ &\le \limsup_{\delta\to 0} 4c_1r_2\int_{(B(x,2h)\times\mathbb{R}^n)\cap B_\delta^3}|a-x|^{-n}\, d\lambda(a,b)\\ &\le \limsup_{\delta\to 0} c_3\int_{B(x,2h)\times\mathbb{R}^n\cap B_\delta^3}|(a,b)-(x,y)|^{-n}\, d\lambda(a,b)\\ &\le c_3\int_{B((x,y),c_4h)}|(x,y)-(a,b)|^{-n}\, d\lambda(a,b).\end{aligned}$$

The last inequality follows from the fact that $(B(x,2h)\times\mathbb{R}^n)\cap B_\delta^3\subset B(x,2h)\times B(y,4h/r_1)$. □

Now we can prove an analogue of [7, Proposition 4.1].

Theorem 3.6. *Let λ be a Radon measure on $\mathbb{R}^n\times\mathbb{R}^n$ with compact support such that $\int|(x,y)-(a,b)|^{-n}\, d\lambda(a,b)<\infty$ for λ-almost all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$. Then for $\theta_n\times\mathcal{L}^1$-almost all $(g,r)\in\mathcal{O}(n)\times(0,\infty)$*

$$\underline{\dim}_{\mathrm{loc}}\pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](x)\ge\underline{\dim}_{\mathrm{loc}}\lambda(x,y)-n=\underline{d}_\lambda^n(x,y)-n$$

and

$$\overline{\dim}_{\mathrm{loc}}\pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](x) \geq \overline{d}^n_\lambda(x,y) - n$$

for λ-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. Let c and $\tilde{c}$ be as in Lemma 3.5. Using Lemma 3.5 we get for λ-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and for all $\varepsilon > 0$

$$\theta_n \times \mathcal{L}^1\{(g,r) \in \mathcal{O}(n) \times [r_1,r_2] : \pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](B(x,h)) > ch^{-n-\varepsilon}\lambda * \psi^n_{\tilde{c}h}(x,y)\} \leq h^\varepsilon.$$

Defining Borel sets

$$A_k = \{(g,r) \in \mathcal{O}(n) \times [r_1,r_2] : \pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](B(x,2^{-k})) > c(2^{-k})^{-n-\varepsilon}\lambda * \psi^n_{\tilde{c}2^{-k}}(x,y)\}$$

and then applying the Borel-Cantelli lemma to A_k we get for $\lambda \times \theta_n \times \mathcal{L}^1$-almost all $(x,y,g,r) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{O}(n) \times [r_1,r_2]$ that

$$\pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](B(x,2^{-k})) \leq c(2^{-k})^{-n-\varepsilon}\lambda * \psi^n_{\tilde{c}2^{-k}}(x,y)$$

for all sufficiently large k. Clearly this holds for sufficiently small h. Thus for $\lambda \times \theta_n \times \mathcal{L}^1$-almost all $(x,y,g,r) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{O}(n) \times [r_1,r_2]$

$$\pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](B(x,h)) \leq ch^{-n-\varepsilon}\lambda * \psi^n_{\tilde{c}h}(x,y)$$

for all sufficiently small h. Now the theorem follows from the definition of local dimensions and from Remark 2.1(1). □

As a corollary we get a result for intersection measures.

Corollary 3.7. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports such that $\int |(x,y)-(a,b)|^{-n}\,d(\mu\times\nu)(a,b) < \infty$ for $\mu\times\nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$*

$$\underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp\nu)(x) \geq \underline{\dim}_{\mathrm{loc}}(\mu\times\nu)(x,y) - n = \underline{d}^n_{\mu\times\nu}(x,y) - n$$

and

$$\overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp\nu)(x) \geq \overline{d}^n_{\mu\times\nu}(x,y) - n$$

for $\mu\times\nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Next we show that Theorem 3.6 holds without assuming the local energy condition. A decomposition of a Radon measure into absolutely continuous and singular parts from [15] is used for this purpose. For all $g \in \mathcal{O}(n)$ and $r \in (0,\infty)$, define

$$E_{g,r} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : (R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2} \text{ is defined}\}.$$

Remark 3.8. If $\int |(x,y)-(a,b)|^{-n}\,d\lambda(a,b) < \infty$ for λ-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, then the measures $(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}$ are defined for $\lambda \times \theta_n \times \mathcal{L}^1$-almost all $(x,y,g,r) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{O}(n) \times (0,\infty)$. This follows from Corollary 3.3 as in [24, Lemma 4.6].

Theorem 3.9. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support. Then for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$*

$$\underline{\dim}_{\rm loc}\pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](x) \geq \underline{\dim}_{\rm loc}\lambda(x,y) - n \geq \underline{d}^n_\lambda(x,y) - n$$

and

$$\overline{\dim}_{\rm loc}\pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](x) \geq \overline{d}^n_\lambda(x,y) - n$$

for λ-almost all $(x,y) \in E_{g,r}$ and moreover $\overline{d}^n_\lambda(x,y) = -\infty$ for λ-almost all $(x,y) \in (\mathbb{R}^n \times \mathbb{R}^n)\backslash E_{g,r}$.

Proof. Since the map $\pi : W_{(z,-z)/2} \to \mathbb{R}^n$, $\pi(x,y) = x$ is bi-Lipschitz, it's enough to consider local dimensions of sliced measures $(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}$. We prove the second inequality. The proof of the first one is similar: one has to use the first inequality in Theorem 3.6 instead of the second one, and the fact that $\underline{\dim}_{\rm loc}\lambda(x,y) - n \geq \underline{d}^n_\lambda(x,y) - n$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Defining a Borel set

$$B = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \int |(x,y) - (a,b)|^{-n}\, d\lambda(a,b) < \infty\}$$

we have by Theorem 3.6 and Remark 2.1(2) for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ and for λ-almost all $(x,y) \in B$

$$\begin{aligned} &\overline{\dim}_{\rm loc}((R_{g\circ\delta_r\sharp}(\lambda|_B))_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})(x, g\circ\delta_r(y)) \\ &= \overline{\dim}_{\rm loc}\pi_\sharp[(R_{g\circ\delta_r\sharp}(\lambda|_B))_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](x) \\ &\geq \overline{d}^n_{\lambda|_B}(x,y) - n \\ &= \overline{d}^n_\lambda(x,y) - n. \end{aligned} \tag{3.5}$$

Moreover,

$$R_{g\circ\delta_r\sharp}(\lambda|_B) = (R_{g\circ\delta_r\sharp}\lambda)|_{B_{g,r}}, \tag{3.6}$$

where $B_{g,r} = R_{g\circ\delta_r}(B)$. Corollary 3.3 gives for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ that $S_{g\circ\delta_r\sharp}(\lambda|_B) \ll \mathcal{L}^n$, which implies

$$P_{W^\perp\sharp}((R_{g\circ\delta_r\sharp}\lambda)|_{B_{g,r}}) \ll \mathcal{H}^n\big|_{W^\perp} \tag{3.7}$$

for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$.

Fix $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ such that (3.5) and (3.7) hold and decompose $\tilde{\lambda} := R_{g\circ\delta_r\sharp}\lambda$ into

$$\tilde{\lambda} = \tilde{\lambda}^W_{\rm abs} + \tilde{\lambda}^W_{\rm sing},$$

where

$$\begin{aligned} &P_{W^\perp\sharp}(\tilde{\lambda}^W_{\rm sing}) \perp \mathcal{H}^n|_{W^\perp}, \\ &P_{W^\perp\sharp}(\tilde{\lambda}^W_{\rm abs}) \ll \mathcal{H}^n|_{W^\perp} \qquad \text{and} \\ &\tilde{\lambda}(A) = \tilde{\lambda}^W_{\rm sing}(A) + \tilde{\lambda}^W_{\rm abs}(A) \end{aligned} \tag{3.8}$$

for all Borel sets $A \subset \mathbb{R}^n \times \mathbb{R}^n$, see [15, Lemma 2.4]. Now by (3.8) and (3.7)

$$\tilde{\lambda}^W_{\text{sing}}(B_{g,r}) = ((R_{g\circ\delta_r\sharp}\lambda)|_{B_{g,r}})^W_{\text{sing}}(B_{g,r}) = 0. \tag{3.9}$$

Moreover, $(\tilde{\lambda}^W_{\text{sing}})_{W,(z,-z)/2} = 0$ for $\mathcal{L}^n$-almost all $z \in \mathbb{R}^n$ by [15, Proposition 2.5(1)]. Thus using also (2.4) we get

$$\begin{aligned}(\tilde{\lambda}|_{B_{g,r}})_{W,(z,-z)/2} &= (\tilde{\lambda}^W_{\text{abs}}|_{B_{g,r}})_{W,(z,-z)/2} \\ &= (\tilde{\lambda}^W_{\text{sing}})_{W,(z,-z)/2}|_{B_{g,r}} + (\tilde{\lambda}^W_{\text{abs}})_{W,(z,-z)/2}|_{B_{g,r}} \\ &= (\tilde{\lambda}^W_{\text{abs}} + \tilde{\lambda}^W_{\text{sing}})_{W,(z,-z)/2}|_{B_{g,r}} \\ &= \tilde{\lambda}_{W,(z,-z)/2}|_{B_{g,r}}\end{aligned} \tag{3.10}$$

for $\mathcal{L}^n$-almost all $z \in \mathbb{R}^n$. Combining this with (3.5), (3.6) and (3.7) we get

$$\overline{\dim}_{\text{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}\big|_{B_{g,r}})(x, g\circ\delta_r(y)) \geq \overline{d}^n_\lambda(x,y) - n$$

for λ-almost all $(x,y) \in B$.

Remark 2.1(2) gives for $\mathcal{L}^n$-almost all $z \in \mathbb{R}^n$

$$\overline{\dim}_{\text{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(z,-z)/2}\big|_{B_{g,r}})(x,y) = \overline{\dim}_{\text{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(z,-z)/2})(x,y) \tag{3.11}$$

for $(R_{g\circ\delta_r\sharp}\lambda)_{W,(z,-z)/2}$-almost all $(x,y) \in B_{g,r}$. Moreover, using (3.7), (3.2) and (3.10) we get for all Borel sets $F \subset B_{g,r}$

$$R_{g\circ\delta_r\sharp}\lambda(F) = \int (R_{g\circ\delta_r\sharp}\lambda)_{W,a}(F)\, d\mathcal{H}^n|_{W^\perp}(a). \tag{3.12}$$

Now defining Borel sets

$$\begin{aligned}F_1 = \{(x,y) \in B : &\overline{\dim}_{\text{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}|_{B_{g,r}})(x, g\circ\delta_r(y)) \\ &\neq \overline{\dim}_{\text{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})(x, g\circ\delta_r(y))\}\end{aligned}$$

and

$$\begin{aligned}F_2 = \{(x,y) \in B_{g,r} : &\overline{\dim}_{\text{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-y,y-x)/2}|_{B_{g,r}})(x,y) \\ &\neq \overline{\dim}_{\text{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-y,y-x)/2})(x,y)\}\end{aligned}$$

we have by (3.11) for $\mathcal{L}^n$-almost all $z \in \mathbb{R}^n$

$$(R_{g\circ\delta_r\sharp}\lambda)_{W,(z-z)/2}(F_2) = 0.$$

Then (3.12) gives

$$\lambda(F_1) = R_{g\circ\delta_r\sharp}\lambda(F_2) = 0.$$

Thus

$$\overline{\dim}_{\text{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})(x, g\circ\delta_r(y)) \geq \overline{d}^n_\lambda(x,y) - n \tag{3.13}$$

for λ-almost all $(x,y) \in B$. By Remark 2.1(3) we have $\overline{d}^n_\lambda(x,y) = -\infty$ for all $(x,y) \notin B$. Thus the inequality (3.13) holds for λ-almost all $(x,y) \in E_{g,r}$.

For the last claim define

$$E_1 = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : (R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2} \text{ is not defined} \\ \text{and } \overline{d}^n_\lambda(x,y) > -\infty\}$$

and

$$E_2 = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : (R_{g\circ\delta_r\sharp}\lambda)_{W,(x-y,y-x)/2} \text{ is not defined} \\ \text{and } \overline{d}^n_{R_{g\circ\delta_r\sharp}\lambda}(x,y) > -\infty\}$$

Then by Lemma 2.2(2)

$$\overline{d}^n_\lambda(x,y) = \overline{d}^n_{R_{g\circ\delta_r\sharp}\lambda}(x,(g\circ\delta_r)(y))$$

and

$$\overline{d}^n_{R_{g\circ\delta_r\sharp}\lambda}(x,y) = \overline{d}^n_\lambda(x,(g\circ\delta_r)^{-1}(y))$$

and so Remark 2.1(3) gives

$$\lambda(E_1) = R_{g\circ\delta_r\sharp}\lambda(E_2) = R_{g\circ\delta_r\sharp}\lambda(E_2 \cap B_{g,r}).$$

Now applying [15, Corollary 2.6] to $R_{g\circ\delta_r\sharp}\lambda$ and using (3.9) yields

$$R_{g\circ\delta_r\sharp}\lambda(E_2 \cap B_{g,r}) = 0.$$

□

As an immediate consequence we get:

Corollary 3.10. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports. Then for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$*

$$\underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp\nu)(x) \geq \underline{\dim}_{\mathrm{loc}}(\mu \times \nu)(x,y) - n \geq \underline{d}^n_{\mu\times\nu}(x,y) - n$$

and

$$\overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp\nu)(x) \geq \overline{d}^n_{\mu\times\nu}(x,y) - n$$

for $\mu \times \nu$-almost all $(x,y) \in E_{g,r}$ and $\overline{d}^n_{\mu\times\nu}(x,y) = -\infty$ for $\mu\times\nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus E_{g,r}$, where

$$E_{g,r} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp\nu \textit{ is defined}\}.$$

Next we consider the validity of the opposite inequalities in Theorem 3.6. The proof is based on results concerning dimensions of sliced measures. For the proof of the lower local dimension we will show that the upper bound of lower local dimensions of sliced measures $\mu_{V,x}$ (see [15, Theorem 2.11]) holds for all planes for which $P_{V^\perp\sharp}\mu \ll \mathcal{H}^m$. The idea of the proof is from [20, Theorem 2.8] where the same question for the average dimension was considered.

Lemma 3.11. *Let μ be a Radon measure on $\mathbb{R}^n$ with compact support such that* $\dim_{\mathrm{H}} \mu > m$*. If $V \in G(n, n-m)$ is such that $P_{V^\perp \sharp}\mu \ll \mathcal{H}^m$, then*

$$\underline{\dim}_{\mathrm{loc}}\mu_{V,x}(x) \le \underline{\dim}_{\mathrm{loc}}\mu(x) - m$$

for μ-almost all $x \in \mathbb{R}^n$.

Proof. Let $\mathcal{D}_k,\ k = 1, 2, \dots$ be the standard half open disjoint dyadic cubes Q of side-lengths $l(Q) = 2^{-k}$. Denote by $\mathcal{D} = \bigcup_{k=1}^\infty \mathcal{D}_k$. Let $\varepsilon > 0$ and $\eta > 0$. For each $Q \in \mathcal{D}$ define

$$A_Q = \{x \in 2Q : \mu_{V,x}(2Q) < \eta\, \mu(2Q)\, l(Q)^{\varepsilon - m}\},$$

where $2Q$ is the cube centred at the same point as Q and with side-length $2l(Q)$. Denote $A = \bigcup_{Q \in \mathcal{D}} A_Q$. Then by (2.3)

$$\begin{aligned} \mu(A) &\le \sum_{Q\in\mathcal{D}} \mu(A_Q) \le \sum_{Q\in\mathcal{D}} \int_{P_{V^\perp}(A_Q)} \mu_{V,a}(2Q)\, d\mathcal{H}^m(a) \\ &\le \sum_{Q\in\mathcal{D}} \eta\, \mu(2Q)\, l(Q)^{\varepsilon-m}\, \mathcal{H}^m(P_{V^\perp}(A_Q)) \le \eta c_1 \sum_{k=1}^\infty 2^{-k\varepsilon} \sum_{Q\in\mathcal{D}_k} \mu(2Q) \qquad (3.14) \\ &\le \eta\, c_2\, \mu(\mathbb{R}^n) \frac{1}{2^\varepsilon - 1} = c_3 \eta, \end{aligned}$$

where c_1, c_2 and c_3 are constants which may depend on m, n and ε.

Let $x \in \mathbb{R}^n \setminus A$ and $0 < r < 1$. Now there exist $Q \in \mathcal{D}$ and a constant $0 < c_4 < 1$ depending only on n such that

$$B(x, c_4 r) \subset 2Q \subset B(x, r).$$

Hence,

$$\mu_{V,x}(B(x,r)) \ge \mu_{V,x}(2Q) \ge \eta\, \mu(2Q)\, l(Q)^{\varepsilon-m} \ge \eta\, c_5\, \mu(B(x, c_4 r))\, r^{\varepsilon-m},$$

where c_5 depends on n, m and ε. Thus

$$\underline{\dim}_{\mathrm{loc}}\mu_{V,x}(x) \le \underline{\dim}_{\mathrm{loc}}\mu(x) - m + \varepsilon$$

for all $x \in \mathbb{R}^n \setminus A$. Since in (3.14) we may choose η as small as we wish, the claim follows. □

The following modification of [19, Lemma 2.4] and of [18, Lemma 5.6] is needed to make sure that results for upper packing dimensions of sliced measures can be used.

Lemma 3.12. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support. If $I_s(\lambda) < \infty$ for some $s > n$, then for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$ there exists for any $\varepsilon > 0$ a compact set $C_\varepsilon \subset \mathbb{R}^n \times \mathbb{R}^n$ with $R_{g\circ\delta_r\sharp}\lambda((\mathbb{R}^n \times \mathbb{R}^n) \setminus C_\varepsilon) < \varepsilon$ and H_ε such that for $\mathcal{H}^n$-almost all $(z, -z)/2 \in W^\perp$ we have*

$$((R_{g\circ\delta_r\sharp}\lambda)\big|_{C_\varepsilon})_{W,(z,-z)/2}(B((x,y),h)) \le ch^{(s-n)/2}$$

for all $(x,y) \in W_{(z,-z)/2}$ *and* $0 < h \le H_\varepsilon$. *Here* c *is a constant depending only on* s *and* n.

Proof. Lemma 3.4 implies that for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ we have

$$I_{s-n}(R_{g\circ\delta_r\sharp}\lambda)_{W,(z,-z)/2}) < \infty \tag{3.15}$$

for $\mathcal{H}^n$-almost all $(z,-z)/2 \in W^\perp$. Consider $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ such that (3.15) holds and $S_{g\circ\delta_r\sharp}\lambda \ll \mathcal{L}^n$ (by Theorem 3.2 this is true for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$). Then using (3.2) and (3.15) we have for $R_{g\circ\delta_r\sharp}\lambda$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\int |(x,y)-(a,b)|^{n-s}\, d(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-y,y-x)/2}(a,b) < \infty.$$

Let $\varepsilon > 0$. For every $i = 1,2,\dots$ define a Borel set

$$B_i = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \int |(x,y)-(a,b)|^{n-s}\, d(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-y,y-x)/2}(a,b) \le i\}.$$

Since

$$\lim_{i\to\infty} R_{g\circ\delta_r\sharp}\lambda((\mathbb{R}^n \times \mathbb{R}^n)\backslash B_i) = 0$$

we find a compact set $C_\varepsilon \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $R_{g\circ\delta_r\sharp}\lambda((\mathbb{R}^n \times \mathbb{R}^n)\backslash C_\varepsilon) < \varepsilon$ and $C_\varepsilon \subset B_{i_\varepsilon}$ for some i_ε.

Let $H_\varepsilon = i_\varepsilon^{\frac{-2}{s-n}}/9$. Consider $(z,-z)/2 \in W^\perp$ such that both $(R_{g\circ\delta_r\sharp}\lambda)_{W,(z,-z)/2}$ and $(R_{g\circ\delta_r\sharp}\lambda|_{C_\varepsilon})_{W,(z,-z)/2}$ are defined. If $(z,-z)/2 \notin P_{W^\perp}(C_\varepsilon)$, then

$$(R_{g\circ\delta_r\sharp}\lambda|_{C_\varepsilon})_{W,(z,-z)/2}(B((x,y),h)) = 0$$

for all $(x,y) \in W_{(z,-z)/2}$ and $h > 0$. This follows from (2.1) and from the fact that $W_{(z,-z)/2}(\delta) \cap C_\varepsilon = \emptyset$ for all small $\delta > 0$, since C_ε is compact. If $(z,-z)/2 \in P_{W^\perp}(C_\varepsilon)$ and $(x,y) \in W_{(z,-z)/2} \cap C_\varepsilon$, then for any $0 < h \le 3H_\varepsilon$, we have

$$\begin{aligned}
&(R_{g\circ\delta_r\sharp}\lambda|_{C_\varepsilon})_{W,(z,-z)/2}(B((x,y),h)) \\
&\le (R_{g\circ\delta_r\sharp}\lambda)_{W,(x-y,y-x)/2}(B((x,y),h)) \\
&\le h^{s-n} \int_{(B((x,y),h))} |(x,y)-(a,b)|^{n-s}\, d(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-y,y-x)/2}(a,b) \\
&\le h^{s-n} i_\varepsilon \le 3^{(n-s)/2} h^{(s-n)/2}.
\end{aligned} \tag{3.16}$$

If $(z,-z)/2 \in P_{W^\perp}(C_\varepsilon)$ and $(x,y) \notin W_{(z,-z)/2} \cap C_\varepsilon$, then there exists $h_{x,y} > 0$ such that $B((x,y),h) \cap C_\varepsilon \cap W_{(z,-z)/2} = \emptyset$ for all $0 < h < h_{x,y}$ and $B((x,y),h) \cap C_\varepsilon \cap W_{(z,-z)/2} \neq \emptyset$ for all $h \ge h_{x,y}$. If $0 < h < h_{x,y}$, then by (2.1)

$$(R_{g\circ\delta_r\sharp}\lambda|_{C_\varepsilon})_{W,(z,-z)/2}(B((x,y),h)) = 0.$$

If $h_{x,y} \le h \le H_\varepsilon$, then $B((x,y),h) \subset B((a,b),3h)$ for some $(a,b) \in W_{(z,-z)/2} \cap C_\varepsilon$, and (3.16) gives the claim. □

Theorem 3.13. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support. Assume that* $\dim_{\mathrm{H}} \lambda > n$. *Then for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$*

(1) $\underline{\dim}_{\mathrm{loc}} \pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](x) = \underline{\dim}_{\mathrm{loc}}\lambda(x,y) - n = \underline{d}^n_\lambda(x,y) - n$

and

(2) $\overline{\dim}_{\mathrm{loc}} \pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](x) = \overline{d}^n_\lambda(x,y) - n$

for λ-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. The lower bounds for local dimensions follow in both cases from Theorem 3.6. In order to prove the upper bounds, it is enough to consider sliced measures $(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}$. Moreover, in both (1) and (2) we have

$$P_{W^\perp\sharp}(R_{g\circ\delta_r\sharp}\lambda) \ll \mathcal{H}^n\big|_{W^\perp} \tag{3.17}$$

for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ by Remark 2.1(3) and Corollary 3.3.

First we prove the upper bound in (1). Since $\dim_{\mathrm{H}}(R_{g\circ\delta_r\sharp}\lambda) = \dim_{\mathrm{H}}\lambda$ we may by (3.17) for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ apply Lemma 3.11 to $R_{g\circ\delta_r\sharp}\lambda$ and W. Thus by Lemma 2.2 and Remark 2.1(1)

$$\begin{aligned} 0 = R_{g\circ\delta_r\sharp}\lambda\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : {}& \underline{\dim}_{\mathrm{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-y,y-x)/2})(x,y) \\ &> \underline{\dim}_{\mathrm{loc}}(R_{g\circ\delta_r\sharp}\lambda)(x,y) - n\} \\ = \lambda\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : {}& \underline{\dim}_{\mathrm{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})(x, g\circ\delta_r(y)) \\ &> \underline{\dim}_{\mathrm{loc}}(R_{g\circ\delta_r\sharp}\lambda)(x, g\circ\delta_r(y)) - n\} \\ = \lambda\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : {}& \underline{\dim}_{\mathrm{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})(x, g\circ\delta_r(y)) \\ &> \underline{\dim}_{\mathrm{loc}}\lambda(x,y) - n\}. \end{aligned}$$

Now we prove the upper bound in (2). Assume first that $I_s(\lambda) < \infty$ for some $s > n$. Then we can use results for packing dimension of sliced measures. It follows from (3.17) that measures $\pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})]$ are defined for $\lambda \times \theta_n \times \mathcal{L}^1$-almost all $(x,y,g,r) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{O}(n) \times (0,\infty)$ (see Remark 3.8).

Assume to the contrary that there are $t_1, t_2 \in \mathbb{R}$ so that

$$\begin{aligned} 0 < \lambda \times \theta_n \times \mathcal{L}^1\{&(x,y,g,r) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{O}(n) \times (0,\infty) : \\ &\overline{\dim}_{\mathrm{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})(x, g\circ\delta_r(y)) > t_1 > t_2 > \overline{d}^n_\lambda(x,y) - n\}. \end{aligned}$$

By Fubini's theorem there exists a Borel set B such that

$$B \subset \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \overline{d}^n_\lambda(x,y) - n < t_2\}$$

with $\lambda(B) > 0$ and for all $(x,y) \in B$ we have

$$\begin{aligned} 0 < \theta_n \times \mathcal{L}^1\{&(g,r) \in \mathcal{O}(n) \times (0,\infty) : \\ &\overline{\dim}_{\mathrm{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})(x) > t_1\}. \end{aligned} \tag{3.18}$$

For every $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ define

$$\lambda_{g,r} = R_{g\circ\delta_r\sharp}(\lambda|_B) = (R_{g\circ\delta_r\sharp}\lambda)|_{B_{g,r}},$$

where $B_{g,r} = R_{g\circ\delta_r}(B)$.

Consider $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ such that both (3.17) and Lemma 3.12 hold. In the proof of [19, Theorem 6.4] it is proved that if k and p are integers with $0 < p < k$ and λ is a Radon measure on $\mathbb{R}^k$ with compact support and $I_{p+d}(\lambda) < \infty$ for some $d > 0$, then

$$\mathcal{H}^p\text{-}\operatorname*{ess\,sup}_{a\in V^\perp} \dim_{\mathrm{p}}^* \lambda_{V,a} \le \lambda\text{-}\operatorname*{ess\,sup}_{x\in\mathbb{R}^k} \overline{d}_\lambda^p(x) - p \tag{3.19}$$

provided that $V \in G(k, k-p)$ is such that [19, Lemma 2.4] holds and $P_{V^\perp\sharp}\lambda \ll \mathcal{H}^p\big|_{V^\perp}$. Now $\lambda_{g,r}$ and W satisfy these assumptions. In other words, we have for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$

$$\mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \dim_{\mathrm{p}}^*(\lambda_{g,r})_{W,(z,-z)/2} \le \lambda_{g,r}\text{-}\operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \overline{d}^n_{\lambda_{g,r}}(x,y) - n. \tag{3.20}$$

On the other hand, using Remark 2.1(2) and Lemma 2.2 and the definition of set B we get for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$

$$\begin{aligned} &\lambda_{g,r}\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \overline{d}^n_{\lambda_{g,r}}(x,y) - n > t_2\} \\ &= \lambda\{(x,y) \in B : \overline{d}^n_{(R_{g\circ\delta_r\sharp}\lambda)|_{B_{g,r}}}(x, g\circ\delta_r(y)) - n > t_2\} \\ &= \lambda\{(x,y) \in B : \overline{d}^n_\lambda(x,y) - n > t_2\} = 0. \end{aligned} \tag{3.21}$$

Thus $\lambda_{g,r}\text{-}\operatorname{ess\,sup}_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \overline{d}^n_{\lambda_{g,r}}(x,y) - n \le t_2$.

Moreover, by (3.18), Remark 2.1(2) and (2.4) we have for $\theta_n \times \mathcal{L}^1$-positively many $(g,r) \in \mathcal{O}(n) \times (0,\infty)$

$$\begin{aligned} 0 &< \lambda\{(x,y) \in B : \overline{\dim}_{\mathrm{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2})(x, g\circ\delta_r(y)) > t_1\} \\ &= \lambda\{(x,y) \in B : \overline{\dim}_{\mathrm{loc}}((R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}|_{B_{g,r}})(x, g\circ\delta_r(y)) > t_1\} \\ &= \lambda_{g,r}\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \overline{\dim}_{\mathrm{loc}}((\lambda_{g,r})_{W,(x-y,y-x)/2})(x,y) > t_1\} \end{aligned}$$

This implies by Corollary 3.3 and (3.2)

$$\begin{aligned} 0 < \mathcal{L}^n\{z \in \mathbb{R}^n : (\lambda_{g,r})_{W,(z,-z)/2}\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n :& \\ \overline{\dim}_{\mathrm{loc}}((\lambda_{g,r})_{W,(z,-z)/2})(x,y) > t_1\} > 0\}.& \end{aligned} \tag{3.22}$$

Therefore

$$\mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \dim_{\mathrm{p}}^*(\lambda_{g,r})_{W,(z,-z)/2} \ge t_1 > t_2 \ge \lambda_{g,r}\text{-}\operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \overline{d}^n_{\lambda_{g,r}}(x,y) - n$$

for $\theta_n \times \mathcal{L}^1$-positively many $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ giving a contradiction with (3.20). Thus the claim holds, if we assume that $I_s(\lambda) < \infty$ for some $s > n$.

Now assume that $\dim_{\mathrm{H}} \lambda > n$. Let $n < s < \dim_{\mathrm{H}} \lambda$ and define for every $i = 1, 2, \dots$

$$B_i = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \int |(x,y) - (a,b)|^{-s}\, d\lambda(a,b) < i\}.$$

Then

$$\lim_{i\to\infty}\lambda((\mathbb{R}^n\times\mathbb{R}^n)\setminus B_i)=\lambda((\mathbb{R}^n\times\mathbb{R}^n)\setminus\bigcup_i B_i)=0. \tag{3.23}$$

Since $I_s(\lambda|_{B_i})<\infty$ we get for $\theta_n\times\mathcal{L}^1$-almost all $(g,r)\in\mathcal{O}(n)\times(0,\infty)$

$$0=\lambda\{(x,y)\in B_i:\\ \overline{\dim}_{\mathrm{loc}}\pi_\sharp[(R_{g\circ\delta_r\sharp}(\lambda|_{B_i}))_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](x)\neq\overline{d}^n_{\lambda|_{B_i}}(x,y)-n\}.$$

Moreover

$$R_{g\circ\delta_r\sharp}(\lambda|_{B_i})=(R_{g\circ\delta_r\sharp}\lambda)|_{R_{g\circ\delta_r}(B_i)}.$$

Using these with (2.4) and Remark 2.1(2) gives

$$0=\lambda\{(x,y)\in B_i:\overline{\dim}_{\mathrm{loc}}\pi_\sharp[(R_{g\circ\delta_r\sharp}\lambda)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}](x)\neq\overline{d}^n_\lambda(x,y)-n\}$$

for $\theta_n\times\mathcal{L}^1$-almost all $(g,r)\in\mathcal{O}(n)\times(0,\infty)$. Now the claim follows by (3.23). □

Remark 3.14. (1) The upper bound for lower local dimensions holds for all $(g,r)\in\mathcal{O}(n)\times(0,\infty)$ for which $P_{W^\perp\sharp}(R_{g\circ\delta_r\sharp}\lambda)\ll\mathcal{H}^n\big|_{W^\perp}$.

(2) In [15] it is shown, that if $I_{m+\varepsilon}(\mu)<\infty$ for some $\varepsilon>0$, then for $\gamma_{n,n-m}$-almost all $(n-m)$-planes V and for μ-almost all $x\in\mathbb{R}^n$

$$\overline{\dim}_{\mathrm{loc}}\mu_{V,x}(x)=\overline{d}^m_\mu(x)-m.$$

Using the methods of the proof of Theorem 3.13 it is possible to show that the equality also holds under the assumption $\dim_{\mathrm{H}}\mu>m$.

Theorem 3.13 implies the following results for intersection measures.

Corollary 3.15. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports. Assume that* $\dim_{\mathrm{H}}(\mu\times\nu)>n$. *Then for $\theta_n\times\mathcal{L}^1$-almost all $(g,r)\in\mathcal{O}(n)\times(0,\infty)$*

(1) $\underline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_{x-g\circ\delta_r(y)}\circ g\circ\delta_r)_\sharp\nu)(x)=\underline{\dim}_{\mathrm{loc}}(\mu\times\nu)(x,y)-n=\underline{d}^n_{\mu\times\nu}(x,y)-n$

and

(2) $\overline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_{x-g\circ\delta_r(y)}\circ g\circ\delta_r)_\sharp\nu)(x)=\overline{d}^n_{\mu\times\nu}(x,y)-n$

for $\mu\times\nu$-almost all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$.

Corollary 3.16. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports. Assume that* $\dim_{\mathrm{H}}(\mu\times\nu)>n$. *Then for $\mathcal{L}^n\times\theta_n\times\mathcal{L}^1$-almost all $(z,g,r)\in\mathbb{R}^n\times\mathcal{O}(n)\times(0,\infty)$*

$$\begin{aligned}(1)\quad \underline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu)(x)&=\underline{\dim}_{\mathrm{loc}}(\mu\times\nu)(x,(\tau_z\circ g\circ\delta_r)^{-1}(x))-n\\&=\underline{d}^n_{\mu\times\nu}(x,(\tau_z\circ g\circ\delta_r)^{-1}(x))-n\end{aligned}$$

and

$$(2)\quad \overline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu)(x)=\overline{d}^n_{\mu\times\nu}(x,(\tau_z\circ g\circ\delta_r)^{-1}(x))-n$$

for $\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu$-almost all $x\in\mathbb{R}^n$.

Proof. We will prove (1). The proof of (2) is similar. By Corollary 3.15(1) and Lemma 2.2 we have for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$

$$\begin{aligned} 0 &= \mu \times \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp \nu)(x) \neq \underline{d}^n_{\mu\times\nu}(x,y) - n\} \\ &= \mu \times (g\circ\delta_r)_\sharp\nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu)(x) \\ &\qquad\qquad \neq \underline{d}^n_{\mu\times(g\circ\delta_r)_\sharp\nu}(x,y) - n\}. \end{aligned}$$

Now $(\mu \times (g\circ\delta_r)_\sharp\nu)_{W,(z,-z)/2}$ is a measure on

$$W_{(z,-z)/2} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x - y = z\}.$$

Thus (3.2) and Lemma 2.2 imply that for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ and for $\mathcal{L}^n$-almost all $z \in \mathbb{R}^n$ we have

$$\begin{aligned} 0 &= (\mu \times (g\circ\delta_r)_\sharp\nu)_{W,(z,-z)/2}\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu)(x) \\ &\qquad\qquad \neq \underline{d}^n_{\mu\times(g\circ\delta_r)_\sharp\nu}(x,y) - n\} \\ &= (\mu \times (g\circ\delta_r)_\sharp\nu)_{W,(z,-z)/2}\{(x,x-z) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{z} \circ g \circ \delta_r)_\sharp \nu)(x) \\ &\qquad\qquad \neq \underline{d}^n_{\mu\times(g\circ\delta_r)_\sharp\nu}(x,x-z) - n\} \\ &= (\mu \times (g\circ\delta_r)_\sharp\nu)_{W,(z,-z)/2}\{(x,x-z) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{z} \circ g \circ \delta_r)_\sharp \nu)(x) \\ &\qquad\qquad \neq \underline{d}^n_{\mu\times\nu}(x,(\tau_z \circ g \circ \delta_r)^{-1}(x)) - n\} \end{aligned}$$

and the claim follows by Remark 2.1(1). □

Remark 3.17. In Corollary 3.15(2) it is not sufficient to assume that $\dim_{\mathrm{p}}(\mu\times\nu) > n$. Thus the upper local dimensions and packing dimensions of intersection measures depend on $\dim_{\mathrm{H}} \mu\times\nu$. Consider example from [20, Remark 3.6]. For $0 < \alpha < \beta < 1$ there are Radon measures μ and ν on $[0,1]$ such that

$$\overline{\dim}_{\mathrm{loc}}(\mu\times\nu)(x,y) \geq \beta + \beta > \alpha + \beta \geq \overline{\dim}_{\mathrm{loc}}\mu(x) + \underline{\dim}_{\mathrm{loc}}\nu(y)$$

for $\mu\times\nu$-almost all $(x,y) \in [0,1]\times[0,1]$. Choosing α and β properly we have $\dim_{\mathrm{p}}(\mu\times\nu) > 1$, but $\underline{\dim}_{\mathrm{loc}}(\mu\times\nu)(x,y) < 1$ in a set of positive $\mu\times\nu$ -measure. Thus Remark 2.1(3) implies that for any such $(x,y) \in [0,1]\times[0,1]$ we have $\overline{d}^1_{\mu\times\nu}(x,y) = -\infty$, which means that $\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp\nu$ is not defined in a set of positive $\mu\times\nu$-measure.

3.3. **Hausdorff and packing dimensions of intersection measures.** Results in Sections 3.1 and 3.2 imply that results concerning Hausdorff and packing dimensions in [18] hold with fewer assumptions. In the case of Hausdorff and upper Hausdorff dimensions we get the following improved versions of [18, Theorem 3.7 and Theorem 4.5].

Theorem 3.18. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports. Assume that $\dim_{\mathrm{H}}(\mu\times\nu) > n$. Then for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n)\times(0,\infty)$*

$$\begin{aligned} \mathcal{L}^n\text{-}\operatorname{ess\,inf}\{\, &\dim_{\mathrm{H}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp\nu) : z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp\nu(\mathbb{R}^n) > 0\} \\ &= \dim_{\mathrm{H}}(\mu\times\nu) - n. \end{aligned}$$

Proof. The proof of [19, Lemma 3.1] shows that for any plane $V \in G(n, n-m)$ such that $P_{V^\perp \sharp}\mu \ll \mathcal{H}^m|_{V^\perp}$ we have

$$\mathcal{H}^m\text{-}\operatorname{ess\,inf}\{\dim_{\mathrm{H}} \mu_{V,a} : a \in V^\perp \text{ with } \mu_{V,a}(\mathbb{R}^n) > 0\} \le \dim_{\mathrm{H}} \mu - m.$$

Since

$$\dim_{\mathrm{H}}(\mu \times (g \circ \delta_r)_\sharp \nu) = \dim_{\mathrm{H}}(\mu \times \nu) > n$$

we may by Corollary 3.3 apply this for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$ to $\mu \times (g \circ \delta_r)_\sharp \nu$ and W to get the upper bound.

For the lower bound define for all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$

$$C_{g,r} = \{z \in \mathbb{R}^n : \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbb{R}^n) > 0\}.$$

It is enough to show that for all $n < t < \dim_{\mathrm{H}}(\mu \times \nu)$

$$\mathcal{L}^n\{z \in C_{g,r} : \dim_{\mathrm{H}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) < t - n\} = 0$$

for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$. Then

$$\mathcal{L}^n\text{-}\operatorname{ess\,inf}\{\dim_{\mathrm{H}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) : z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbb{R}^n) > 0\} \ge t - n$$

for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$, and the result follows by taking a sequence $t_i \nearrow \dim_{\mathrm{H}}(\mu \times \nu)$.

Fix $(g, r) \in \mathcal{O}(n) \times (0, \infty)$ such that $S_{g \circ \delta_r \sharp}(\mu \times \nu) \ll \mathcal{L}^n$ and Corollary 3.15 holds. If we have $\mathcal{L}^n(E_{g,r}) > 0$, where

$$E_{g,r} = \{z \in C_{g,r} : \dim_{\mathrm{H}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) < t - n\},$$

then

$$\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu\{x \in \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu)(x) < t - n\} > 0$$

for $z \in E_{g,r}$, and it follows by using (3.2) that

$$\mu \times (g \circ \delta_r)_\sharp \nu\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu)(x) < t - n\} > 0.$$

Then Corollary 3.15 implies that

$$\begin{aligned}
&\mu \times \nu\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \times \nu)(x, y) < t\} \\
&= \mu \times \nu\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x - g \circ \delta_r(y)} \circ g \circ \delta_r)_\sharp \nu)(x) < t - n\} \\
&= \mu \times (g \circ \delta_r)_\sharp \nu\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu)(x) < t - n\} \\
&> 0.
\end{aligned}$$

Thus $t > \dim_{\mathrm{H}}(\mu \times \nu)$ which is a contradiction. □

Theorem 3.19. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports. Assume that $\dim_{\mathrm{H}}(\mu \times \nu) > n$. Then for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$*

$$\mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z \in \mathbb{R}^n} \dim_{\mathrm{H}}^*(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) = \dim_{\mathrm{H}}^*(\mu \times \nu) - n.$$

Proof. The proof of [19, Lemma 4.1] shows that for any plane $V \in G(n, n-m)$ such that $P_{V^\perp \sharp}\mu \ll \mathcal{H}^m|_{V^\perp}$ we have

$$\mathcal{H}^m\text{- }\operatorname*{ess\,sup}_{a \in V^\perp} \dim_{\mathrm{H}}^* \mu_{V,a} \le \dim_{\mathrm{H}}^* \mu - m.$$

Since

$$\dim_{\mathrm{H}}(\mu \times (g \circ \delta_r)_\sharp \nu) = \dim_{\mathrm{H}}(\mu \times \nu) > n$$

and

$$\dim_{\mathrm{H}}^*(\mu \times (g \circ \delta_r)_\sharp \nu) = \dim_{\mathrm{H}}^*(\mu \times \nu)$$

we may by Corollary 3.3 apply this for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$ to $\mu \times (g \circ \delta_r)_\sharp \nu$ and W to get the upper bound.

For the lower bound let $0 < t < \dim_{\mathrm{H}}^*(\mu \times \nu) - n$. Fix $(g, r) \in \mathcal{O}(n) \times (0, \infty)$ such that $S_{g \circ \delta_r \sharp}(\mu \times \nu) \ll \mathcal{L}^n$ and Corollary 3.15 holds. Then

$$\begin{aligned} &\mu \times (g \circ \delta_r)_\sharp \nu \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu)(x) > t\} \\ &= \mu \times \nu \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x - g \circ \delta_r(y)} \circ g \circ \delta_r)_\sharp \nu)(x) > t\} \\ &= \mu \times \nu \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \underline{\dim}_{\mathrm{loc}}(\mu \times \nu)(x, y) > t + n\} > 0. \end{aligned}$$

Thus Corollary 3.3 and (3.2) imply that

$$\begin{aligned} \mathcal{L}^n\{z \in \mathbb{R}^n : (\mu \times (g \circ \delta_r)_\sharp \nu)_{W,(z,-z)/2}\{(x, x - z) \in \mathbb{R}^n \times \mathbb{R}^n : \\ \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu)(x) > t\} > 0\} > 0 \end{aligned}$$

which gives

$$\begin{aligned} \mathcal{L}^n\{z \in \mathbb{R}^n : \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \{x \in \mathbb{R}^n : \\ \underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu)(x) > t\} > 0\} > 0. \end{aligned}$$

So we have

$$\mathcal{L}^n\text{- }\operatorname*{ess\,sup}_{z \in \mathbb{R}^n} \dim_{\mathrm{H}}^*(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) \ge t$$

for $\theta_n \times \mathcal{L}^1$-almost all $(g, r) \in \mathcal{O}(n) \times (0, \infty)$, and the theorem follows by taking a sequence $t_i \nearrow \dim_{\mathrm{H}}^*(\mu \times \nu) - n$. □

Remark 3.20. In previous theorems it is necessary to consider dimension of product measure instead of dimensions of μ and ν separately. That is, there does not exist a function of $\dim_{\mathrm{H}} \mu$ and $\dim_{\mathrm{H}} \nu$ that could replace the right hand side in Theorem 3.18 and Theorem 3.19. In [20, Example 3.5] there is an example of measures μ and ν on $\mathbb{R}$ such that $\dim_{\mathrm{H}}(\mu \times \nu) > \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu$. Let $\tilde{\mu}$ and $\tilde{\nu}$ be measures on self-similar Cantor sets in $\mathbb{R}$ such that $\dim_{\mathrm{H}} \tilde{\mu} = \dim_{\mathrm{H}} \mu$ and $\dim_{\mathrm{H}} \tilde{\nu} = \dim_{\mathrm{H}} \nu$. Then

$$\dim_{\mathrm{H}} \tilde{\mu} \times \tilde{\nu} = \dim_{\mathrm{H}} \tilde{\mu} + \dim_{\mathrm{H}} \tilde{\nu} < \dim_{\mathrm{H}}(\mu \times \nu).$$

For packing and upper packing dimension we get that [18, Theorem 5.9 and Theorem 6.3] hold under assumption $\dim_{\mathrm{H}}(\mu \times \nu) > n$.

Theorem 3.21. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports such that $\dim_{\mathrm{H}}(\mu \times \nu) > n$. Then for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ we have*

$$\begin{aligned}\mathcal{L}^n\text{-}\operatorname{ess\,inf}\{\, \dim_{\mathrm{p}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) : z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbb{R}^n) > 0\} \\ = \mu \times \nu\text{-} \operatorname*{ess\,inf}_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \overline{d}^n_{\mu\times\nu}(x,y) - n.\end{aligned}$$

Proof. The proof of the lower bound goes as in the proof of Theorem 3.18.

For the upper bound let $t > \mu \times \nu\text{-}\operatorname{ess\,inf}_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \overline{d}^n_{\mu\times\nu}(x,y)$. Fix $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ such that $S_{g\circ\delta_r\sharp}(\mu \times \nu) \ll \mathcal{L}^n$ and Corollary 3.15 holds. Then

$$\begin{aligned}&\mu \times (g \circ \delta_r)_\sharp \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu)(x) < t - n\} \\ &= \mu \times \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp \nu)(x) < t - n\} \\ &= \mu \times \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \overline{d}^n_{\mu\times\nu}(x,y) < t\} > 0.\end{aligned}$$

Thus (3.2) implies that

$$\begin{aligned}0 < \mathcal{L}^n\{z \in \mathbb{R}^n : (\mu \times (g \circ \delta_r)_\sharp \nu)_{W,(z,-z)/2}\{(x, x-z) \in \mathbb{R}^n \times \mathbb{R}^n : \\ \overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu)(x) < t - n\} > 0\}\end{aligned}$$

and further

$$\mathcal{L}^n\{z \in \mathbb{R}^n : \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbb{R}^n) > 0 \text{ and } \dim_{\mathrm{p}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) \le t - n\} > 0.$$

So we have for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$

$$\begin{aligned}&\mathcal{L}^n\text{-}\operatorname{ess\,inf}\{\quad \dim_{\mathrm{p}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) : z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbb{R}^n) > 0\} \\ &\le t - n,\end{aligned}$$

and the claim follows. □

Theorem 3.22. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports such that $\dim_{\mathrm{H}}(\mu \times \nu) > n$. Then for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ we have*

$$\mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \dim^*_{\mathrm{p}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu) = \mu \times \nu\text{-} \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \overline{d}^n_{\mu\times\nu}(x,y) - n.$$

Proof. The lower bound can be proven as in the proof of Theorem 3.19.

For the upper bound let

$$t < \mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \dim^*_{\mathrm{p}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu).$$

Fix $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ such that $S_{g\circ\delta_r\sharp}(\mu \times \nu) \ll \mathcal{L}^n$ and Corollary 3.15 holds. Then

$$\mathcal{L}^n\{z \in \mathbb{R}^n : \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu\{x : \overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu)(x) > t\} > 0\} > 0,$$

which means that

$$\begin{aligned}\mathcal{L}^n\{z \in \mathbb{R}^n : (\mu \times (g \circ \delta_r)_\sharp \nu)_{W,(z,-z)/2}\{(x, x-z) : \\ \overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu)(x) > t\} > 0\} > 0.\end{aligned}$$

Thus (3.2) gives

$$\begin{aligned}0 &< \mu \times (g \circ \delta_r)_\sharp \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \overline{\dim}_{\text{loc}}(\mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu)(x) > t\} \\ &= \mu \times \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \overline{\dim}_{\text{loc}}(\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp \nu)(x) > t\} \\ &= \mu \times \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \overline{d}^n_{\mu\times\nu}(x,y) - n > t\}.\end{aligned}$$

So we have for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$

$$t \leq \mu \times \nu\text{-} \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \overline{d}^n_{\mu\times\nu}(x,y) - n,$$

and thus by taking a sequence $t_i \nearrow \mathcal{L}^n\text{-}\operatorname{ess\,sup}_{z\in\mathbb{R}^n} \dim^*_{\text{p}}(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu)$ the claim follows. □

3.4. **Average dimensions.** Next we will show that the average dimension of intersection measures behaves like the lower local dimension. This improves a result by Llorente [20, Theorem 4.7 and Theorem 4.10].

Theorem 3.23. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports and let $\dim_{\text{H}}(\mu \times \nu) > n$. Then for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$*

$$\dim_{\text{A}}(\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp \nu)(x) = \dim_{\text{A}}(\mu \times \nu)(x,y) - n$$

for $\mu \times \nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. First we prove the upper bound. The proof of [20, Theorem 2.8] shows that for any plane $V \in G(n, n-m)$ such that $P_{V^\perp\sharp}\mu \ll \mathcal{H}^m|_{V^\perp}$ we have

$$\dim_{\text{A}} \mu_{V,x}(x) \leq \dim_{\text{A}} \mu(x) - m$$

for μ-almost all $x \in \mathbb{R}^n$. Since

$$\dim_{\text{H}}(\mu \times (g \circ \delta_r)_\sharp \nu) = \dim_{\text{H}}(\mu \times \nu)$$

we may by Corollary 3.3 apply this for $\theta_n \times \mathcal{L}^1$-almost all $(g,r) \in \mathcal{O}(n) \times (0,\infty)$ to $\mu \times (g \circ \delta_r)_\sharp \nu$ and W. Thus

$$\begin{aligned}&\mu \times (g \circ \delta_r)_\sharp \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \dim_{\text{A}}(\mu \times (g\circ\delta_r)_\sharp \nu)_{W,(x-y,y-x)/2}(x,y) \\ &\quad > \dim_{\text{A}} \mu \times (g \circ \delta_r)_\sharp \nu(x,y) - n\} = 0\end{aligned}$$

and further

$$\begin{aligned}&\mu \times \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \dim_{\text{A}}(\mu \times (g\circ\delta_r)_\sharp \nu)_{W,(x-g\circ\delta_r(y),g\circ\delta_r(y)-x)/2}(x, g \circ \delta_r(y)) \\ &\quad > \dim_{\text{A}} \mu \times (g \circ \delta_r)_\sharp \nu(x, g \circ \delta_r(y)) - n\} = 0,\end{aligned}$$

which gives by Lemma 2.2(3)

$$\begin{aligned}&\mu \times \nu\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \dim_{\text{A}}(\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp \nu)(x) \\ &\quad > \dim_{\text{A}}(\mu \times \nu)(x,y) - n\} = 0.\end{aligned}$$

For the lower bound let

$$0 < s < \dim_{\text{A}}(\mu \times \nu)(x,y) - n. \tag{3.24}$$

We want to show that for $\mu\times\nu\times\theta_n\times\mathcal{L}^1$-almost all $(x,y,g,r)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{O}(n)\times(0,\infty)$

$$D^s_{\mu\cap(\tau_{x-g\circ\delta_r(y)}\circ g\circ\delta_r)_\sharp\nu}(x)=0.$$

Let $0<r_1<r_2<\infty$ and let $c_1,\dots,c_5$ be constants which may depend on r_1,r_2,s and n. Now using Lemma 3.5, Fubini's theorem and a change of variable we get for $\mu\times\nu$-almost all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$

$$\begin{aligned}
&\int_{r_1}^{r_2}\int \mu\cap(\tau_{x-g\circ\delta_r(y)}\circ g\circ\delta_r)_\sharp\nu(B(x,h))\,d\theta_n(g)\,d\mathcal{L}^1(r)\\
&\le c_1\int_{B((x,y),c_2h)}|(x,y)-(a,b)|^{-n}\,d(\mu\times\nu)(a,b)\\
&=c_1\int_0^\infty \mu\times\nu\{(a,b)\in B((x,y),c_2h):|(x,y)-(a,b)|^{-n}\ge u\}\,d\mathcal{L}^1(u)\\
&=c_1(c_2h)^{-n}\mu\times\nu(B((x,y),c_2h)+c_1\int_0^{c_2h}t^{-n-1}\mu\times\nu(B((x,y),t)\,d\mathcal{L}^1(t)
\end{aligned}$$

Since

$$\begin{aligned}
&c_1(c_2h)^{-n}\mu\times\nu(B((x,y),c_2h)\\
&=c_1n\,\mu\times\nu(B((x,y),c_2h)\big[\int_{c_2h}^{2c_2h}t^{-n-1}\,d\mathcal{L}^1(t)+\int_{2c_2h}^{\infty}t^{-n-1}\,d\mathcal{L}^1(t)\big]\\
&=c_1n\int_{c_2h}^{2c_2h}\mu\times\nu(B((x,y),c_2h))t^{-n-1}\,d\mathcal{L}^1(t)+c_1(2c_2h)^{-n}\mu\times\nu(B((x,y),c_2h))
\end{aligned}$$

we get

$$c_1(c_2h)^{-n}\mu\times\nu(B((x,y),c_2h)\le c_3\int_{c_2h}^{2c_2h}\mu\times\nu(B((x,y),t))t^{-n-1}\,d\mathcal{L}^1(t),$$

and thus

$$\begin{aligned}
&\int_{r_1}^{r_2}\int \mu\cap(\tau_{x-g\circ\delta_r(y)}\circ g\circ\delta_r)_\sharp\nu(B(x,h))\,d\theta_n(g)\,d\mathcal{L}^1(r)\\
&\le c_4\int_0^{2c_2h}t^{-n-1}\mu\times\nu(B((x,y),t))\,d\mathcal{L}^1(t).
\end{aligned}\tag{3.25}$$

Then, since $\dim_{\mathrm{H}}(\mu \times \nu) > n$, by using Fatou's lemma, Fubini's theorem,(3.24), (3.25) and [20, Lemma 2.7] we get for $\mu \times \nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\begin{aligned}
&\int_{r_1}^{r_2} \int D^s_{\mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp \nu}(x)\, d\theta_n(g)\, d\mathcal{L}^1(r) \\
&\le \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 p^{-s-1} \int_{r_1}^{r_2} \int \mu \cap (\tau_{x-g\circ\delta_r(y)} \circ g \circ \delta_r)_\sharp \nu(B(x,p)) \\
&\qquad \times \, d\theta_n(g)\, d\mathcal{L}^1(r)\, d\mathcal{L}^1(p) \\
&\le c_4 \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 p^{-s-1} \int_0^{2c_2 p} t^{-n-1} \mu \times \nu(B((x,y),t))\, d\mathcal{L}^1(t)\, d\mathcal{L}^1(p) \\
&= c_4 \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \Big[\int_{2c_2\delta}^{2c_2} \int_{t/2c_2}^1 p^{-s-1} t^{-n-1} \mu \times \nu(B((x,y),t))\, d\mathcal{L}^1(p)\, d\mathcal{L}^1(t) \\
&\qquad + \int_0^{2c_2\delta} \int_\delta^1 p^{-s-1} t^{-n-1} \mu \times \nu(B((x,y),t))\, d\mathcal{L}^1(p)\, d\mathcal{L}^1(t) \Big] \\
&\le c_5 \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \Big[\int_{2c_2\delta}^1 \frac{\mu \times \nu(B((x,y),t))}{t^{s+n+1}}\, d\mathcal{L}^1(t) \\
&\qquad + \int_1^{2c_2} \frac{\mu \times \nu(B((x,y),t))}{t^{s+n+1}}\, d\mathcal{L}^1(t) + \delta^{-s} \int_0^{2c_2\delta} \frac{\mu \times \nu(B((x,y),t))}{t^{n+1}}\, d\mathcal{L}^1(t) \Big] \\
&= c_5 \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \Big[\int_{2c_2\delta}^1 \frac{\mu \times \nu(B((x,y),t))}{t^{s+n+1}}\, d\mathcal{L}^1(t) \\
&\qquad + \delta^{-s} \int_0^{2c_2\delta} \frac{\mu \times \nu(B((x,y),t))}{t^{n+1}}\, d\mathcal{L}^1(t) \Big] \\
&= 0
\end{aligned}$$

□

Corollary 3.24. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports such that $\dim_{\mathrm{H}}(\mu \times \nu) > n$. Then for $\mathcal{L}^n \times \theta_n \times \mathcal{L}^1$-almost all $(z,g,r) \in \mathbb{R}^n \times \mathcal{O}(n) \times (0,\infty)$*

$$\dim_A(\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu)(x) = \dim_A(\mu \times \nu)(x, (\tau_z \circ g \circ \delta_r)^{-1}(x)) - n$$

for $\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu$-almost all $x \in \mathbb{R}^n$.

Proof. As the proof of Corollary 3.16. □

4. Linear maps and continuously differentiable functions

We study intersection measures in the case where similarities are replaced by linear mappings. We only need to consider invertible linear mappings, since almost all linear mappings $L : \mathbb{R}^n \to \mathbb{R}^n$ are invertible.

We denote by $GL(n)$ the group of invertible linear mappings $L : \mathbb{R}^n \to \mathbb{R}^n$. Now every $L \in GL(n)$ has a unique representation, the QR-decomposition, as

$$L = g \circ T,$$

where $g \in \mathcal{O}(n)$ and $T \in T(n)_+$, which is the group of upper triangular matrices with strictly positive diagonal entries, see for example [8, Theorem 1.6.1]. Thus we consider intersection measures

$$\mu \cap (\tau_z \circ g \circ T)_\sharp \nu = \pi_\sharp\big[(\mu \times (g \circ T)_\sharp \nu)_{W,(z,-z)/2}\big].$$

Denote the singular values of L, that is, the lengths of the semiaxes of the image of the unit ball, by $\varrho_1^L, \dots, \varrho_n^L$. Let $\varrho_1^L \geq \dots \geq \varrho_n^L$ and define

$$GL(n)^{r_1,r_2} = \{L \in GL(n) : r_1 \leq \varrho_n^L \text{ and } \varrho_1^L \leq r_2\}$$

and

$$T(n)_+^{r_1,r_2} = \{T \in T(n)_+ : r_1 \leq \varrho_n^T \text{ and } \varrho_1^T \leq r_2\}.$$

By [8, Proposition 5.3.2] there exists a measure α on $GL(n)$ such that for every integrable Borel function $f : GL(n)^{r_1,r_2} \to \mathbb{R}$ we have

$$\int f(L)\, d\alpha(L) = \iint_{T(n)_+} f(g \circ T)\, d\theta_n(g)\, \mathcal{L}^{\frac{n}{2}(n+1)}(T).$$

Moreover, this measure is mutually absolutely continuous with the Haar measure on $GL(n)$ (and with $\mathcal{L}^{n^2}$).

We also consider the case where similarities are replaced by continuously differentiable functions, that is, we consider intersection measures

$$\mu \cap (\tau_z \circ f)_\sharp \nu = \pi_\sharp\big[(\mu \times f_\sharp \nu)_{W,(z,-z)/2}\big],$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Now it is not clear what we mean by 'almost every continuously differentiable function', since there is no analogue for Lebesgue measure or Haar measure in the infinite-dimensional space $C^1(\mathbb{R}^n, \mathbb{R}^n)$ of continuously differentiable functions. We use a notion of prevalence from [14] instead.

Definition 4.1. Let V be a complete metric linear space. A Borel measure μ on V is transverse to a Borel set $S \subset V$ if

(1) there exists a compact set $U \subset V$ for which $0 < \mu(U) < \infty$ and
(2) $\mu(S + v) = 0$ for every $v \in V$.

A Borel set $P \subset V$ is prevalent if there exist a measure transverse to the complement of P.

In our application we take μ to be the measure $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$. We begin by proving results for continuously differentiable functions. Then as a corollary we get results for almost all linear maps by choosing $f = 0$. Proofs in this chapter are often just slight modifications of those in Chapter 3 and in these cases the details of the proofs are omitted.

4.1. **Properties of intersection measures.** As before we need to consider measures $(R_{g\circ T+f\sharp}\lambda)_{W,(z,-z)/2}$ instead of product measures. Here λ is a Radon measure on $\mathbb{R}^n\times\mathbb{R}^n$ with compact support. The following analogue of Lemma 3.1 is important in proving properties of these measures.

Lemma 4.2. *Let λ be a Radon measure on $\mathbb{R}^n\times\mathbb{R}^n$ and let $f\in C^1(\mathbb{R}^n,\mathbb{R}^n)$. Assume that $I_s(\lambda)<\infty$ for some $s\geq n$. Then for all $0<r_1<r_2<\infty$*

$$\begin{aligned}&\limsup_{\delta\to 0}\delta^{-n}\iint_{T(n)_+^{r_1,r_2}}\theta_n\{g\in\mathcal{O}(n)\colon |a-x-g\circ T(b-y)-(f(b)-f(y))|\leq\delta\}\\&\qquad\times\, d\mathcal{L}^{\frac{n}{2}(n+1)}(T)|(a,b)-(x,y)|^{-s+n}d(\lambda\times\lambda)(a,b,x,y)\\&\leq cI_s(\lambda),\end{aligned}$$

where c is a constant depending only on n, s, r_1 and r_2.

In the proof of the lemma we need to integrate in polar coordinates. We denote by S^{n-1} the surface of the unit ball, that is,

$$S^{n-1}=\{x\in\mathbb{R}^n: |x|=1\},$$

and for $x\in\mathbb{R}^n\backslash\{0\}$ we define

$$r=|x| \text{ and } \hat{x}=\frac{x}{r}\in S^{n-1}.$$

Then there is a unique Borel measure σ on S^{n-1} such that for every integrable Borel function $f:\mathbb{R}^n\to[0,\infty)$

$$\int f(x)\,d\mathcal{L}^n(x)=\int_{S^{n-1}}\int_0^\infty f(r\hat{x})r^{n-1}\,d\mathcal{L}^1(r)\,d\sigma(\hat{x}). \tag{4.1}$$

For the proof see for example [9, Theorem 2.49].

Proof of Lemma 4.2. Denote by $c_1,\dots,c_9$ constants which may depend on n, s, r_1 and r_2. Define for all $(a,b,x,y)\in(\mathbb{R}^n)^4$

$$\begin{aligned}I_\delta(a,b,x,y)=&\int_{T(n)_+^{r_1,r_2}}\theta_n\{g\in\mathcal{O}(n):\\&\qquad |a-x-g\circ T(b-y)-(f(b)-f(y))|\leq\delta\}\,d\mathcal{L}^{\frac{n}{2}(n+1)}(T).\end{aligned}$$

As is the proof of Lemma 3.1 we get

$$\begin{aligned}I_\delta(a,b,x,y)=&\int_{\{T\in T(n)_+^{r_1,r_2}:\left||a-x-(f(b)-f(y))|-|T(b-y)|\right|\leq\delta\}}\theta_n\{g\in\mathcal{O}(n):\\&|a-x-g\circ T(b-y)-(f(b)-f(y))|\leq\delta\}\,d\mathcal{L}^{\frac{n}{2}(n+1)}(T),\end{aligned}$$

and further,

$$\int I_\delta(a,b,x,y)|(a,b)-(x,y)|^{-s+n}\, d(\lambda\times\lambda)(a,b,x,y)$$
$$= \int_{A_\delta} I_\delta(a,b,x,y)|(a,b)-(x,y)|^{-s+n}\, d(\lambda\times\lambda)(a,b,x,y),$$

where

$$A_\delta = \{(a,b,x,y)\in(\mathbb{R}^n)^4 : r_1|b-y|-\delta \le |a-x-(f(b)-f(y))| \le r_2|b-y|+\delta\}.$$

Let

$$A_\delta^1 = \{(a,b,x,y)\in A_\delta : |a-x-(f(b)-f(y))|\le 2\delta\},$$
$$A_\delta^2 = \{(a,b,x,y)\in A_\delta : r_1|b-y|\le 2\delta\}$$

and

$$A_\delta^3 = \{(a,b,x,y)\in A_\delta : |a-x-(f(b)-f(y))| > 2\delta,\ r_1|b-y|>2\delta\},$$

in which case $A_\delta = A_\delta^1\cup A_\delta^2\cup A_\delta^3$.

If $(a,b,x,y)\in A_\delta^1$, then $|a-x-(f(b)-f(y))|\le 2\delta$ and $r_1|b-y|\le 3\delta$. Moreover, if $(a,b,x,y)\in \operatorname{spt}\lambda\times\operatorname{spt}\lambda$, the facts that λ has a compact support and $f\in C^1(\mathbb{R}^n,\mathbb{R}^n)$ imply $|f(b)-f(y)|\le M|b-y|$, where M is independent of b and y. Thus we have for $(a,b,x,y)\in A_\delta^1\cap\operatorname{spt}\lambda\times\operatorname{spt}\lambda$

$$|a-x|\le 2\delta + |f(b)-f(y)|\le c_1\delta.$$

So

$$|(a,b)-(x,y))| = \sqrt{|a-x|^2+|b-y|^2}\le c_2\delta,$$

and using finiteness of the s-energy, we get

$$\begin{aligned}&\limsup_{\delta\to0}\delta^{-n}\int_{A_\delta^1} I_\delta(a,b,x,y)|(a,b)-(x,y)|^{-s+n}\, d(\lambda\times\lambda)(a,b,x,y)\\ &\le c_2^n\mathcal{L}^{\frac{n}{2}(n+1)}(T(n)_+^{r_1,r_2})\limsup_{\delta\to0}\int_{A_\delta^1}|(a,b)-(x,y)|^{-s}\, d(\lambda\times\lambda)(a,b,x,y) = 0.\end{aligned} \tag{4.2}$$

Similarly, if $(a,b,x,y)\in A_\delta^2$, then $r_1|b-y|\le 2\delta$ and $|a-x-(f(b)-f(y))|\le (1+\frac{2r_2}{r_1})\delta$. So

$$\limsup_{\delta\to0}\delta^{-n}\int_{A_\delta^2} I_\delta(a,b,x,y)|(a,b)-(x,y)|^{-s+n}\, d(\lambda\times\lambda)(a,b,x,y) = 0. \tag{4.3}$$

Finally, let $(a,b,x,y)\in A_\delta^3$. Then

$$\frac{1}{2}r_1|b-y|\le|a-x-(f(b)-f(y))|\le 2r_2|b-y| \tag{4.4}$$

and thus for $(a,b,x,y)\in A_\delta^3\cap\operatorname{spt}\lambda\times\operatorname{spt}\lambda$

$$|a-x|\le 2r_2|b-y|+|f(b)-f(y)|\le c_3|b-y|,$$

which implies

$$|(a,b)-(x,y)| = \sqrt{|a-x|^2+|b-y|^2} \le c_4|b-y|. \tag{4.5}$$

We write every $T \in T(n)_+^{r_1,r_2}$ in polar coordinates, that is $T = r\hat{T}$, where $r = \sqrt{\sum_{i,j}|T_{i,j}|^2}$ and $\hat{T} = \frac{T}{r}$. Then $\varrho_i^{\hat{T}} = r^{-1}\varrho_i^T$ for every $i = 1,\dots,n$. We also have the following relation

$$\varrho_1^T \le \sqrt{\sum_{i,j}|T_{i,j}|^2} \le \sqrt{n}\varrho_1^T.$$

For the proof of this, see for example [10, (2.2-9) and (2.3-6)]. Thus we get that, if $T \in T_+^{r_1,r_2}$, then $\hat{T} \in T_+^{c_5,r_2/r_1}$, where $c_5 = \frac{r_1}{\sqrt{n}r_2}$.

Now by (4.1), binomial formula and (4.4) we have

$$\begin{aligned}
&\mathcal{L}^{\frac{n}{2}(n+1)}\{T \in T(n)_+^{r_1,r_2} : \big||a-x-(f(b)-f(y))| - |T(b-y)|\big| \le \delta\} \\
&\le \int_{S^{\frac{n}{2}(n+1)-1}\cap T_+^{c_5,r_2/r_1}} \int_{\frac{|a-x-(f(b)-f(y))|-\delta}{|\hat{T}(b-y)|}}^{\frac{|a-x-(f(b)-f(y))|+\delta}{|\hat{T}(b-y)|}} r^{\frac{n}{2}(n+1)-1}\, d\mathcal{L}^1(r)\, d\sigma(\hat{T}) \\
&= \big(\frac{n}{2}(n+1)\big)^{-1}\big[(|a-x-(f(b)-f(y))|+\delta)^{\frac{n}{2}(n+1)} - (|a-x-(f(b)-f(y))| \\
&\qquad -\delta)^{\frac{n}{2}(n+1)}\big] \int_{S^{\frac{n}{2}(n+1)-1}\cap T_+^{c_5,r_2/r_1}} |\hat{T}(b-y)|^{-\frac{n}{2}(n+1)}\, d\sigma(\hat{T}) \\
&\le \big(\frac{n}{2}(n+1)\big)^{-1} c_6\delta|a-x-(f(b)-f(y))|^{\frac{n}{2}(n+1)-1} \int_{S^{\frac{n}{2}(n+1)-1}\cap T_+^{c_5,r_2/r_1}} c_5^{-\frac{n}{2}(n+1)} \\
&\qquad \times |b-y|^{-\frac{n}{2}(n+1)}\, d\sigma(\hat{T}) \\
&\le c_7\delta|b-y|^{-1}.
\end{aligned}$$

Using this with [25, Lemma 3.8] gives

$$\begin{aligned}
&I_\delta(a,b,x,y) \\
&\le c_8\delta^{n-1}|a-x-(f(b)-f(y))|^{1-n}c_7\delta|b-y|^{-1} \\
&\le c_8c_7\delta^n|b-y|^{-n}.
\end{aligned}$$

Thus, using also (4.5), we get

$$\begin{aligned}
&\limsup_{\delta\to 0}\delta^{-n}\int_{A_\delta^3} I_\delta(a,b,x,y)|(a,b)-(x,y)|^{-s+n}\, d(\lambda\times\lambda)(a,b,x,y) \\
&\le c_7c_8\limsup_{\delta\to 0}\int_{A_\delta^3}|b-y|^{-n}|(a,b)-(x,y)|^{-s+n}\, d(\lambda\times\lambda)(a,b,x,y) \\
&\le c_9\limsup_{\delta\to 0}\int_{A_\delta^3}|(a,b)-(x,y)|^{-s}\, d(\lambda\times\lambda)(a,b,x,y)
\end{aligned}$$

and the lemma follows by (4.2) and (4.3). □

Now we can prove an analogue of Theorem 3.2.

Theorem 4.3. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support such that $I_n(\lambda) < \infty$ and let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g, T) \in \mathcal{O}(n) \times T(n)_+$*

$$S_{(g\circ T+f)\sharp}\lambda \ll \mathcal{L}^n.$$

Proof. As the proof of Theorem 3.2 using Lemma 4.2 instead of Lemma 3.1. □

As in the case of similarities we only need to assume the following local energy condition.

Corollary 4.4. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support such that $\int |(x, y) - (a, b)|^{-n} \, d\lambda(a, b) < \infty$ for λ-almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$S_{(g\circ T+f)\sharp}\lambda \ll \mathcal{L}^n$$

for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g, T) \in \mathcal{O}(n) \times T(n)_+$.

Proof. As the proof of Corollary 3.3. □

4.2. **Local dimensions of intersection measures.** We prove results analogous to results in Section 3.2. The following analogue of Lemma 3.5 is needed for this.

Lemma 4.5. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support. Fix $0 < r_1 < r_2 < \infty$ and $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. If $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ is such that $\int |(x, y) - (a, b)|^{-n} \, d\lambda(a, b) < \infty$, then*

$$\int_{T(n)_+^{r_1,r_2}} \int \pi_\sharp[(R_{(g\circ T+f)\sharp}\lambda)_{W,(x-(g\circ T+f)(y),(g\circ T+f)(y)-x)/2}] \, d\theta_n(g) \, d\mathcal{L}^{\frac{n}{2}(n+1)}(T)$$
$$\leq c \int_{B((x,y),\tilde{c}h)} |(x, y) - (a, b)|^{-n} \, d\lambda(a, b),$$

where c and $\tilde{c}$ are constants depending on n, r_1 and r_2.

Proof. As the proof of Lemma 3.5 using similar modifications as was used in the proof of Lemma 4.2. □

Now we can prove a result for intersection measures.

Theorem 4.6. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports such that $\int |(x, y) - (a, b)|^{-n} \, d(\mu \times \nu)(a, b) < \infty$ for $\mu \times \nu$-almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g, T) \in \mathcal{O}(n) \times T(n)_+$*

$$\underline{\dim}_{\text{loc}}(\mu \cap (\tau_{x-(g\circ T+f)(y)} \circ (g \circ T + f))_\sharp \nu)(x) \geq \underline{\dim}_{\text{loc}}(\mu \times \nu)(x, y) - n$$
$$= \underline{d}^n_{\mu\times\nu}(x, y) - n$$

and

$$\overline{\dim}_{\text{loc}}(\mu \cap (\tau_{x-(g\circ T+f)(y)} \circ (g \circ T + f))_\sharp \nu)(x) \geq \overline{d}^n_{\mu\times\nu}(x, y) - n$$

for $\mu \times \nu$-almost all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. As the proof of Theorem 3.6 using Lemma 4.5 instead of Lemma 3.5. □

In order to prove that Theorem 4.6 holds without assuming the local energy condition we have to make an extra assumption that $g \circ T + f$ is injective. We also need the following analogue of Lemma 2.2 for the proof.

Lemma 4.7. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support and let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$*

$$(1) \quad \underline{d}^n_{R_{(g\circ T+f)\sharp}\lambda}(x,(g\circ T+f)(y)) \le \underline{d}^n_\lambda(x,y),$$
$$(2) \quad \overline{d}^n_{R_{(g\circ T+f)\sharp}\lambda}(x,(g\circ T+f)(y)) \le \overline{d}^n_\lambda(x,y)$$

and

$$(3) \quad \dim_{\mathrm{A}}(R_{(g\circ T+f)\sharp}\lambda)(x,(g\circ T+f)(y)) \le \dim_{\mathrm{A}}\lambda(x,y)$$

for λ-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Equalities hold in (1), (2) and (3) for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$ for which $g \circ T + f$ is injective.

Proof. The proof of the upper bounds in both (1) and (2) goes as in the proof of Lemma 2.2, since for fixed $(g,T) \in \mathcal{O}(n) \times T(n)_+$ we have

$$|(x,(g\circ T+f)(y)) - (a,(g\circ T+f)(b))| \le c_1|(x,y)-(a,b)|$$

for all $(x,y),(a,b) \in \operatorname{spt}\lambda$. Here c_1 is a constant which does not depend on (x,y) or (a,b). This also implies

$$\lambda(B((x,y),h/c_1)) \le R_{(g\circ T+f)\sharp}\lambda(B((x,(g\circ T+f)(y)),h))$$

which gives the upper bound in (3).

For the opposite inequalities fix $(x,y) \in \operatorname{spt}\lambda$ and let $(g,T) \in \mathcal{O}(n) \times T(n)_+$ be such that $g \circ T + f$ is injective and $|(Df(y) + g\circ T)(b)| \ge c_2|b|$ for all $b \in \mathbb{R}^n$, where c_2 is a constant. Now

$$f(b) - f(y) = Df(y)(b-y) + |b-y|\varepsilon(b-y),$$

where ε is a function such that $\varepsilon(b-y) \to 0$ as $|b-y| \to 0$.

Let $|b-y|$ be so small that $|\varepsilon(b-y)| \le c_2/2$. Then

$$\begin{aligned}
&|(g\circ T+f)(b) - (g\circ T+f)(y)| = |f(b)-f(y) + (g\circ T)(b-y)| \\
&= |(Df(y)+g\circ T)(b-y) + |b-y|\varepsilon(b-y)| \\
&\ge \big||(Df(y)+g\circ T)(b-y)| - |b-y||\varepsilon(b-y)|\big| \\
&\ge \frac{1}{2}c_2|b-y|,
\end{aligned}$$

which implies

$$(4.6) \qquad |(x,y)-(a,b)| \le c_3|(x,(g\circ T+f)(y)) - (a,(g\circ T+f)(b))|$$

for all $(a,b) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $|(a,b)-(x,y)|$ is small enough. Further, since $g \circ T + f$ is injective, we get

$$\begin{aligned}
&\int_{B((x,(g\circ T+f)(y)),h)} |(a,b)-(x,(g\circ T+f)(y))|^{-n}\, dR_{(g\circ T+f)\sharp}\lambda(a,b) \\
&\leq \int_{B((x,y),c_3h)} |(a,(g\circ T+f)(b))-(x,(g\circ T+f)(y))|^{-n}\, d\lambda(a,b) \\
&\leq c_3^n \int_{B((x,y),c_3h)} |(a,b)-(x,y)|^{-n}\, d\lambda(a,b),
\end{aligned}$$

if h is small enough. The equalities (1) and (2) follow then by using Fubini's theorem, since

$$\begin{aligned}
&\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}(\mathcal{O}(n)\times T(n)_+) \\
&\quad = \theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}\Big(\bigcup_i \{(g,T) : |(Df(y)+g\circ T)(b)| \geq 2^{-i}|b| \text{ for all } b \in \mathbb{R}^n\}\Big).
\end{aligned}$$

Combining this with (4.6) also implies that if h is small enough, then for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n)\times T(n)_+$ for which $g\circ T + f$ is injective

$$R_{(g\circ T+f)\sharp}\lambda(B((x,(g\circ T+f)(y)),h)) \leq \lambda(B((x,y),h/c_3))$$

for λ-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Thus the equality in (3) follows. □

Corollary 4.8. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support and let $f \in C^1(\mathbb{R}^n,\mathbb{R}^n)$. If $\dim_{\mathrm{H}} \lambda > n$, then*

$$\dim_{\mathrm{H}}(R_{(g\circ T+f)\sharp}\lambda) = \dim_{\mathrm{H}} \lambda$$

and

$$\dim_{\mathrm{H}}^*(R_{(g\circ T+f)\sharp}\lambda) = \dim_{\mathrm{H}}^* \lambda$$

for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n)\times T(n)_+$ for which $g\circ T+f$ is injective.

Proof. Follows from Lemma 4.7(1) and Remark 2.1(1). □

Now we can prove the theorem. As before we define for all $g \in \mathcal{O}(n)$, $T \in T(n)_+$ and $f \in C^1(\mathbb{R}^n,\mathbb{R}^n)$

$$E_{g,T,f} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \mu \cap (\tau_{x-(g\circ T+f)(y)} \circ (g\circ T+f))_\sharp \nu \text{ is defined}\}.$$

Remark 4.9. If $\int |(x,y)-(a,b)|^{-n}\, d(\mu\times\nu)(a,b) < \infty$ for $\mu\times\nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, then intersection measures $\mu \cap (\tau_{x-(g\circ T+f)(y)} \circ (g\circ T+f))_\sharp \nu$ are defined for all $f \in C^1(\mathbb{R}^n,\mathbb{R}^n)$ and for $\mu\times\nu\times\theta_n\times\mathcal{L}^{\frac{n}{n}(n+1)}$-almost all $(x,y,g,T) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{O}(n) \times T(n)_+$. Again, this follows from Corollary 4.4 as in [24, Lemma 4.6].

Theorem 4.10. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports and let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$ for which $g \circ T + f$ is injective*

$$\underline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-(g\circ T+f)(y)} \circ (g \circ T + f))_\sharp \nu)(x) \geq \underline{\dim}_{\mathrm{loc}}(\mu \times \nu)(x,y) - n \\ \geq \underline{d}^n_{\mu\times\nu}(x,y) - n$$

and

$$\overline{\dim}_{\mathrm{loc}}(\mu \cap (\tau_{x-(g\circ T+f)(y)} \circ (g \circ T + f))_\sharp \nu)(x) \geq \overline{d}^n_{\mu\times\nu}(x,y) - n$$

for $\mu \times \nu$-almost all $(x,y) \in E_{g,T,f}$ and $\overline{d}^n_{\mu\times\nu}(x,y) = -\infty$ for $\mu \times \nu$-almost all $(x,y) \in (\mathbb{R}^n \times \mathbb{R}^n)\backslash E_{g,T,f}$.

Proof. As the proof of Theorem 3.9 using Corollary 4.4, Theorem 4.6 and Lemma 4.7. □

In order to prove upper bounds for local dimensions of intersection measures we again have to make an assumption that $g \circ T + f$ is injective. In the case of the lower local dimension we can replace this assumption by the assumption that $\dim_{\mathrm{H}}(\mu \times \nu) = \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu$. We need the next lemma for that.

Lemma 4.11. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports. If $\dim_{\mathrm{H}}(\mu \times \nu) = \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu$, then in a prevalent set of functions $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$*

$$\dim_{\mathrm{H}}(\mu \times f_\sharp \nu) = \dim_{\mathrm{H}}(\mu \times \nu)$$

and

$$\dim^*_{\mathrm{H}}(\mu \times f_\sharp \nu) = \dim^*_{\mathrm{H}}(\mu \times \nu).$$

Proof. Since $(x,y) \mapsto (x, f(y))$ is Lipschitz on the compact set $\mathrm{spt}(\mu \times \nu)$, we have $\dim_{\mathrm{H}}(\mu \times f_\sharp \nu) \leq \dim_{\mathrm{H}}(\mu \times \nu)$. It is proved in [13] that if ν is a Radon measure on $\mathbb{R}^n$ with compact support, then in a prevalent set of functions $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\dim_{\mathrm{H}}(f_\sharp \nu) = \dim_{\mathrm{H}} \nu.$$

Thus by (2.7) in a prevalent set of functions $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\dim_{\mathrm{H}}(\mu \times f_\sharp \nu) \geq \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}}(f_\sharp \nu) = \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu = \dim_{\mathrm{H}}(\mu \times \nu).$$

The same proof works for $\dim^*_{\mathrm{H}}$. □

For the upper bound of upper local dimensions of intersection measures we need the following lemma, which is an analogue of Lemma 3.12.

Lemma 4.12. *Let λ be a Radon measure on $\mathbb{R}^n \times \mathbb{R}^n$ with compact support such that $\dim_H \lambda = s > n$ and let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$ there exists for any $\varepsilon > 0$ a compact set $C_\varepsilon \subset \mathbb{R}^n \times \mathbb{R}^n$ with*

$R_{(g\circ T+f)\sharp}\lambda((\mathbb{R}^n \times \mathbb{R}^n)\backslash C_\varepsilon) < \varepsilon$ *and* H_ε *such that for* $\mathcal{H}^n$*-almost all* $(z,-z)/2 \in W^\perp$ *we have*

$$((R_{(g\circ T+f)\sharp}\lambda)\big|_{C_\varepsilon})_{W,(z,-z)/2}(B((x,y),h)) \le ch^{(s-n)/2}$$

for all $(x,y) \in W_{(z,-z)/2}$ *and* $0 < h \le H_\varepsilon$*. Here* c *is a constant depending only on* s *and* n*.*

Proof. Theorem 4.6 implies that for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$ we can find for $R_{(g\circ T+f)\sharp}\lambda$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ a constant H depending on (x,y) such that for all $h < H$

$$(R_{(g\circ T+f)\sharp}\lambda)_{W,(x-y,y-x)/2}(B(x,y),h) \le h^{(s-n)/2}.$$

Let $\varepsilon > 0$. For every $i = 1,2,\dots$ define a Borel set

$$B_i = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \\ (R_{(g\circ T+f)\sharp}\lambda)_{W,(x-y,y-x)/2}(B(x,y),h) \le h^{(s-n)/2} \text{ for all } h \le 2^{-i}\}.$$

Since

$$\lim_{i\to\infty} R_{(g\circ T+f)\sharp}\lambda((\mathbb{R}^n \times \mathbb{R}^n)\backslash B_i) = 0$$

we find a compact set $C_\varepsilon \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $R_{(g\circ T+f)\sharp}\lambda((\mathbb{R}^n \times \mathbb{R}^n)\backslash C_\varepsilon) < \varepsilon$ and $C_\varepsilon \subset B_{i_\varepsilon}$ for some i_ε.

Let $H_\varepsilon = 2^{-i_\varepsilon}/3$. Consider $(z,-z)/2 \in W^\perp$ such that both $(R_{(g\circ T+f)\sharp}\lambda)_{W,(z,-z)/2}$ and $(R_{(g\circ T+f)\sharp}\lambda|_{C_\varepsilon})_{W,(z,-z)/2}$ are defined. If $(z,-z)/2 \notin P_{W^\perp}(C_\varepsilon)$, then

$$(R_{(g\circ T+f)\sharp}\lambda|_{C_\varepsilon})_{W,(z,-z)/2}(B((x,y),h)) = 0$$

for all $(x,y) \in W_{(z,-z)/2}$ and $h > 0$. This follows from (2.1) and from the fact that $W_{(z,-z)/2}(\delta) \cap C_\varepsilon = \emptyset$ for all small $\delta > 0$, since C_ε is compact. If $(z,-z)/2 \in P_{W^\perp}(C_\varepsilon)$ and $(x,y) \in W_{(z,-z)/2} \cap C_\varepsilon$, then for any $0 < h \le 3H_\varepsilon$, we have

$$\begin{aligned}&(R_{(g\circ T+f)\sharp}\lambda|_{C_\varepsilon})_{W,(z,-z)/2}(B((x,y),h)) \\ &\le (R_{(g\circ T+f)\sharp}\lambda)_{W,(x-y,y-x)/2}(B((x,y),h)) \\ &\le h^{(s-n)/2}.\end{aligned} \tag{4.7}$$

If $(z,-z)/2 \in P_{W^\perp}(C_\varepsilon)$ and $(x,y) \notin W_{(z,-z)/2} \cap C_\varepsilon$, then there exists $h_{x,y} > 0$ such that $B((x,y),h) \cap C_\varepsilon \cap W_{(z,-z)/2} = \emptyset$ for all $0 < h < h_{x,y}$ and $B((x,y),h) \cap C_\varepsilon \cap W_{(z,-z)/2} \ne \emptyset$ for all $h \ge h_{x,y}$. If $0 < h < h_{x,y}$, then by (2.1)

$$(R_{(g\circ T+f)\sharp}\lambda|_{C_\varepsilon})_{W,(z,-z)/2}(B((x,y),h)) = 0.$$

If $h_{x,y} \le h \le H_\varepsilon$, then $B((x,y),h) \subset B((a,b),3h)$ for some $(a,b) \in W_{(z,-z)/2} \cap C_\varepsilon$, and (4.7) gives the claim. □

Now we are ready to prove the theorem.

Theorem 4.13. *Let* μ *and* ν *be Radon measures on* $\mathbb{R}^n$ *with compact supports.*

(1) *If* $\dim_{\mathrm{H}}(\mu\times\nu)>n$, *then for all* $f\in C^1(\mathbb{R}^n,\mathbb{R}^n)$ *and for* $\theta_n\times\mathcal{L}^{\frac{n}{2}(n+1)}$*-almost all* $(g,T)\in\mathcal{O}(n)\times T(n)_+$ *for which* $g\circ T+f$ *is injective*

$$\begin{aligned}\underline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_{x-(g\circ T+f)(y)}\circ(g\circ T+f))_\sharp\nu)(x)&=\underline{\dim}_{\mathrm{loc}}(\mu\times\nu)(x,y)-n\\&=\underline{d}^n_{\mu\times\nu}(x,y)-n\end{aligned}$$

and

$$\overline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_{x-(g\circ T+f)(y)}\circ(g\circ T+f))_\sharp\nu)(x)=\overline{d}^n_{\mu\times\nu}(x,y)-n$$

for $\mu\times\nu$*-almost all* $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$.

(2) *If* $\dim_{\mathrm{H}}(\mu\times\nu)=\dim_{\mathrm{H}}\mu+\dim_{\mathrm{H}}\nu>n$, *then in a prevalent set of functions* $f\in C^1(\mathbb{R}^n,\mathbb{R}^n)$

$$\underline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_{x-f(y)}\circ f)_\sharp\nu)(x)=\underline{\dim}_{\mathrm{loc}}(\mu\times\nu)(x,y)-n=\underline{d}^n_{\mu\times\nu}(x,y)-n$$

for $\mu\times\nu$*-almost all* $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$.

Proof. The lower bounds in both cases follow from Theorem 4.6. In order to get the upper bound for the lower local dimensions in both cases we apply Lemma 3.11 to $\mu\times(g\circ T+f)_\sharp\nu$ (or to $\mu\times f_\sharp\nu$) as in the proof Theorem 3.13. This can be done by Corollary 4.4, Lemma 4.7, Remark 2.1(1) and Corollary 4.8 (or Lemma 4.11). Note, that from Lemma 4.7 we only need the part which holds in a prevalent set.

The upper bound for the upper local dimension can be proven as in the proof of Theorem 3.13 using Lemma 4.4, Lemma 4.7, Corollary 4.8 and Lemma 4.12. □

Corollary 4.14. *Let* μ *and* ν *be Radon measures on* $\mathbb{R}^n$ *with compact supports. If* $\dim_{\mathrm{H}}(\mu\times\nu)>n$, *then for all* $f\in C^1(\mathbb{R}^n,\mathbb{R}^n)$ *and for* $\mathcal{L}^n\times\theta_n\times\mathcal{L}^{\frac{n}{2}(n+1)}$*-almost all* $(z,g,T)\in\mathbb{R}^n\times\mathcal{O}(n)\times T(n)_+$ *for which* $g\circ T+f$ *is injective*

$$\begin{aligned}(1)\quad\underline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_z\circ(g\circ T+f))_\sharp\nu)(x)&=\underline{\dim}_{\mathrm{loc}}(\mu\times(g\circ T+f)_\sharp\nu)(x,x-z)-n\\&=\underline{d}^n_{\mu\times(g\circ T+f)_\sharp\nu}(x,x-z)-n\end{aligned}$$

and

$$(2)\quad\overline{\dim}_{\mathrm{loc}}(\mu\cap(\tau_z\circ g\circ T)_\sharp\nu)(x)=\overline{d}^n_{\mu\times(g\circ T+f)_\sharp\nu}(x,x-z)-n$$

for $\mu\cap(\tau_z\circ(g\circ T+f))_\sharp\nu$*-almost all* $x\in\mathbb{R}^n$.

Proof. As the proof of Corollary 3.16. □

4.3. **Hausdorff, packing and average dimensions.** Using Theorem 4.13 we get results for Hausdorff and packing dimensions. We also consider average dimensions of intersection measures. All these results are analogous to the similarity case in Sections 3.3 and 3.4.

Theorem 4.15. *Let* μ *and* ν *be Radon measures on* $\mathbb{R}^n$ *with compact supports.*

(1) *If* $\dim_{\mathrm{H}}(\mu \times \nu) > n$*, then for all* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and for* $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$*-almost all* $(g, T) \in \mathcal{O}(n) \times T(n)_+$

$$\begin{aligned}&\mathcal{L}^n\text{-}\operatorname{ess\,inf}\{\dim_{\mathrm{H}}(\mu \cap (\tau_z \circ (g \circ T + f))_\sharp \nu) :\\ &\qquad\qquad z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ (g \circ T + f))_\sharp \nu(\mathbb{R}^n) > 0\}\\ &\geq \dim_{\mathrm{H}}(\mu \times \nu) - n.\end{aligned}$$

The equality holds for all $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and for* $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$*-almost all* $(g, T) \in \mathcal{O}(n) \times T(n)_+$ *for which* $g \circ T + f$ *is injective.*

(2) *If* $\dim_{\mathrm{H}}(\mu \times \nu) = \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu > n$*, then in a prevalent set of functions* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\begin{aligned}&\mathcal{L}^n\text{-}\operatorname{ess\,inf}\{\dim_{\mathrm{H}}(\mu \cap (\tau_z \circ f)_\sharp \nu) :\\ &\qquad\qquad z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ f)_\sharp \nu(\mathbb{R}^n) > 0\}\\ &= \dim_{\mathrm{H}}(\mu \times \nu) - n.\end{aligned}$$

Proof. As the proof of Theorem 3.18 using Corollary 4.4, Corollary 4.8, Lemma 4.11 and Theorem 4.13. □

Theorem 4.16. *Let* μ *and* ν *be Radon measures on* $\mathbb{R}^n$ *with compact supports.*

(1) *If* $\dim_{\mathrm{H}}(\mu \times \nu) > n$*, then for all* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and for* $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$*-almost all* $(g, T) \in \mathcal{O}(n) \times T(n)_+$

$$\mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z \in \mathbb{R}^n} \dim^*_{\mathrm{H}}(\mu \cap (\tau_z \circ (g \circ T + f))_\sharp \nu) \geq \dim^*_{\mathrm{H}}(\mu \times \nu) - n.$$

The equality holds for all $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and for* $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$*-almost all* $(g, T) \in \mathcal{O}(n) \times T(n)_+$ *for which* $g \circ T + f$ *is injective.*

(2) *If* $\dim_{\mathrm{H}}(\mu \times \nu) = \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu > n$*, then in a prevalent set of functions* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z \in \mathbb{R}^n} \dim^*_{\mathrm{H}}(\mu \cap (\tau_z \circ f)_\sharp \nu) = \dim^*_{\mathrm{H}}(\mu \times \nu) - n.$$

Proof. As the proof of Theorem 3.19 using Corollary 4.4, Corollary 4.8, Lemma 4.11 and Theorem 4.13. □

Theorem 4.17. *Let* μ *and* ν *be Radon measures on* $\mathbb{R}^n$ *with compact supports. If* $\dim_{\mathrm{H}}(\mu \times \nu) > n$*, then for all* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and for* $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$*-almost all* $(g, T) \in \mathcal{O}(n) \times T(n)_+$

$$\begin{aligned}&\mathcal{L}^n\text{-}\operatorname{ess\,inf}\{\dim_{\mathrm{p}}(\mu \cap (\tau_z \circ (g \circ T + f))_\sharp \nu) :\\ &\qquad\qquad z \in \mathbb{R}^n \text{ with } \mu \cap (\tau_z \circ (g \circ T + f))_\sharp \nu(\mathbb{R}^n) > 0\}\\ &\geq \mu \times \nu\text{-}\operatorname*{ess\,inf}_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n} \overline{d}^n_{\mu \times \nu}(x, y) - n.\end{aligned}$$

The equality holds for all $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and for* $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$*-almost all* $(g, T) \in \mathcal{O}(n) \times T(n)_+$ *for which* $g \circ T + f$ *is injective.*

Proof. As the proof of Theorem 3.21 using Corollary 4.4 and Theorem 4.13. □

Theorem 4.18. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports. If $\dim_{\mathrm{H}}(\mu \times \nu) > n$, then for all $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$*

$$\mathcal{L}^n\text{-}\operatorname*{ess\,sup}_{z\in\mathbb{R}^n} \dim^*_{\mathrm{p}}(\mu \cap (\tau_z \circ (g \circ T + f))_\sharp \nu) \geq \mu \times \nu\text{-} \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \overline{d}^n_{\mu\times\nu}(x,y) - n.$$

The equality holds for all $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$ for which $g \circ T + f$ is injective.

Proof. As the proof of Theorem 3.22 using Corollary 4.4 and Theorem 4.13. □

For the average dimension of intersection measures we get the following results.

Theorem 4.19. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports.*

(1) *If $\dim_{\mathrm{H}}(\mu \times \nu) > n$, then for all $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$*

$$\dim_{\mathrm{A}}(\mu \cap (\tau_{x-(g\circ T+f)(y)} \circ (g \circ T + f))_\sharp \nu)(x) \geq \dim_{\mathrm{A}}(\mu \times \nu)(x,y) - n$$

for $\mu \times \nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. The equality holds for all $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and for $\theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(g,T) \in \mathcal{O}(n) \times T(n)_+$ for which $g \circ T + f$ is injective.

(2) *If $\dim_{\mathrm{H}}(\mu \times \nu) = \dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu > n$, then in a prevalent set of functions $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$*

$$\dim_{\mathrm{A}}(\mu \cap (\tau_{x-f(y)} \circ f)_\sharp \nu)(x) = \dim_{\mathrm{A}}(\mu \times \nu)(x,y) - n$$

for $\mu \times \nu$-almost all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. For the upper bounds we may apply [20, Theorem 2.8] to $\mu \times (g \circ T + f)_\sharp \nu$ (or to $\mu \times f_\sharp \nu$) as in the proof of Theorem 3.23. This can be done by Corollary 4.4, Lemma 4.7 and Corollary 4.8 (or Lemma 4.11). Note, that from Lemma 4.7 we only need the part which holds in a prevalent set.

The lower bounds can be proven as in the proof of Theorem 3.23 using Lemma 4.5. □

Corollary 4.20. *Let μ and ν be Radon measures on $\mathbb{R}^n$ with compact supports. If $\dim_{\mathrm{H}}(\mu \times \nu) > n$, then for all $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and for $\mathcal{L}^n \times \theta_n \times \mathcal{L}^{\frac{n}{2}(n+1)}$-almost all $(z,g,T) \in \mathbb{R}^n \times \mathcal{O}(n) \times T(n)_+$ for which $g \circ T + f$ is injective*

$$\dim_A(\mu \cap (\tau_z \circ (g \circ T + f))_\sharp \nu)(x) = \dim_A(\mu \times (g \circ T + f)_\sharp \nu)(x, x-z) - n$$

for $\mu \cap (\tau_z \circ (g \circ T + f))_\sharp \nu$-almost all $x \in \mathbb{R}^n$.

Proof. As the proof of Corollary 3.16. □

Remark 4.21. It remains an open question whether equalities in results for upper local dimension and packing dimension (i.e. in Theorem 4.14(2), Theorem 4.17 and Theorem 4.18) hold in a prevalent set of functions $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

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