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SUOMALAINEN TIEDEAKATEMIA

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Antti V. Vähäkangas

Contents

1. Introduction

The foundations for the modern treatment of singular integrals were provided by Calderón and Zygmund in their seminal work [CZ52] and these operators have been studied extensively in various settings [Chr90, HS94, NTV03, Tor91]. The principal objective of this monograph is to explore certain weakly singular integral operators which are related to singular integrals and originate in pseudodifferential operators and PDE's. To be more precise, assume that $\Omega \subset \mathbb{R}^n$ is a domain and $K : \Omega \times \Omega \to \mathbb{C}$ is a kernel that defines a linear integral operator

$$Tf(x) = \int_\Omega K(x,y)f(y)dy, \quad x \in \Omega. \tag{1.1}$$

We establish conditions on this domain and kernel under which the weak derivatives

$$\partial^\alpha T = f \mapsto \partial^\alpha \int_\Omega K(\cdot,y)f(y)dy \tag{1.2}$$

of prescribed order $|\alpha| = m > 0$ have bounded extension to the spaces $L^p(\Omega)$ for $1 < p < \infty$ or to some other natural function spaces on Ω. Results to this direction are available in the case $\Omega = \mathbb{R}^n$ [Tor91, p. 141] but apparently not in the context of proper domains.

Inhomogeneous Dirichlet problem. Weakly singular integral operators (WSIO's) as in (1.1) arise naturally in connection with elliptic PDE's on domains. In order to illustrate our main result, consider the following inhomogeneous Dirichlet problem in the ball $B = B(0,1) \subset \mathbb{R}^n$ for $n \geq 3$,

$$\begin{cases} -\Delta u = f \in L^p(B), \\ u \in W_0^{1,p}(B). \end{cases} \tag{1.3}$$

We restrict ourselves to the exponents $1 < p < \infty$. Then (1.3) has a unique solution and it satisfies $||u||_{W^{2,p}(B)} \leq C_{n,p}||f||_{L^p(B)}$ [ADN59], [JK95, Theorem 0.3], [GT83, Theorem 8.12]. Certain Hölder-regularity estimates combined with our main result can be used to deduce this $W^{2,p}(B)$-regularity and we sketch this argument for illustration. The solution of (1.3) can be expressed in terms of a weakly singular integral operator

$$u(x) = \mathbf{G}f(x) = \int_B G(x,y)f(y)dy, \quad x \in B. \tag{1.4}$$

Here $G : B \times B \setminus \{(x,x)\} \to \mathbb{R}$ is the *Green's function*, which is defined for $y \neq 0$ by

$$G(x,y) = \frac{|y|^{2-n}|x-\bar{y}|^{2-n} - |x-y|^{2-n}}{\omega_{n-1}(2-n)}, \quad \bar{y} = |y|^{-2}y,$$

and $G(x,0) = (\omega_{n-1}(2-n))^{-1}(1-|x|^{2-n})$, where ω_{n-1} is the $(n-1)$-measure of $\partial B = S^{n-1}$. The Green's function is symmetric so that $G(x,y) = G(y,x)$ if $x \neq y$. Using the definition it is straightforward to verify the estimate

$$|\partial_x^\alpha G(x,y)| \le C_{n,\alpha}|x-y|^{2-n-|\alpha|} \quad \text{if } \alpha \in \mathbb{N}_0^n \text{ and } x \neq y. \tag{1.5}$$

As a consequence, the kernel G belongs to a certain class K_B^{-2} of standard kernels of order -2 which is defined in connection with our main result.

Let $k \in \mathbb{N}_0$ and $0 < \delta < 1$. Then $C^{k,\delta}(\bar{B})$ denotes the Hölder space of functions $f \in C^k(\bar{B})$ satisfying

$$\sum_{|\alpha|\le k} ||\partial^\alpha f||_{L^\infty(B)} + \sum_{|\alpha|=k} \sup\left\{\frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\delta} : x,y \in B, x \neq y\right\} < \infty.$$

We use the Hölder-regularity estimate that, if $f \in C^{0,\delta}(\bar{B}) \subset L^p(B)$, the solution (1.4) satisfies $u = \mathbf{G}f \in C^{2,\delta}(\bar{B})$ [GT83, Corollary 4.14]. This estimate is used for

$$\mathbf{G}1 = \mathbf{G}\chi_B \in C^{2,\delta}(\bar{B}), \quad \text{if } 0 < \delta < 1. \tag{1.6}$$

Using the properties (1.5) and (1.6) with our main result, Theorem 1.20, it is possible to verify that the operators $\partial^\alpha \mathbf{G}$ are bounded on $L^p(B)$ if $|\alpha| = 2$. As a consequence, the solution (1.4) satisfies $u \in W^{2,p}(B)$ and also the norm-estimate

$$||u||_{W^{2,p}(B)} = ||\mathbf{G}f||_{W^{2,p}(B)} \le C_{n,p}||f||_{L^p(B)}.$$

To recapitulate, the kernel size-estimates with the Hölder-regularity estimate on $\mathbf{G}1$ imply $W^{2,p}(B)$-regularity for the solution of (1.3). For details, see Example 6.21.

Integral operators on $\mathbb{R}^n$. Next we consider operators of convolution type on $\mathbb{R}^n$. These integral operators, treated in Section 2 and restricted to kernels of the form $K(x,y) = k(x-y)$, can be applied in solving the problem $-\Delta u = f$ with data f in homogeneous Triebel–Lizorkin or Besov spaces.

The basic ingredients of the following result are available in the literature but the novelty lies in their combination, showing how concrete characterizations are available in our situation. Here $\mathcal{S}_\infty$ is the space of Schwartz functions with all vanishing moments and $\mathcal{S}'/\mathcal{P}$ is its topological dual space; $\dot{B}_p^{\alpha,q}$'s form the scale of homogeneous Besov spaces, to be defined in Section 2.1.

Theorem 1.7. *Let* $\alpha \in \mathbb{R}$, $p \in \{1, \infty\}$, *and* $1 \leq q \leq \infty$. *Let* $k \in \mathcal{S}'/\mathcal{P}$. *Then the convolution operators* $f \mapsto \partial^\alpha k \star f : \mathcal{S}_\infty \to \mathcal{S}'/\mathcal{P}$, $|\alpha| = 1$, *have bounded extensions to* $\dot{B}_p^{\alpha,q}$ *if, and only if,* k *has an extension to a regular tempered distribution* $K \in \mathcal{S}'$ *satisfying the integral estimate*

$$\sup_{y \neq 0} \left\{ |y|^{-1} \int_{\mathbb{R}^n} |K(x-y) + K(x+y) - 2K(x)| dx \right\} < \infty. \tag{1.8}$$

The endpoint spaces $\dot{B}_1^{\alpha,q}$ and $\dot{B}_\infty^{\alpha,q}$ are classical in the following cases. If $\alpha > 0$, then $\dot{B}_\infty^{\alpha,\infty} \approx \dot{\mathcal{C}}^\alpha$ is the homogeneous Hölder–Zygmund space. If $0 < \alpha < 1$, then this space consists of complex-valued functions f on $\mathbb{R}^n$, satisfying a modulus of continuity estimate

$$|f(x) - f(y)| \leq C|x-y|^\alpha, \quad \text{if } x, y \in \mathbb{R}^n.$$

The norm $||f||_{\dot{\mathcal{C}}^\alpha}$ is the infimum over all constants $C > 0$ for which the above inequality holds and this space is defined modulo polynomials. If $1 \leq \alpha < 2$, then the first order difference is replaced with the second order difference, and so on. In the other endpoint, the space $\dot{B}_1^{0,1}$ is a so called minimal Banach space that is discussed later. Because Theorem 1.7 is a characterization for the boundedness in the endpoint spaces, it can be used to obtain boundedness results on the whole scale of Besov spaces via interpolation. In particular, if the condition (1.8) holds true, then the convolution operators $f \mapsto \partial^\alpha k \star f$, $|\alpha| = 1$, have bounded extensions to the space $\dot{B}_2^{0,2} \approx L^2(\mathbb{R}^n)$. Further results about boundedness of weakly singular integral operators on homogeneous Triebel–Lizorkin and Besov spaces can be found in [Tor91, Väh08].

Then we advance beyond convolution operators. We assume that T is a WSIO as in (1.1) with $\Omega = \mathbb{R}^n$. We also assume that the associated kernel K is a standard kernel of order $-m$; this condition is quantified later. Then the operator $\partial^\alpha T$ as in (1.2) is a so called SK(δ)-type operator. These operators generalize the classical singular integral operators and their $L^p(\mathbb{R}^n)$-boundedness is characterized in the seminal work of David and Journé [DJ84]. Cancellation properties captured by the quantities $T1$ and T^t1, along with certain weak boundedness property, play a decisive role therein. Generalizations of the $T1$ theorem are numerous but the conditions and conclusions often share the same spirit [Chr90, DJ84, HS94, NTV97, NTV03, Tor91, Väh08, Wan99]. These include results about certain WSIO's [Tor91, Väh08]. There is also a very general Tb theorem about SK(δ)'s due to Nazarov, Treil, and Volberg [NTV03]. This result is targeted at certain metric measure spaces and, in particular, it applies in domains.

For the convenience of the reader we formulate the fundamental result of David and Journé. Fix any $m \in \mathbb{N}$ and define the *normalized bump functions* consisting of those smooth functions $\Phi : \mathbb{R}^n \to \mathbb{C}$, supported in the unit ball, that satisfy $||\partial^\alpha \Phi||_\infty \leq 1$ if $0 \leq |\alpha| \leq m$. For each ball $B(x_0, R) \subset \mathbb{R}^n$ we write

$$\Phi^{R,x_0}(x) = \Phi\left(\frac{x - x_0}{R}\right).$$

A linear and continuous operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ has the *weak boundedness property*, denoted $T \in \mathrm{WBP}$, if there exists a constant $A > 0$ such that for all pairs of normalized bump functions Φ, Ψ, it satisfies

$$|\langle T\Phi^{R,x_0}, \Psi^{R,x_0}\rangle| \leq AR^n$$

for all $R > 0$ and $x_0 \in \mathbb{R}^n$. Here the constant A may not depend on the functions Φ^{R,x_0} or Ψ^{R,x_0}. The weak boundedness property is a natural condition since if T has a bounded extension to $L^2(\mathbb{R}^n)$ then $T \in \mathrm{WBP}$.

A function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\} \to \mathbb{C}$ is a *Calderón–Zygmund standard kernel* if there exists $\delta \in (0,1)$ such that $|K(x,y)| \leq C_K|x-y|^{-n}$ and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C_K|x-x'|^\delta|x-y|^{-n-\delta} \tag{1.9}$$

if $2|x-x'| \leq |x-y|$. A continuous linear operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is associated with a Calderón–Zygmund standard kernel K if

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)f(y)g(x)dydx \tag{1.10}$$

given that $f, g \in C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ have disjoint supports. We denote this by $T \in \mathrm{SK}(\delta)$. The transpose $T^t \in \mathrm{SK}(\delta)$ is defined by $\langle T^t f, g\rangle = \langle f, Tg\rangle$ for $f, g \in \mathcal{S}(\mathbb{R}^n)$, and it is associated with the transpose kernel $(x,y) \mapsto K(y,x)$.

For instance, if $\alpha \in \mathbb{N}_0^n$, the operator $f \mapsto \partial^\alpha f$ belongs to the class $\mathrm{SK}(\delta)$ since the associated kernel is identically zero and other requirements are also satisfied. The additional condition $T \in \mathrm{WBP}$ excludes such pathological cases for $\alpha \neq 0$. Intuitively, this condition states that

$$\left| \fint_B T\chi_B \right| \leq A \tag{1.11}$$

for every ball $B \subset \mathbb{R}^n$. The integral (1.11) is not defined, a priori, but if T has a bounded extension to $L^2(\Omega)$, then the extension satisfies (1.11).

We define $T1$ as a continuous linear functional on $\mathcal{S}_0 \cap C_0^\infty$ ($\mathcal{S}_0$ consists of Schwartz functions satisfying $\int \varphi = 0$). For this purpose we assume that $T \in \mathrm{SK}(\delta)$. Fix $f \in \mathcal{S}_0$ so that $\operatorname{supp} f \subset B(0,R)$, $R = R_f > 0$. Fix a cut-off function $\eta = \eta_f \in C_0^\infty(\mathbb{R}^n)$ so that $\eta(x) = 1$ for every $x \in B(0,2R)$. Define $\langle T1, f\rangle = \langle T\eta, f\rangle + \langle 1-\eta, T^t f\rangle$. The quantity $\langle T\eta, f\rangle$ is defined because $\eta, f \in \mathcal{S}(\mathbb{R}^n)$. Because $\int f = 0$, we have

$$T^t f(y) = \int_{\mathbb{R}^n} (K(x,y) - K(0,y))f(x)dx$$

for $y \notin B(0,2R)$. This allows us to define

$$\langle 1-\eta, T^t f\rangle = \int_{\mathbb{R}^n} (1-\eta)(y) \int_{\mathbb{R}^n} (K(x,y) - K(0,y))f(x)dxdy.$$

The Hölder–Zygmund-condition (1.9) shows that this integral converges absolutely. Applying (1.10) it is simple to verify that the quantity $\langle T1, f\rangle$ is independent of the cut-off function η_f. Because $\mathcal{S}_0 \cap C_0^\infty \subset H^1(\mathbb{R}^n)$ is dense it is reasonable to denote $T1 \in \mathrm{BMO}(\mathbb{R}^n) = (H^1(\mathbb{R}^n))^*$ if

$$|\langle T1, f\rangle| \le C||f||_{H^1(\mathbb{R}^n)}, \quad \text{if } f \in \mathcal{S}_0 \cap C_0^\infty$$

holds with C independent of f. Here is the $T1$ theorem of David and Journé [DJ84].

Theorem 1.12. *Assume that $T \in \mathrm{SK}(\delta)$. The following conditions are equivalent*

- $T1, T^t1 \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $T \in \mathrm{WBP}$,
- $T \in \mathrm{CZO}$, *that is, T has a bounded extension to $L^2(\mathbb{R}^n)$,*
- *T has a bounded extension to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

To indicate a possible usage of this theorem, let T be as in (1.1) such that $\partial^\alpha T \in \mathrm{SK}(\delta)$ if $|\alpha| = m$. The $T1$ theorem above applies and it provides a characterization for the L^p-boundedness of $\partial^\alpha T$. However, the verification of the assumptions is not necessarily feasible and in some cases it is more effective to work directly with the operator T. The non-homogeneous Tb theorem of Nazarov, Treil, and Volberg [NTV03] can be invoked to localize the situation to domains but the aforementioned difficulties remain.

WSIO's on domains. In this monograph we suggest a geometric approach to WSIO's on domains and the implied connection between integral operators and geometry is apparently a new one. The main result of ours is somewhat reminiscent to the $T1$ theorem of David and Journé, but modified to the direction of Theorem 1.7 so that the conditions involved concern T instead of, say, $\partial^\alpha T$.

In what follows, we confine ourselves to this main result of ours, applicable on so called admissible domains. To define this class of domains, we use the notions of uniformity and (co)plumpness.

A path $\gamma : [0, L] \to \Omega \subset \mathbb{R}^n$ is *rectifiable*, if

$$\ell(\gamma) = \sup\left\{ \sum_{j=1}^{k} |\gamma(t_j) - \gamma(t_{j-1})| \,:\, 0 = t_0 < t_1 < \cdots < t_k = L \right\} < \infty.$$

A rectifiable path $\gamma : [0, L] \to \Omega$ is *parametrized by the arc length* if $\ell(\gamma|[0, s]) = s$ for every $s \in [0, L]$. In particular, $L = \ell(\gamma)$. Next we pose the definition of uniform domains [Mar80, p. 198] and the definition of (co)plumpness [MV93, p. 251].

Definition 1.13. Assume that $n \ge 2$. A domain $\emptyset \ne \Omega \subset \mathbb{R}^n$ is *uniform* if there exists a *uniformity constant* $a \in [1, \infty)$ with the following property. Each pair of points $x, y \in \Omega$ can be joined by a path $\gamma : [0, \ell(\gamma)] \to \Omega$, parametrized by the arc length, such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = y$, and

- $\ell(\gamma) \le a|x - y|$,
- $\min(t, \ell(\gamma) - t) \le a \,\mathrm{dist}(\gamma(t), \partial\Omega)$ for every $t \in [0, \ell(\gamma)]$.

Definition 1.14. Assume that $n \ge 2$ and $c \ge 1$. A set $A \subset \mathbb{R}^n$ is *c-plump* if for all $x \in \bar{A}$ and $0 < r < \mathrm{diam}(A)$ there is $z \in \bar{B}(x, r)$ with $B(z, r/c) \subset A$. A set $A \subset \mathbb{R}^n$ is *c-coplump if $\mathbb{R}^n \setminus A$ is c-plump.*

Now we can define Whitney coplump and admissible domains.

Definition 1.15. Assume that $n \geq 2$. A domain $\emptyset \neq \Omega \subsetneq \mathbb{R}^n$ is *Whitney coplump* if $\operatorname{diam}(\mathbb{R}^n \setminus \Omega) = \infty$ and Ω is c-coplump for $c \geq 1$. The Euclidean space $\Omega = \mathbb{R}^n$ is Whitney coplump. A domain is *admissible* if it is both uniform and Whitney coplump.

The family consisting of admissible domains is invariant under quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ as follows: if $\Omega \subset \mathbb{R}^n$ is an admissible domain, then the image $f\Omega \subset \mathbb{R}^n$ is also admissible. The unit ball $B(0,1) \subset \mathbb{R}^n$ is admissible and, as a consequence, the images $fB(0,1)$ of the unit ball under quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ are admissible, see Section 6.

We make use of certain sequence spaces $\dot{f}^{m,2}_{\infty}(\Omega)$ on domains; these sequence spaces are defined later. However, if we formally set $m = 0$ and $\Omega = \mathbb{R}^n$ then we recover the space $\dot{f}^{0,2}_{\infty}(\mathbb{R}^n)$ that is related to the space $\mathrm{BMO}(\mathbb{R}^n)$ [Ste93, pp. 140–141]. This relation is established using the Carleson's condition [Mey92, p. 151–156].

Next we define local kernel classes on domains by generalizing condition (1.9). For this purpose we need certain difference operators $y \mapsto \Delta^{\ell}_h(f, D, y) : \mathbb{R}^n \to \mathbb{C}$ that are parametrized by $\ell \in \mathbb{N}$, $h \in \mathbb{R}^n$, and $D \subset \mathbb{R}^n$. These operate on functions $f : D \to \mathbb{C}$ according to the rule

$$\Delta^{\ell}_h(f, D, y) = \begin{cases} \sum_{k=0}^{\ell} (-1)^{\ell+k} \binom{\ell}{k} f(y + kh), & \text{if } \{y, y+h, \dots, y+\ell h\} \subset D, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain. Let $m \in \mathbb{N}$, $0 < m < n$, and $0 < \delta < 1$. Consider a continuous kernel $K : \Omega \times \Omega \setminus \{(x,x)\} \to \mathbb{C}$ satisfying

- kernel size estimate

$$|K(x,y)| \leq C_K |x-y|^{m-n}, \quad x, y \in \Omega, \tag{1.16}$$

- semilocal integral estimate

$$\sup_{|h| \leq \operatorname{diam}(Q)} \frac{1}{|Q|^{1+(m+\delta)/n}} \int_Q |\Delta^{m+1}_h(K(x,\cdot), Q, y)|\,dy \leq C_K |x - x^Q|^{-n-\delta}, \tag{1.17}$$

  if $x \in \Omega$ and $Q \subset\subset \Omega$ is a cube[1], centered at x^Q and $C_K \operatorname{diam}(Q) \leq |x - x^Q|$. We assume the same estimate with $K(x,\cdot)$ replaced by $K(\cdot,x)$.

In the case that (1.16) and (1.17) hold true, we say that K is a *standard kernel of order* $-m$ and denote this by $K \in \mathrm{K}^{-m}_{\Omega}(\delta)$. It is simple to verify that (1.17) is implied by the following semilocal condition

- estimate on the order $(m+1)$ differences

$$|\Delta^{m+1}_h(K(x,\cdot), Q, y)| \leq C_K |h|^{m+\delta} |x-y|^{-n-\delta} \tag{1.18}$$

  if $x, y \in \Omega$, $Q \subset\subset \Omega$ is a cube, and $2(m+1)|h| \leq |x-y|$. We also assume the same estimate but with $K(x,\cdot)$ replaced by $K(\cdot,x)$.

[1]In this work *cubes* have sides parallel to the coordinate axes.

The motivation for these kernel classes is that, if we formally set $m = 0$ and $\Omega = \mathbb{R}^n$, the conditions (1.16) and (1.18) reduce to the defining conditions for the Calderón–Zygmund standard kernels. Thus the kernel classes $\mathrm{K}^{-m}_{\mathbb{R}^n}(\delta)$ extend these standard kernels in a natural fashion. Assume that $K \in \mathrm{K}^{-1}_{\mathbb{R}^n}(\delta)$ satisfies (1.18) and $K(x,y) = k(x-y)$ for some $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$. Then the size-condition (1.16) reduces to

$$|k(x)| \leq c_k |x|^{-n+1}$$

and (1.18) reduces to

$$|k(x+2h) - 2k(x+h) + k(x)| \leq c_k |h|^{1+\delta}|x|^{-n-\delta}, \quad \text{if } 4|h| \leq |x|.$$

Applying these reduced conditions and changing the variables $\omega = x + h$, we see that k satisfies the integral condition (1.8). That is, if we consider operators of convolution type, the resulting conditions here are stronger than in Theorem 1.7.

There is also technical motivation for the kernel classes above. Condition (1.17) resembles the condition on certain local smoothness space $\mathscr{C}^{m+\delta}_{\infty}(\Omega)$, see Section 5. Indeed, the kernel classes $\mathrm{K}^{-m}_{\Omega}(\delta)$ have a close connection to these and other Hölder-type spaces, and such relations are exploited in solving a natural kernel extension problem by reducing it to the Hölder extension on the product domain $\Omega \times \Omega$. These extension results, applicable on uniform domains, reflect the *local-to-global* type function theoretic phenomenon that is emerging in connection with various classes of functions defined on uniform domains [Geh87].

A *weakly singular integral operator* (abbreviated WSIO) of order $-m$ on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is defined for $f \in C_0(\Omega)$ *pointwise* by

$$Tf(x) = \int_{\Omega} K(x,y) f(y) dy, \quad x \in \Omega, \tag{1.19}$$

where $K \in \mathrm{K}^{-m}_{\Omega}(\delta)$. We denote this by $T \in \mathrm{SK}^{-m}_{\Omega}(\delta)$, and we say that the operator T is associated with the kernel K. The *adjoint operator* $T^* \in \mathrm{SK}^{-m}_{\Omega}(\delta)$ is the uniquely defined operator which is associated with the *adjoint kernel*

$$K^* = (x,y) \mapsto \overline{K(y,x)}.$$

The adjoint operator satisfies $\langle T^* f \mid g \rangle = \langle f \mid Tg \rangle$ if $f, g \in C_0(\Omega)$.

The integral (1.19) exists since the function $y \mapsto K(x,y)$ is locally integrable for every $x \in \mathbb{R}^n$. This is unlike with Calderón–Zygmund operators, where the singularity causes the need to define T as a continuous operator $\mathcal{S} \to \mathcal{S}'$ with the weak boundedness property. The order of standard kernels and corresponding WSIO's is related to the estimate (1.16). Indeed, if a kernel K satisfies $K(x,y) = |x-y|^{-n+m}$, then the corresponding integral operator T is $-m$ homogeneous so that

$$T(f(\lambda \cdot)) = \lambda^{-m}(Tf)(\lambda \cdot), \quad \lambda > 0.$$

This justifies the terminology regarding the order of standard kernels and operators.

$T1$ theorem on admissible domains. The following theorem is our main result. It is similar in spirit to Theorem 1.12.

Theorem 1.20. *Assume that $\Omega \subset \mathbb{R}^n$ is an admissible domain and $T \in \mathrm{SK}_\Omega^{-m}(\delta)$, where $m \in \{1, 2, \ldots, n-1\}$. Then the following conditions are equivalent*

- $T\chi_\Omega, T^*\chi_\Omega \in \dot{f}_\infty^{m,2}(\Omega)$,
- $\partial^\alpha T, \partial^\alpha T^*$ *have a bounded extension to* $L^2(\Omega)$ *if* $|\alpha| = m$,
- $\partial^\alpha T, \partial^\alpha T^*$ *have a bounded extension to* $L^p(\Omega)$ *if* $|\alpha| = m$ *and* $1 < p < \infty$.

The derivatives are understood here in the weak sense.

This theorem is reformulated and proven in Section 6. Here we outline parts of its proof: Assume that $\Omega \subset \mathbb{R}^n$ is admissible, $T \in \mathrm{SK}_\Omega^{-m}(\delta)$ is associated with a kernel $K \in \mathrm{K}_\Omega^{-m}(\delta)$, and

$$T\chi_\Omega, T^*\chi_\Omega \in \dot{f}_\infty^{m,2}(\Omega). \tag{1.21}$$

We wish to establish the $L^2(\Omega)$-boundedness of $\partial^\alpha T, \partial^\alpha T^*$ for $|\alpha| = m$. The proof proceeds with a construction of an extension $\hat{T} \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta')$ so that $\partial^\alpha \hat{T}$ and $\partial^\alpha \hat{T}^*$ have a bounded extension to $L^2(\mathbb{R}^n)$ if $|\alpha| = m$, and also

$$\langle f \mid T^* g\rangle = \langle Tf \mid g\rangle = \langle \hat{T} f \mid g\rangle = \langle f \mid \hat{T}^* g\rangle, \tag{1.22}$$

if $f, g \in C_0(\Omega) \subset C_0(\mathbb{R}^n)$. The existence of such an extension $\hat{T}$ implies that $\partial^\alpha T$ and $\partial^\alpha T^*$ both have a bounded extension to $L^2(\Omega)$; this is seen by applying (1.22) with $g = (-1)^{|\alpha|}\partial^\alpha h$, $h \in C_0^\infty(\Omega)$, and using that $C_0^\infty(\Omega) \subset L^2(\Omega)$ is dense.

The extension $\hat{T}$ is constructed as follows. First consider the corresponding kernel extension problem. That is, how to construct a kernel $\hat{K} \in \mathrm{K}_{\mathbb{R}^n}^{-m}(\delta')$ so that

$$K = \hat{K} | \Omega \times \Omega \setminus \{(x, x)\}, \tag{1.23}$$

where T is associated with the kernel $K \in \mathrm{K}_\Omega^{-m}(\delta)$. For this purpose we establish an atomic decomposition for kernels of the class $\mathrm{K}_\Omega^{-m}(\delta)$, where $\Omega \subset \mathbb{R}^n$ is uniform. Using this decomposition, the kernel extension reduces to Hölder extension on the product domain $\Omega \times \Omega \subset \mathbb{R}^{2n}$. The described kernel extension procedure, treated in sections 4 and 5, immediately leads to the extension of the corresponding weakly singular interal operator: the operator $\hat{T} \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta')$, associated with the extended kernel $\hat{K}$, satisfies the condition (1.22) because of (1.23).

There is no reason why $\partial^\alpha \hat{T}$ and $\partial^\alpha \hat{T}^*$ would now have a bounded extension to $L^2(\mathbb{R}^n)$ if $|\alpha| = m$ or, equivalently, $\hat{T}1, \hat{T}^*1 \in \dot{f}_\infty^{m,2}(\mathbb{R}^n)$. We solve this problem by modifying $\hat{K}$ outside of $\Omega \times \Omega \setminus \{(x, x)\}$ in order to obtain a *bounded* extension, still satisfying (1.22). Within Section 3 we prove a $T\chi_\Omega$ theorem which states that this modification is possible if, and only if, the condition (1.21) holds. This modification is established with the aid of so called reflected paraproduct operators which are used, in addition to the standard reduction, in propagating certain error terms near the boundary to the complement of the domain. This is the place where the Whitney coplumpness of the domain is utilized.

2. WSIO's of convolution type

In this section we provide a proof for Theorem 1.7. That is, we obtain a concrete integral characterization of the boundedness of $f \mapsto \nabla k \star f$ in terms of the convolving kernel k and this characterization applies, for instance, in the homogeneous Hölder–Zygmund spaces and in the so called minimal Banach space. The proof relies on the Littlewood–Paley theory [FJW91]. We also construct unbounded operators that are counterexamples to certain natural questions related to the themes of this work.

2.1. **Homogeneous Besov spaces.** Before the definition of homogeneous Besov spaces, let us illustrate their connection to our problem and introduce the so called minimal Banach space $\dot{B}^{0,1}_1$. Consider a Banach space B that contains $\mathcal{S}_0(\mathbb{R}^n)$, the Schwartz functions with zeroth vanishing moment, and that is continuously embedded in $\mathcal{S}_0'(\mathbb{R}^n)$. Assume also that the B-norm is translation and L^1-dilatation invariant. The Banach space $B = \dot{B}^{0,1}_1$ satisfies the aforementioned properties and it is actually the minimal such space according to the following theorem, whose proof can be found in [FJW91, pp. 25–32].

Theorem 2.1. *Let $(B, ||\cdot||_B)$ be a Banach space that is continuously embedded in $\mathcal{S}_0'(\mathbb{R}^n)$ and that contains $S_0(\mathbb{R}^n)$. Assume that if $f \in B$, $h \in \mathbb{R}^n$, and $\lambda > 0$, then $\tau_h f = f(\cdot - h), \lambda^n f(\lambda \cdot) \in B$ and*

$$||\tau_h f||_B = ||f||_B, \quad ||\lambda^n f(\lambda \cdot)||_B = ||f||_B.$$

Then $\dot{B}^{0,1}_1 \subset B$ and this inclusion is bounded.

We take this result for granted and it gives a description of $\dot{B}^{0,1}_1(\mathbb{R}^n)$ for the purposes of the present discussion. The maximal characterization of Hardy spaces [Ste93, p. 90–91] implies that $H^1(\mathbb{R}^n)$ satisfies the assumptions of Theorem 2.1. As a consequence, the following inclusions are bounded

$$\dot{B}^{0,1}_1(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n).$$

Theorem 1.7 characterizes the boundedness of $f \mapsto \nabla k \star f$ in the minimal Banach space. This result is somewhat anticipated as the boundedness of singular integrals depend on delicate cancellations that manifest only if the function that we are integrating has the zeroth vanishing moment. And this is true, in particular, for functions in the minimal Banach space $\dot{B}^{0,1}_1(\mathbb{R}^n)$ as a subspace of the Hardy space. However, one should be careful here because the assumptions of Theorem 1.7 do not suffice for the boundedness of $f \mapsto \nabla k \star f$ in the Hardy space $H^1(\mathbb{R}^n)$, let alone in $L^1(\mathbb{R}^n)$.

Next we justify the $\dot{B}^{0,1}_1(\mathbb{R}) \to L^1(\mathbb{R})$ boundedness of those operators characterized in Theorem 1.7. This justification is not completely rigorous as is later indicated. This one-dimensional heuristics, based on a special atom decomposition, provides insight how integrals associated with the second order differences emerge. The following discussion regarding this decomposition is from [FJW91, pp. 25–32] and details can also be found in [Mey92, pp. 192–197]. For a finite interval $I = [a, b] \subset \mathbb{R}$

we denote its left and right halves by $I_l = [a,(a+b)/2]$ and $I_r = [(a+b)/2, b]$. The *special atom associated with* I is the function $h_I = |I|^{-1}(\chi_{I_l} - \chi_{I_r})$. Clearly $||h_I||_{L^1(\mathbb{R})} = 1$ and

$$\frac{d}{dx} h_J = |a-b|^{-1}(\delta_a + \delta_b - 2\delta_{(a+b)/2}) \tag{2.2}$$

in the sense of distributions, where δ_x is the Dirac's delta located at the point x. Let B consist of all those distributions $f \in \mathcal{S}_0'(\mathbb{R})$ having a *special atom representation*

$$f = \sum_{j=1}^{\infty} c_j h_{I_j} \tag{2.3}$$

where the convergence is in $\mathcal{S}_0'(\mathbb{R})$, the complex coefficients satisfy $\sum_{j=1}^{\infty} |c_j| < \infty$, and h_{I_j}'s are special atoms as described above. If we let

$$||f||_B = \inf \left\{ \sum_{j=1}^{\infty} |c_j| \,:\, f = \sum_{j=1}^{\infty} c_j h_{I_j} \text{ is a special atom representation} \right\},$$

then $(B, ||\cdot||_B)$ is a Banach space, known as the *special atom space*. Related spaces are studied in [OSS86] but the following result is taken from [FJW91, p. 32].

Theorem 2.4. *The special atom space* $(B, ||\cdot||_B)$ *coincides with* $\dot{B}_1^{0,1}(\mathbb{R})$ *and there exists a constant* $c > 0$ *so that*

$$c^{-1}||f||_B \le ||f||_{\dot{B}_1^{0,1}(\mathbb{R})} \le c||f||_B, \qquad \text{if } \ f \in B_1^{0,1}(\mathbb{R}).$$

Having this atomic decomposition at our disposal we can now proceed. Assume that k satisfies the integral condition (1.8).[2] Fix also $f \in \dot{B}_1^{0,1}(\mathbb{R})$ and apply (2.2) and (2.3) in order to justify the following manipulations

$$\begin{aligned} \frac{d}{dx}(k \star f)(x) &= \sum_{j=1}^{\infty} c_j \frac{d}{dx}(k \star h_{I_j})(x) \\ &= \sum_{j=1}^{\infty} c_j |a_j - b_j|^{-1}\big(k(x-a_j) + k(x-b_j) - 2k(x-(a_j+b_j)/2)\big). \end{aligned}$$

Integrating this identity with respect to the x-variable, yields

$$\Big|\Big| \frac{d}{dx}(k \star f) \Big|\Big|_{L^1(\mathbb{R})} \le C_k \sum_{j=1}^{\infty} |c_j|$$

where C_k denotes the left-hand side of (1.8). Then infimizing the right-hand side over all atomic decompositions (2.3) and applying Theorem 2.4 yields the boundedness of $f \mapsto \frac{d}{dx} k \star f$ from $\dot{B}_1^{0,1}(\mathbb{R})$ to $L^1(\mathbb{R})$.

[2] This requires a more specific quantification. A trivial one is that $k \in \mathcal{S}_0(\mathbb{R})$. Perhaps a more realistic one is to assume that k can be approximated by a sequence $(k_j)_{j\in\mathbb{N}} \subset \mathcal{S}_0(\mathbb{R})$ satisfying the integral estimates (1.8) uniformly.

A rigorous proof of Theorem 1.7 is based on the Littlewood–Paley theory. This is where we are heading from now on.

Definition of Besov spaces. The topic here is the scale of homogeneous Besov spaces and our treatment follows [FJ90, FJW91], see also [Tri83, Chapter 5]. These spaces allow us to measure the size and smoothness of a given distribution in high precision.

First we introduce some notation. The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is defined pointwise by $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ if $\xi \in \mathbb{R}^n$. The Fourier transform extends to $\mathcal{S}'$ by the duality $\langle \mathcal{F}\Lambda, \varphi\rangle = \langle \Lambda, \mathcal{F}\varphi\rangle$, where $\Lambda \in \mathcal{S}'$ is a tempered distribution and $\varphi \in \mathcal{S} \subset L^1(\mathbb{R}^n)$ is a Schwartz function [Rud91, p. 192]. A Schwartz function $\varphi : \mathbb{R}^n \to \mathbb{C} \in \mathcal{S}$ is a *Littlewood–Paley function* if it satisfies:

- $\hat{\varphi}$ is a real-valued function,
- $\operatorname{supp}\hat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 < |\xi| < 2\}$. In particular $\varphi \in \mathcal{S}_\infty$, that is, φ is a Schwartz function whose all moments vanish,
- $|\hat{\varphi}(\xi)| \geq c > 0$ if $3/5 \leq |\xi| \leq 5/3$.

The function ψ, $\hat{\psi} = \hat{\varphi}/\eta$, $\eta = \sum_{\nu\in\mathbb{Z}}(\hat{\varphi}(2^{-\nu}\cdot))^2$, is a *Littlewood–Paley dual function related to* φ. It is a Littlewood–Paley function itself, satisfying

$$\sum_{\nu\in\mathbb{Z}} \hat{\varphi}(2^{-\nu}\xi)\hat{\psi}(2^{-\nu}\xi) = 1, \quad \text{if } \xi \neq 0. \tag{2.5}$$

The topological dual of the closed subspace $\mathcal{S}_\infty \subset \mathcal{S}$ is isomorphic to $\mathcal{S}'/\mathcal{P}$, the space of tempered distributions modulo polynomials. The convolution of $\Lambda \in \mathcal{S}'/\mathcal{P}$ and a test function $\varphi \in \mathcal{S}$, denoted by $\Lambda \star \varphi = \varphi \star \Lambda$, is the element of $\mathcal{S}'/\mathcal{P}$ defined by $\langle \Lambda \star \varphi, \psi\rangle = \langle \Lambda, \tilde{\varphi} \star \psi\rangle$ for $\psi \in \mathcal{S}_\infty$. Here $\tilde{\varphi} = \varphi(-\cdot)$ is the reflection of φ. Furthermore, $\Lambda \star \varphi$ is regular and it coincides with the smooth function $x \mapsto \Lambda \star \varphi(x) = \langle \Lambda, \varphi(x - \cdot)\rangle$. The identity (2.5) yields the following *Calderón reproducing formula* for $\Lambda \in \mathcal{S}'/\mathcal{P}$,

$$\Lambda = \sum_{\nu\in\mathbb{Z}} \varphi_\nu \star \psi_\nu \star \Lambda \tag{2.6}$$

with convergence in the weak*-topology of $\mathcal{S}'/\mathcal{P}$ [FJW91, pp. 120–125]. Choose a Littlewood–Paley function ρ, depending on a priori function φ, so that $\hat{\rho}(\xi) = 1$ if $\xi \in \operatorname{supp}\hat{\varphi}$. Then

$$\varphi_\nu \star \rho_\nu = (\varphi \star \rho)_\nu = \varphi_\nu = 2^{\nu n}\varphi(2^\nu\cdot). \tag{2.7}$$

Having a fixed but arbitrary Littlewood–Paley function φ at our disposal, we define the homogeneous Besov spaces as follows.

Definition 2.8. Let $1 \leq p, q \leq \infty$ and $\alpha \in \mathbb{R}$. The *homogeneous Besov space* $\dot{B}^{\alpha,q}_p$ is the Banach space of all $f \in \mathcal{S}'/\mathcal{P}$ satisfying

$$||f||_{\dot{B}^{\alpha,q}_p} = ||f||_{\dot{B}^{\alpha,q}_p(\varphi)} = \left(\sum_{\nu\in\mathbb{Z}} \left(2^{\nu\alpha}||\varphi_\nu \star f||_{L^p}\right)^q\right)^{1/q} < \infty.$$

If $q = \infty$, we use sup-norm instead of ℓ^q-norm.

The index α is related to regularity or smoothness and the indices p, q are related to size. Higher value of the regularity index α implies higher local regularity properties. The size index p is more significant than the so called fine-index q.

The spaces $\dot{B}_p^{\alpha,q}$ are included in $\mathcal{S}'/\mathcal{P}$ by definition. However, the convergence in the Calderón reproducing formula (2.6) can be further analyzed to show that the formula

$$f \mapsto \sum_{\nu\in\mathbb{Z}} \varphi_\nu \star \psi_\nu \star f$$

defines a linear extension operator $\dot{B}_p^{\alpha,q} \to \mathcal{S}'_k$ if $k = \max\{[\alpha - n/p], -1\}$ [Kyr03]. Here $\mathcal{S}'_k$, $k \in \mathbb{N}_0$, denotes the topological dual of $\mathcal{S}_k = \{\varphi \in \mathcal{S} : \int x^\sigma \varphi = 0 \text{ if } |\sigma| \leq k\}$, which is a closed subspace of $\mathcal{S}$. Denote also $\mathcal{S}'_{-1} = \mathcal{S}'$. Furthermore, $[\alpha - n/p]$ denotes the greatest integer less than or equal to $\alpha - n/p$.

A non-linear extension operator $\mathcal{S}'/\mathcal{P} \to \mathcal{S}'$ exists and it allows us to compute the Fourier transform of $\Lambda \in \mathcal{S}'/\mathcal{P}$. This Fourier transform is uniquely defined outside the origin. There are also implications to regularity. Indeed, assume that $\alpha > 0$, $1 \leq p, q \leq \infty$, and $f \in \dot{B}_p^{\alpha,q}$. The Hahn–Banach theorem [Rud91, p. 61] allows us to extend f to a Schwartz distribution, that is, there is $\Lambda \in \mathcal{S}'$ such that $\Lambda|\mathcal{S}_\infty = f$. Let φ, ψ be a dual pair of Littlewood–Paley functions that are even (this is achieved by choosing φ so that its Fourier transform is even). Define a Schwartz function Φ,

$$\hat{\Phi}(\xi) = \begin{cases} 1, & \xi = 0 \\ \sum_{\nu=-\infty}^{0} \hat{\varphi}(2^{-\nu}\xi)\hat{\psi}(2^{-\nu}\xi), & \xi \neq 0. \end{cases}$$

Then $\Lambda = \Phi \star \Lambda + \sum_{\nu=1}^\infty \varphi_\nu \star \psi_\nu \star \Lambda$ in the weak* topology of $\mathcal{S}'$. Here $\Phi \star \Lambda$ is a smooth polynomially bounded function, according to Paley–Wiener theorem [Rud91, pp. 199–202], and the series $\sum_{\nu=1}^\infty \varphi_\nu \star \psi_\nu \star f$ converges absolutely in $L^p(\mathbb{R}^n)$ since $f \in \dot{B}_p^{\alpha,\infty}$ and $\alpha > 0$. Thus Λ is regular so that

$$\Lambda \in C^\infty_{\mathcal{P}}(\mathbb{R}^n) + L^p(\mathbb{R}^n), \tag{2.9}$$

where $C^\infty_{\mathcal{P}}(\mathbb{R}^n)$ is the space of smooth polynomially bounded functions.

Definition 2.8 is *φ-independent* so that the resulting norms associated with two different Littlewood–Paley functions are equivalent [FJ90, Remark 2.6.]. This provides the means to interprete Littlewood–Paley functions as eigenfunctions for elliptic partial differential operators. To illustrate this further consider powers of the Laplacian

$$(-\Delta)^m \varphi_\nu = 2^{2m\nu}((-\Delta)^m\varphi)_\nu, \quad m \in \mathbb{N}.$$

Here $(-\Delta)^m\varphi = \mathcal{F}^{-1}[(4\pi^2)^m|\xi|^{2m}\hat{\varphi}]$ remains a Littlewood–Paley function. In this sense the functions φ_ν are eigenfunctions with corresponding eigenvalues $2^{2m\nu}$. One can consider more general homogeneous elliptic pseudodifferential operators like fractional powers of the Laplacian provided by Riesz potentials,

$$(\sqrt{-\Delta})^\alpha \varphi = \mathcal{I}^{-\alpha}\varphi = \mathcal{F}^{-1}[(2\pi|\xi|)^\alpha \hat{\varphi}], \quad \alpha \in \mathbb{R}. \tag{2.10}$$

Hence the Littlewood–Paley functions are eigenfunctions whose eigenvalues are $2^{\alpha\nu}$. It is not difficult to show that Riesz potentials $\mathcal{I}^\alpha$ map $\mathcal{S}_\infty$ to itself continuously and they map $\mathcal{S}'/\mathcal{P}$ to itself when defined via the duality $\langle \mathcal{I}^\alpha \Lambda, \varphi\rangle = \langle \Lambda, \mathcal{I}^\alpha \varphi\rangle$. Using homogeneities of the Fourier transform it is easy to verify that, if $0 < \beta < n$, then

$$\mathcal{I}^\beta \varphi(x) = C_{n,\beta} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-\beta}} dy, \quad \text{if } x \in \mathbb{R}^n, \tag{2.11}$$

given that φ has some regularity, say, if $\varphi \in C_0(\mathbb{R}^n)$. Hereby we are lead to weakly singular integral operators as $\mathcal{I}^m \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ if $m \in \{1, 2, \ldots, n-1\}$ and $0 < \delta < 1$.

The following result establishes the fact that different means to quantify the boundedness of $\nabla k \star f$ on homogeneous Besov spaces are equivalent. Notice that the distribution derivative ∂^α, $\alpha \in \mathbb{N}_0^n$, induces a linear operator, mapping $\mathcal{S}'/\mathcal{P}$ to itself.

Proposition 2.12. *Let $\alpha, s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Then $\mathcal{I}^s$ is an isomorphism of $\dot{B}^{\alpha,q}_p$ onto $\dot{B}^{s+\alpha,q}_p$ and the Riesz transforms $R_j = -\mathcal{I}^1 \partial_j$, $j = 1, 2, \ldots, n$, are bounded on $\dot{B}^{\alpha,q}_p$. In particular, if $k \in \mathcal{S}'/\mathcal{P}$, the following are equivalent*

- *$f \mapsto k \star f$ is bounded $\dot{B}^{\alpha,q}_p \to \dot{B}^{1+\alpha,q}_p$,*
- *$f \mapsto \mathcal{I}^{-1} k \star f = \sum_{j=1}^n R_j(\partial_j k \star f)$ is bounded on $\dot{B}^{\alpha,q}_p$,*
- *$f \mapsto \partial_j k \star f$ is bounded on $\dot{B}^{\alpha,q}_p$ for every $j = 1, 2, \ldots, n$.*

That is, the initial domain of definition of these operators is $\mathcal{S}_\infty$ and, in the positive case, they admit a bounded extension to the corresponding Besov space. If $1 \le p, q < \infty$, then this bounded extension is unique.

The verification of this result is straightforward by using the φ-independence and the reproducing formula (2.7). The specific details are omitted.

Various homogeneous Besov spaces have a characterization in terms of relevant and concrete function spaces [FJW91, Gra04]. The minimal Banach space $\dot{B}^{0,1}_1$ was mentioned. The homogeneous Sobolev space $\dot{B}^{\alpha,2}_2 \approx \dot{W}^{\alpha,2}(\mathbb{R}^n)$ consists of $f \in \mathcal{S}'/\mathcal{P}$ with $\mathcal{I}^{-\alpha} f \in L^2(\mathbb{R}^n)$. This space is normed with

$$||f||_{\dot{W}^{\alpha,2}} = ||\mathcal{I}^{-\alpha} f||_{L^2}.$$

Furthermore, $\dot{B}^{\alpha,\infty}_\infty \approx \dot{\mathcal{C}}^\alpha(\mathbb{R}^n)$ is the homogeneous Hölder–Zygmund space. If $\alpha = m + \delta$, where $m \in \mathbb{N}_0$ and $0 < \delta < 1$, this space consists of continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$|\Delta^{m+1}_h (f, x)| \le C|h|^{m+\delta}, \quad x, h \in \mathbb{R}^n,$$

where $\Delta^{m+1}_h(f, \cdot) = (\tau_{-h} - \mathrm{id})^{m+1} f$ is the order $(m + 1)$ difference of f. These differences provide a similar characterization of all those homogeneous Besov spaces for which $\alpha > 0$. This is what we study next and our focus is on the space $\dot{B}^{1,\infty}_1$.

2.2. **Boundedness of convolution operators.** The purpose of this section is to provide a proof of Theorem 1.7. We begin with a characterization of $\dot{B}^{1,\infty}_1$ in terms of the integral condition (1.8). Then we characterize the boundedness of $f \mapsto \mathcal{I}^{-1} k \star f$ in terms of test functions. Removing this test function dependence in the endpoint exponents yields Theorem 1.7 as the resulting condition is precisely that $k \in \dot{B}^{1,\infty}_1$.

Characterization of $\dot{B}_1^{1,\infty}$. Recall that $\Delta_h^m(f,\cdot) = (\tau_{-h} - \mathrm{id})^m f$ is the order $m \in \mathbb{N}$ difference with offset $h \in \mathbb{R}^n$ of $f : \mathbb{R}^n \to \mathbb{C}$. The difference operator extends to $\mathcal{S}'/\mathcal{P}$ via duality

$$\langle \Delta_h^m(\Lambda,\cdot),\varphi\rangle = \langle \Lambda, \Delta_{-h}^m(\varphi,\cdot)\rangle, \quad \Lambda \in \mathcal{S}'/\mathcal{P}.$$

These difference operators yield a characterization of homogeneous Besov spaces with positive regularity. To be more precise, we fix $1 \le p,q \le \infty$ and $\alpha = m+\delta > 0$, where $m \in \mathbb{N}_0$ and $0 \le \delta < 1$. Then $m+1 > m+\delta$ and the following norm-equivalence holds true [Tri83, p. 242]

$$||k||_{\dot{B}_p^{\alpha,q}} \approx \begin{cases} \Big(\int_{\mathbb{R}^n} |h|^{-\alpha q} ||\Delta_h^{m+1}(k,\cdot)||_{L^p}^q \frac{dh}{|h|^n}\Big)^{1/q}, & q < \infty, \\ \sup_{h\neq 0} |h|^{-\alpha} ||\Delta_h^{m+1}(k,\cdot)||_{L^p}, & q = \infty. \end{cases} \tag{2.13}$$

The restriction $\alpha > 0$ in (2.13) is crucial. For instance, the Dirac's delta satisfies $\delta \in \dot{B}_1^{0,\infty}$ but $\Delta_h^{m+1}(\delta,\cdot)$ remains only a measure if $m \in \mathbb{N}_0$ and $h \neq 0$. However, Proposition 2.12 implies that $\mathcal{I}^1(\delta) = C_n|x|^{-n+1} \in \dot{B}_1^{1,\infty}$.

The right-hand side of (2.13) requires an interpretation. The series in the Calderón reproducing formula (2.6) converges *a priori* in the weak* topology of $\mathcal{S}'/\mathcal{P}$. But there is more to this if we apply the formula to $k \in \dot{B}_p^{\alpha,q}$ with indices as specified above. Fix a Littlewood–Paley function φ and its dual function ψ. Then there exists polynomials $\{P_N : N \in \mathbb{N}\} \subset \mathcal{P}_\kappa$, where $\kappa = \max\{[\alpha - n/p], -1\} \le m$ and $\mathcal{P}_{-1} = \{0\}$, and a tempered distribution $K \in \mathcal{S}'$ such that

$$K = \lim_{N\to\infty} \Big(\sum_{\nu=-N}^{\infty} \varphi_\nu \star \psi_\nu \star k + P_N \Big), \tag{2.14}$$

with convergence in the weak* topology of $\mathcal{S}'$ [Kyr03]. The Calderón reproducing formula states that $K|\mathcal{S}_\infty = k$ and (2.9) shows that $K \in C_{\mathcal{P}}^\infty(\mathbb{R}^n) + L^p(\mathbb{R}^n)$. In particular, if $h \in \mathbb{R}^n$, we define

$$||\Delta_h^{m+1}(k,\cdot)||_{L^p} = ||\Delta_h^{m+1}(K,\cdot)||_{L^p}$$

and these quantities are well-defined. We also have the identification

$$\int_{\mathbb{R}^n} \Delta_h^{m+1}(K,x)\varphi(x)dx = \sum_{\nu=-\infty}^{\infty} \langle \Delta_h^{m+1}(\varphi_\nu \star \psi_\nu \star k,\cdot),\varphi\rangle, \quad \varphi \in \mathcal{S} \tag{2.15}$$

by using (2.14) with the identity $\Delta_h^{m+1}(P,\cdot) \equiv 0$ if $P \in \mathcal{P}_m$ (this follows from the representation (A.1) for the order $m+1$ difference). Using the identification (2.15) with an approximation of the identity, we see that $\Delta_h^{m+1}(K,\cdot)$ is independent of the renormalizing polynomials.

We have settled the interpretation of (2.13) and then we continue the proof of this norm-equivalence under the assumptions $p = 1$, $q = \infty$, and $\alpha = 1$. The case $p = \infty$ is treated in the same manner but the general case requires further estimates that we omit.

Lemma 2.16. *Let $k \in \dot{B}_1^{1,\infty}$ and φ, ψ be a dual pair of Littlewood–Paley functions. Then k satisfies the estimate*

$$\sup_{y\neq 0}\left\{|y|^{-1}\sum_{\nu\in\mathbb{Z}}||\Delta_y^2(\varphi_\nu\star\psi_\nu\star k,\cdot)||_{L^1}\right\}<\infty. \tag{2.17}$$

Proof. Fix $y \in \mathbb{R}^n \setminus \{0\}$ and choose $\nu_0 \in \mathbb{Z}$ so that $2^{\nu_0} \leq |y|^{-1} < 2^{\nu_0+1}$. Split the summation in (2.17) as $\sum_{\nu\in Z} = \sum_{\nu\geq\nu_0} + \sum_{\nu<\nu_0} = \Sigma_1 + \Sigma_2$. Applying the triangle-inequality and Young's inequality, we obtain

$$|\Sigma_1| \leq 4\sum_{\nu\geq\nu_0}||\psi_\nu||_{L^1}||\varphi_\nu\star k||_{L^1} \leq C_\psi\sum_{\nu\geq\nu_0}2^{-\nu}2^\nu||\varphi_\nu\star k||_{L^1} \leq C_\psi||k||_{\dot{B}_1^{1,\infty}}|y|. \tag{2.18}$$

Next we estimate $|\Sigma_2|$. Applying the translation invariance and linearity of the convolution, we obtain the identity

$$\Delta_y^2(\varphi_\nu\star\psi_\nu\star k,x) = (\Delta_y^2(\psi_\nu,\cdot))\star\varphi_\nu\star k(x), \quad x\in\mathbb{R}^n.$$

Using this together with Young's inequality, we get the estimate

$$|\Sigma_2| \leq \sum_{\nu<\nu_0}||\Delta_y^2(\psi_\nu,\cdot)||_{L^1}||\varphi_\nu\star k||_{L^1} = ||k||_{\dot{B}_1^{1,\infty}}\sum_{\nu<\nu_0}2^{-\nu}||\Delta^2_{(2^\nu y)}(\psi,\cdot)||_{L^1}. \tag{2.19}$$

To estimate further, we use the following integral representation

$$\begin{aligned}\Delta_h^2(\psi,x) &= (\tau_{-h}-\mathrm{id})^2\psi(x)\\ &= \sum_{j_1=1}^n\sum_{j_2=1}^n h_{j_1}h_{j_2}\int_0^1\int_0^1(\partial_{j_1}\partial_{j_2}\psi)(x+(\theta_1+\theta_2)h)d\theta_1 d\theta_2,\end{aligned}$$

which follows from applying the identity $\psi(x+h)-\psi(x) = \int_0^1 h\cdot\nabla\psi(x+\theta h)d\theta$ twice for $h\in\mathbb{R}^n\setminus\{0\}$. Taking the absolute values in this representation, then using triangle-inequality and Fubini's theorem yields

$$||\Delta_h^2(\psi,\cdot)||_{L^1} \leq |h|^2\sum_{|\alpha|=2}c_\alpha||\partial^\alpha\psi||_{L^1}.$$

Applying this estimate with $h = 2^\nu y$ allows us to continue the estimate (2.19) as follows

$$|\Sigma_2| \leq C_\psi||k||_{\dot{B}_1^{1,\infty}}\sum_{\nu<\nu_0}2^{-\nu}(2^\nu|y|)^2 \leq C_\psi||k||_{\dot{B}_1^{1,\infty}}|y|. \tag{2.20}$$

The estimate (2.17) follows by combining (2.18) and (2.20). □

We are ready for the characterization of $\dot{B}_1^{1,\infty}$. Recall that, by definition, $k\in\dot{B}_1^{1,\infty}$ means that $k\in\mathcal{S}'/\mathcal{P}$ satisfies the uniform estimate $\sup_{\nu\in\mathbb{Z}}2^\nu||\varphi_\nu\star k||_{L^1}<\infty$ for a Littlewood–Paley function $\varphi\in\mathcal{S}_\infty$.

Theorem 2.21. *Let $k \in \mathcal{S}'/\mathcal{P}$. Then k extends to to a regular tempered distribution $K \in \mathcal{S}'$ which satisfies the integral estimate*

$$\sup_{y\neq 0}\left\{|y|^{-1}\int_{\mathbb{R}^n}|K(x-y)+K(x+y)-2K(x)|dx\right\}=B<\infty \tag{2.22}$$

if, and only if, $k \in \dot{B}^{1,\infty}_1$.

Proof. Fix a dual pair of Littlewood–Paley functions φ, ψ, satisfying $\varphi(x) = \varphi(-x)$ if $x \in \mathbb{R}^n$. This is achieved by choosing φ so that its Fourier transform is even. Assume first that k is a restriction of a regular tempered distribution K, satisfying the integral condition (2.22). Fix $\nu \in \mathbb{Z}$. Because $\varphi \in \mathcal{S}_\infty$ is even and $\int \varphi = 0$, we obtain for almost every $x \in \mathbb{R}^n$,

$$|\varphi_\nu \star k(x)| = |\varphi_\nu \star K(x)| \leq 2^{-1}\int_{\mathbb{R}^n}|\varphi_\nu(y)||K(x-y)+K(x+y)-2K(x)|dy.$$

Integrating this estimate with Fubini's theorem and (2.22), yields

$$||\varphi_\nu \star k||_{L^1} \leq B\int_{\mathbb{R}^n}|\varphi_\nu(y)||y|dy = B2^{-\nu}\int_{\mathbb{R}^n}2^{\nu n}|\varphi(2^\nu y)||2^\nu y|dy = C_\varphi B 2^{-\nu}.$$

This implies that $k \in \dot{B}^{1,\infty}_1$ as desired. Then assume that $k \in \dot{B}^{1,\infty}_1$. At this stage we could apply (2.14) and (2.15) with Lemma 2.16 to reach $K \in \mathcal{S}'$ and identify $\Delta^2_y(K,\cdot)$, $y \neq 0$, with the integrable function $\sum_{\nu\in\mathbb{Z}}\Delta^2_y(\varphi_\nu \star \psi_\nu \star k,\cdot)$. We include further details in the easy case $n > 1$. Fix $\nu \in \mathbb{Z}$. Then

$$||\mathcal{F}(\varphi_\nu \star \psi_\nu \star k)||_{L^\infty} \leq ||\varphi_\nu \star \psi_\nu \star k||_{L^1} \leq C_\psi||\varphi_\nu \star k||_{L^1} \leq C_{\psi,k}2^{-\nu}$$

and properties of Littlewood-Paley functions imply that $\operatorname{supp}\mathcal{F}(\varphi_\nu \star \psi_\nu \star k)$ is contained in the annulus $B(0,2^{\nu+1}) \setminus B(0,2^{\nu-1})$. Combining the observations above and using the assumption $n > 1$, we see that the formula

$$\mathcal{F}K = \sum_{\nu=-\infty}^{\infty}\mathcal{F}(\varphi_\nu \star \psi_\nu \star k) \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) \subset \mathcal{S}'$$

defines a Fourier transform of $K \in \mathcal{S}'$ with $K = \sum_{\nu=-\infty}^{\infty}\varphi_\nu \star \psi_\nu \star k$ in the weak* topology of $\mathcal{S}'$. The Calderón reproducing formula (2.6) implies that K is an extension of k, that is $K|\mathcal{S}_\infty = k$. Estimating as in (2.9), we see that $K \in C^\infty_{\mathcal{P}}(\mathbb{R}^n) + L^1(\mathbb{R}^n)$. Then, assuming that $y \in \mathbb{R}^n \setminus \{0\}$, we apply Lemma 2.16 to see that $\Delta^2_y(K,\cdot) \in L^1(\mathbb{R}^n)$ satisfies the estimate

$$||\Delta^2_y(K,\cdot)||_{L^1} \leq \sum_{\nu=-\infty}^{\infty}||\Delta^2_y(\varphi_\nu \star \psi_\nu \star k,\cdot)||_{L^1} \leq C|y|.$$

By the change of variables $w = x + y$ we replace the integrand $\Delta^2_y(K,\cdot)$ on the left-hand side by the absolute value of the symmetric second order difference, which yields the desired estimate (2.22). □

Characterization of the boundedness. Next we characterize the boundedness of the operator $f \mapsto \nabla k \star f$ on homogenenous Besov spaces in terms of the convolving kernel. In certain endpoints the condition is that $k \in \dot{B}_1^{1,\infty}$. Moreover, this function space admits a concrete characterization given in Theorem 2.21.

Our proof is an adaptation of Peetre's characterization of the boundedness of $f \mapsto k \star f$ [Pee76, pp. 132–136] where the corresponding endpoint condition reads as $k \in \dot{B}_1^{0,\infty}$. This function space does not have a concrete integral characterization as was pointed out in connection with (2.13).

Theorem 2.23. *Let $\alpha \in \mathbb{R}$, $1 \le p, q \le \infty$, and $k \in \mathcal{S}'/\mathcal{P}$. Then the convolution operator $f \mapsto k \star f$ has a bounded extension $\dot{B}_p^{\alpha,q} \to \dot{B}_p^{1+\alpha,q}$ if, and only if,*

$$\sup_{\nu \in \mathbb{Z}} 2^\nu ||\varphi_\nu \star k \star f||_{L^p} \le C_k ||f||_{L^p}, \quad f \in \mathcal{S}_\infty, \tag{2.24}$$

where φ is any Littlewood–Paley function.

Proof. We prove the case $\alpha = 0$. The general case then follows from Proposition 2.12 and the identity $\mathcal{I}^\alpha(k \star \mathcal{I}^{-\alpha} f) = k \star f$ for $f \in \mathcal{S}_\infty$. Fix Littlewood–Paley functions φ, ρ as in (2.7), satisfying $\varphi \star \rho = \varphi$.

First we prove necessity. Fix $f \in \mathcal{S}_\infty$ and $\nu \in \mathbb{Z}$. Using the assumption about boundedness, we get

$$\begin{aligned} 2^\nu ||\varphi_\nu \star k \star f||_{L^p} = 2^\nu ||\varphi_\nu \star k \star \rho_\nu \star f||_{L^p} &\le ||k \star \rho_\nu \star f||_{\dot{B}_p^{1,q}(\varphi)} \\ &\le C_k ||\rho_\nu \star f||_{\dot{B}_p^{0,q}(\varphi)} \le C_k ||\rho_\nu \star f||_{\dot{B}_p^{0,1}(\varphi)}. \end{aligned} \tag{2.25}$$

In order to estimate further, notice that the supports of $\hat{\varphi}$ and $\hat{\rho}$ are contained in $\{\xi \in \mathbb{R}^n : 1/2 < |\xi| < 2\}$. Thus, using the Fourier transform, we see that $\varphi_\mu \star \rho_\nu = 0$ if $|\mu - \nu| > 1$. Applying this and the Young's inequality, we obtain

$$||\rho_\nu \star f||_{\dot{B}_p^{0,1}(\varphi)} = \sum_{\mu : |\mu - \nu| \le 1} ||\varphi_\mu \star \rho_\nu \star f||_{L^p} \le C_{\varphi,\rho} ||f||_{L^p}. \tag{2.26}$$

Combine estimates (2.25) and (2.26) in order to obtain the estimate (2.24).

Then we prove sufficiency. Let $f \in \mathcal{S}_\infty$ and $\nu \in \mathbb{Z}$. Then using (2.24), we get

$$2^\nu ||\varphi_\nu \star k \star f||_{L^p} = 2^\nu ||\varphi_\nu \star k \star \rho_\nu \star f||_{L^p} \le C_k ||\rho_\nu \star f||_{L^p}.$$

Taking the $\ell^q(\mathbb{Z})$-norms we get the estimate $||k \star f||_{\dot{B}_p^{1,q}(\varphi)} \le C_k ||f||_{\dot{B}_p^{0,q}(\rho)}$. The boundedness result follows from this and the φ-independence. □

Theorem 2.23 has the following convenient interpretation in the case $p \in \{1, \infty\}$, connecting the boundedness properties of $f \mapsto k \star f$ to the condition $k \in \dot{B}_1^{1,\infty}$.

Theorem 2.27. *Let $k \in \mathcal{S}'/\mathcal{P}$, $\alpha \in \mathbb{R}$, $p \in \{1, \infty\}$, and $1 \le q \le \infty$. Then the convolution operator $f \mapsto k \star f$ has a bounded extension $\dot{B}_p^{\alpha,q} \to \dot{B}_p^{1+\alpha,q}$ if, and only if, $k \in \dot{B}_1^{1,\infty}$, that is, $k \in \mathcal{S}'/\mathcal{P}$ satisfies*

$$\sup_{\nu \in \mathbb{Z}} 2^\nu ||\varphi_\nu \star k||_{L^1} < \infty$$

for any Littlewood–Paley function φ.

Proof. Fix Littlewood–Paley functions φ, ρ as in (2.7), satisfying $\varphi \star \rho = \varphi$. We begin with *sufficiency for $p \in \{1, \infty\}$*. For $f \in \mathcal{S}_\infty$ and $\nu \in \mathbb{Z}$, apply Young's inequality to obtain the estimate

$$2^\nu ||\varphi_\nu \star k \star f||_{L^p} \leq 2^\nu ||\varphi_\nu \star k||_{L^1} ||f||_{L^p} \leq C||f||_{L^p}.$$

Thus k satisfies (2.24) and Theorem 2.23 implies the boundedness result. *Then we prove necessity for $p = 1$.* Applying Theorem 2.23 we obtain the following estimate

$$2^\nu ||\varphi_\nu \star k||_{L^1} = 2^\nu ||\varphi_\nu \star k \star \rho_\nu||_{L^1} \leq C_k ||\rho_\nu||_{L^1} = C_k ||\rho||_{L^1}, \quad \nu \in \mathbb{Z}.$$

The right-hand side is independent of ν and therefore we have $k \in \dot{B}_1^{1,\infty}$. *Finally we consider necessity for $p = \infty$.* For a fixed $\nu \in \mathbb{Z}$ it suffices to show the same estimate as before,

$$||2^\nu \varphi_\nu \star k||_{L^1} \leq C||\rho_\nu||_{L^1} = C||\rho||_{L^1} \tag{2.28}$$

with C independent of ν. Fix $R > 0$ and choose a sequence of smooth functions $(g_j)_{j \in \mathbb{N}}$ such that $|g_j| \leq \chi_{B(0,R+1)}$ and $\lim_{j\to\infty} g_j = \exp(-i \arg(2^\nu \varphi_\nu \star k))\chi_{B(0,R)}$, $\arg(0) = 0$, pointwise almost everywhere. Denote also $h_j(x) = g_j(-x)$ for every $j \in \mathbb{N}$. Now using the identity $\varphi = \varphi \star \rho$, the Dominated convergence theorem, and Theorem 2.23 we have the estimate

$$\begin{aligned}
&\int_{B(0,R)} |2^\nu \varphi_\nu \star k(x)| dx = \lim_{j\to\infty} \left| \int_{\mathbb{R}^n} (2^\nu \varphi_\nu \star k)(x) g_j(x) dx \right| \\
&\leq \limsup_{j\to\infty} 2^\nu ||\varphi_\nu \star k \star \rho_\nu \star h_j||_{L^\infty} \leq \limsup_{j\to\infty} C_k ||\rho_\nu||_{L^1} ||h_j||_{L^\infty} \leq C_k ||\rho||_{L^1}.
\end{aligned}$$

We inserted ρ_ν to ensure that Theorem 2.23 applies. Indeed, it may be that $h_j \notin \mathcal{S}_\infty$ but in any case we have $\rho_\nu \star h_j \in \mathcal{S}_\infty$. The estimate (2.28) follows from Fatou's lemma and the estimate above as the right-hand side there is independent of R. □

The following concrete characterization for the boundedness of $f \mapsto k \star f$ follows by combining Theorem 2.21 and Theorem 2.27.

Theorem 2.29. *Let $\alpha \in \mathbb{R}$, $p \in \{1, \infty\}$, and $1 \leq q \leq \infty$. Assume that $k \in \mathcal{S}'/\mathcal{P}$. Then the operator $f \mapsto k \star f : \mathcal{S}_\infty \to \mathcal{S}'/\mathcal{P}$ has a bounded extension $\dot{B}_p^{\alpha,q} \to \dot{B}_p^{1+\alpha,q}$ if, and only if, k has an extension to a regular tempered distribution $K \in \mathcal{S}'$ satisfying the integral estimate*

$$\sup_{y \neq 0} \left\{ |y|^{-1} \int_{\mathbb{R}^n} |K(x-y) + K(x+y) - 2K(x)| dx \right\} < \infty. \tag{2.30}$$

In Theorem 1.7 we proposed a somewhat different formulation; its proof follows from Theorem 2.29 and Proposition 2.12.

Remark 2.31. (i) Let $\alpha, s \in \mathbb{R}$, $1 \leq q \leq \infty$, and $p \in \{1, \infty\}$. Using the Riesz potentials and Theorem 2.27 it is simple to verify that the convolution operator $f \mapsto k \star f$ has a bounded extension $\dot{B}_p^{\alpha,q} \to \dot{B}_p^{s+\alpha,q}$ if, and only if,

$k \in \dot{B}_1^{s,\infty}$. Assuming that $m+1 > s > 0$ then, according to (2.13), the condition $k \in \dot{B}_1^{s,\infty}$ is equivalent to the integral condition

$$\sup_{y \neq 0} \left\{ |y|^{-s} \int_{\mathbb{R}^n} |\Delta_y^{m+1}(k,x)| dx \right\} < \infty;$$

(ii) Assuming that $k \in \dot{B}_1^{s,\infty}$, the operator $f \mapsto k \star f$ has a bounded extension $\dot{B}_p^{\alpha,q} \to \dot{B}_p^{s+\alpha,q}$ if $\alpha \in \mathbb{R}$ and $1 \leq p,q \leq \infty$. This follows from the Young's inequality and φ-independence of the homogeneous Besov norm;

(iii) An inhomogenous counterpart of Theorem 2.27 exists. That is, the corresponding result holds within the scale of inhomogenous Besov spaces $B_p^{\alpha,q}$. For the definition of inhomogenous spaces, see [Tri92, p. 28]. For the proof of the result, see [Tai65, p. 827].

2.3. **Unbounded operators.** Theorem 1.7 characterizes the boundedness of the operator $f \mapsto \nabla k \star f$ on spaces $\dot{B}_1^{0,1}$ and $\dot{\mathcal{C}}^\alpha$. The characterizing integral condition (1.8), equivalent to that $k \in \dot{B}_1^{1,\infty}$, is also sufficient for the $L^2(\mathbb{R}^n)$ boundedness of the operator in question; this follows from (ii) in Remark 2.31. Here we give a counterexample which implies the following negative result:

- *The condition $k \in \dot{B}_1^{1,\infty}$ is insufficient for the boundedness of the operator $f \mapsto \nabla k \star f$ in the Hardy space $H^1(\mathbb{R}^n)$ or even in $L^p(\mathbb{R}^n)$, $p \in (1,\infty) \setminus \{2\}$.*

We don't know if the L^p-boundedness holds with a kernel $k \in \dot{B}_1^{1,\infty}$ satisfying the pointwise size condition $|k(x)| \leq c_k |x|^{-n+1}$. At least this size condition itself is insufficient for the L^p-boundedness, as our second counterexample implies:

- *The L^2-boundedness of the operator $f \mapsto \nabla k \star f$ can fail under the size condition $|k(x)| \leq c_k |x|^{-n+1}$. This is so even when combined with a natural pointwise Lipschitz regularity condition.*

To complement this negative result, let us also state a closely connected positive result here. Assume that a locally integrable kernel satisfies the size condition $|k(x)| \leq c_k |x|^{-n+1}$ and the $(1+\delta)$-Hölder condition for $0 < \delta < 1$, that is,

$$|k(x+2h) - 2k(x+h) + k(x)| \leq c_k |h|^{1+\delta} |x|^{-n-\delta}, \quad 4|h| \leq |x|.$$

Then we gain boundedness of the operator $f \mapsto \nabla k \star f$ in the spaces $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$. Indeed, Theorem 3.40 shows that the operator $T = f \mapsto k \star f \in \mathrm{SK}_{\mathbb{R}^n}^{-1}(\delta)$ satisfies the assumptions of Theorem 1.20. Further analysis yields boundedness also in the Hardy space [Väh08]. It is also interesting to notice that k here satisfies (1.8).

Failure of the L^p-boundedness, $p \neq 2$. We show that the condition $k \in \dot{B}_1^{1,\infty}$ itself does not suffice for the boundedness of $f \mapsto \nabla k \star f$ on L^p-spaces, $p \neq 2$. The construction here is due to Triebel [Tri79] and it exploits certain intricate cancellations captured by the Littlewood–Paley inequality. This inequality implies that the relevant cancellation effects manifest already within dyadic frequency ranges and two separate Littlewood–Paley projections $\varphi_\nu \star f$ and $\varphi_\mu \star f$, $\mu \neq \nu$, do not interact by producing significant cancellations. That is, if $1 < p < \infty$, we have

$$C^{-1}||f||_{L^p} \le \Big|\Big|\Big(\sum_{\nu\in\mathbb{Z}} |\varphi_\nu \star f|^2\Big)^{1/2}\Big|\Big|_{L^p} \le C||f||_{L^p}, \tag{2.32}$$

where φ is any Littlewood–Paley function and C is independent of $f \in L^p(\mathbb{R}^n)$ [FJW91, p. 42]. Then consider the following functions $G_N, F_N : \mathbb{R}^n \to \mathbb{C}$ defined in the Fourier variable

(A) $\mathcal{F}(G_N)(\xi) = \sum_{m=1}^N \pi(\xi - 2^m e_1)$ is the sum of translations of a fixed bump function $\pi \in C_0^\infty(\mathbb{R}^n)$ at dyadic points $2^m e_1$ for $m = 1, 2, \ldots, N$,

(B) $\mathcal{F}(F_N)(\xi) = \sum_{m=1}^N e^{-2\pi i 2^m \xi_1}\pi(\xi - 2^m e_1)$ is as in (A) but with the m'th translation modulated by $\xi \mapsto e^{-2\pi i 2^m \xi_1}$.

The inequality (2.32) implies that the L^p-norm of the function G_N is roughly $N^{1/2}$. To reach this estimate, it suffices to notice that

$$|\varphi_\nu \star G_N| = |\varphi_\mu \star G_N|, \quad \text{if } 1 \le \mu, \nu \le N.$$

The functions F_N are uniformly bounded with respect to N; indeed, due to modulation in the Fourier variable, the m'th bump function is concentrated near to the point $2^m e_1$ in the spatial variable. Interpolation between the endpoint spaces L^2 and L^∞ implies that the L^p-norm of the function F_N is roughly $N^{1/p}$ if $2 < p < \infty$. *The punchline is as follows:* an operator, transforming F_N to G_N independent of N, is unbounded in L^p for $2 < p < \infty$ because $\lim_{N\to\infty} N^{1/2-1/p} = \infty$. We shall realize the program above in what follows.

First we construct a kernel $k \in \dot{B}_1^{1,\infty}$. For this purpose, fix a Littlewood–Paley function φ satisfying

$$\mathcal{F}(\varphi)|B(e_1,\varepsilon) \equiv 1, \quad \mathcal{F}(\varphi_{-1})|B(e_1,\varepsilon) \equiv 0 \equiv \mathcal{F}(\varphi_1)|B(e_1,\varepsilon), \tag{2.33}$$

for some $\varepsilon \in (0,1)$. Let $\pi \ne 0$ be a real-valued Schwartz function satisfying $\operatorname{supp}\pi \subset B(0,\varepsilon)$ and define $\hat{f}_m(\xi) = e^{2\pi i 2^m \xi_1}\pi(\xi - 2^m e_1)$ for all $m \in \mathbb{N}$. Using (2.33), we see that $\mathcal{F}(\varphi_\nu)\hat{f}_m = \delta_{\nu m}\hat{f}_m$ if $m \in \mathbb{N}$ and $\nu \in \mathbb{Z}$. Define $g(\xi) = \sum_{m=1}^\infty 2^{-m}\hat{f}_m(\xi)$. We claim that $k = \mathcal{F}^{-1}g$ has the desired properties. By the properties of φ, we have $\varphi_\nu \star k = 0$ if $\nu < 1$. For $\nu \ge 1$, we have

$$2^\nu||\varphi_\nu \star k||_{L^1} = ||f_\nu||_{L^1} = ||\tau_{-2^\nu e_1}(e^{2\pi i 2^\nu x_1}\mathcal{F}^{-1}\pi)||_{L^1} = ||\mathcal{F}^{-1}\pi||_{L^1} < \infty.$$

Thus $k \in \dot{B}_1^{1,\infty}$, as desired. In particular, according to Theorem 2.21, the kernel k satisfies the integral condition (1.8). Applying the Fourier transform we see that, in the sense of distributions,

$$k(x) = \sum_{m=1}^\infty 2^{-m} e^{2\pi i 2^m e_1\cdot(x+2^m e_1)}(\mathcal{F}^{-1}\pi)(x + 2^m e_1), \quad x \in \mathbb{R}^n,$$

so it is immediate, for instance, that the integral $\int_{\mathbb{R}^n} |k(x)|dx$ converges and therefore the kernel has decay at the infinity. The following result shows that *size+cancellation* is a delicate combination when understood as conditions that might or might not imply the boundedness of $f \mapsto \nabla k \star f$ on L^p.

Proposition 2.34. *Let $k \in \dot{B}^{1,\infty}_{\infty}$ be as constructed above. Then the convolution operator $f \mapsto \mathcal{I}^{-1}k \star f = \sum_{j=1}^{n} R_j(\partial_j k \star f)$ has no bounded extension to $L^p(\mathbb{R}^n)$ if $1 < p < \infty$ and $p \neq 2$. In particular, the operator $f \mapsto \nabla k \star f$ has no bounded extension to these function spaces.*

Proof. Let F_N be given by its Fourier transform $\mathcal{F}(F_N) = \sum_{m=1}^{N} \overline{\hat{f}_m}$. Then $F_N \in \mathcal{S}_\infty$ for all $N \in \mathbb{N}$ and using the basic properties of Fourier transform, we have the explicit formula

$$F_N(x) = \sum_{m=1}^{N} e^{2\pi i 2^m e_1 \cdot (x - 2^m e_1)} (\mathcal{F}^{-1}\pi)(x - 2^m e_1).$$

From this we can conclude that

$$||F_N||_\infty \leq \sup_{x \in \mathbb{R}^n} \left\{ \sum_{m=1}^{\infty} |(\mathcal{F}^{-1}\pi)(x - 2^m e_1)| \right\} < \infty$$

for all $N \in \mathbb{N}$. Also $||F_N||_{L^2} = ||\mathcal{F}(F_N)||_{L^2} = CN^{1/2}$, where C is independent of N. Thus, if $2 < p < \infty$, we have the inequality

$$||F_N||_{L^p} \leq ||F_N||_\infty^{1-2/p} ||f_N||_{L^2}^{2/p} \leq CN^{1/p}, \tag{2.35}$$

where C is independent of N. On the other hand, applying the Littlewood–Paley inequality (2.32) with φ-independence, yields the estimate

$$\begin{aligned} ||\mathcal{I}^{-1}k \star F_N||_{L^p} &\geq C_\varphi \Bigg|\Bigg| \Bigg(\sum_{\nu=1}^{N} (2^\nu |\varphi_\nu \star k \star F_N|)^2 \Bigg)^{1/2} \Bigg|\Bigg|_{L^p} \\ &= C_\varphi \Bigg|\Bigg| \Bigg(\sum_{\nu=1}^{N} |e^{2\pi i 2^\nu x_1} (\mathcal{F}^{-1}\pi \star \mathcal{F}^{-1}\pi)|^2 \Bigg)^{1/2} \Bigg|\Bigg|_{L^p} \\ &= C_\varphi N^{1/2} ||\mathcal{F}^{-1}\pi \star \mathcal{F}^{-1}\pi||_{L^p}. \end{aligned} \tag{2.36}$$

Combining the estimates (2.35) and (2.36) we get the following estimate, with constant C independent of N,

$$N^{1/2} \leq C||\mathcal{I}^{-1}k \star F_N||_{L^p} \leq C||F_N||_{L^p} \leq CN^{1/p}.$$

Hence the operator $f \mapsto \mathcal{I}^{-1}k \star f$ is unbounded on $L^p(\mathbb{R}^n)$ if $p > 2$. This conclusion for exponents $1 < p < 2$ follows from duality. Unboundedness of $f \mapsto \nabla k \star f$ follows now from the boundedness of Riesz transforms R_j, $j = 1, 2, \ldots, n$, on L^p-spaces for $1 < p < \infty$. □

Failure of the L^2-boundedness. Here we show that natural size condition on k, combined with Lipschitz regularity, does not suffice for the boundedness of $f \mapsto \mathcal{I}^{-1}k \star f$ on $L^p(\mathbb{R}^n)$ if $1 < p < \infty$. Using the boundedness of the Riesz transforms on these spaces, combined with the identity

$$\mathcal{I}^{-1}k \star f = \sum_{j=1}^{n} R_j(\partial_j k \star f), \quad f \in \mathcal{S}_\infty,$$

we see that the boundedness of $f \mapsto \nabla k \star f$ on $L^p(\mathbb{R}^n)$ does not either follow from these size and smoothness conditions on k.

The following construction is joint work with M. A. Vähäkangas [Väh05]. Define $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by $k(x) = \sin(x_1)|x|^{-n}$. We claim that $f \mapsto \mathcal{I}^{-1}k \star f$ is unbounded on $L^2(\mathbb{R}^n)$ and the kernel k satisfies

- the size condition

$$|k(x)| \le c_k |x|^{1-n} \text{ for all } x \in \mathbb{R}^n, \tag{2.37}$$

- the Lipschitz condition

$$|k(x+h) - k(x)| \le c_k |h||x|^{-n} \text{ for } |x| \ge 2|h|. \tag{2.38}$$

Clearly k satisfies (2.37) and it is continuously differentiable outside the origin with

$$\partial_1 k(x) = \frac{\cos(x_1)}{|x|^n} - \frac{n \sin(x_1) x_1}{|x|^{n+2}}; \quad \partial_j k(x) = \frac{-n \sin(x_1) x_j}{|x|^{n+2}}, \text{ if } j \ne 1.$$

Thus $|\nabla k(x)| \le C_n |x|^{-n}$ and k satisfies (2.38). It suffices to prove that the Fourier transform of k fails to be bounded near $\pm(2\pi)^{-1}e_1$. Then the operator $f \mapsto \mathcal{I}^{-1}k \star f$ has no bounded extension to $L^2(\mathbb{R}^n)$ and an interpolation argument, combined with duality, shows that $\mathcal{I}^{-1}k \star f$ has no bounded extension to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Proposition 2.39. *Let $k(x) = \sin(x_1)|x|^{-n}$, $x \in \mathbb{R}^n \setminus \{0\}$. Then the Fourier transform $\mathcal{F}k \in \mathcal{S}'$ is a regular distribution, given by the formula*

$$\mathcal{F}k(\xi) = C_n \log \frac{|\xi - (1/2\pi)e_1|}{|\xi + (1/2\pi)e_1|}, \quad \xi \in \mathbb{R}^n. \tag{2.40}$$

As a consequence, the convolution operator $f \mapsto \mathcal{I}^{-1}k \star f$ has no bounded extension to $L^p(\mathbb{R}^n)$ if $1 < p < \infty$.

Proof. We prove the claim only when $n \ge 3$; similar computations apply in dimensions $n = 1$ (the derivative of $\log|\xi|$ is p.v. $1/\xi$) and $n = 2$ (recall Laplace's fundamental solution). Because the Fourier transform of $x \mapsto \sin x_1$ is compactly supported, we have for $j \in \{1, 2, \ldots, n\}$ the identity

$$\partial_j \mathcal{F}k = \mathcal{F}\left(\frac{-2\pi i x_j \sin x_1}{|x|^n}\right) = -2\pi i \mathcal{F}(x_j |x|^{-n}) \star \mathcal{F}(\sin x_1).$$

We also have the identity

$$\begin{aligned} \mathcal{F}(x_j|x|^{-n}) &= C_n \mathcal{F}(\partial_j |x|^{2-n}) \\ &= C_n \xi_j \mathcal{F}(|x|^{2-n}) = C_n \xi_j |\xi|^{-2} = C_n \partial_j \log|\xi|. \end{aligned}$$

Combining the two previous identities, we obtain

$$\partial_j \mathcal{F}k = C_n \partial_j \log|\xi| \star \left(\delta_{(1/2\pi)e_1} - \delta_{-(1/2\pi)e_1}\right) = C_n \partial_j \log \frac{|\xi - (1/2\pi)e_1|}{|\xi + (1/2\pi)e_1|}.$$

Because the partial derivatives coincide and $k \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, so that the Fourier transform of k cannot converge to a nonzero constant at the infinity, we have the desired identity (2.40). $\square$

3. WSIO's with global kernels

In this section we study global WSIO's – operators $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ whose associated kernels are globally defined. The main result here is a so called $T\chi_\Omega$ theorem, see (3.1), characterizing the desired boundedness property of global WSIO's restricted to a Whitney coplump domain $\Omega \subset \mathbb{R}^n$. The following ingredients are utilized in the proof whose main obstruction is comprised of terms associated with so called boundary cubes:

- A reduced $T1$ theorem, formally stated as
$$T1 = 0 = T^*1 \Rightarrow \{\partial^\alpha T, \partial^\alpha T^* \,:\, |\alpha| = m\} \subset \mathscr{L}(L^2(\mathbb{R}^n)),$$
where $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$. If T is of convolution type, then the condition $T1 = 0 = T^*1$ is satisfied. The main application of this result lies in the proof of the $T\chi_\Omega$ theorem.
- BMO-type spaces $\dot{f}^{m,2}_\infty(\Omega)$ and $\dot{F}^{m,2}_\infty(\Omega)$ defined on general domains $\Omega \subset \mathbb{R}^n$. These are large spaces because certain terms associated with boundary cubes in the domain are omitted from the definitions.
- A geometric characterization of Whitney coplump domains in terms of a reflection. This implies that boundary cubes in the domain can be reflected to the complement of the domain, so that the diameter of the cubes is preserved. Also the mutual distance between the boundary and reflected cube should be bounded by a constant multiple of the common diameter of these two cubes.
- Reflected paraproduct operators on Whitney coplump domains $\Omega \subset \mathbb{R}^n$. Such an operator depends heavily on the reflection occuring in the characterization of the Whitney coplump domains. The purpose of these operators is twofold: they are used in a reduction to the reduced $T1$ theorem (in a standard manner) but they also modify the associated kernel K outside of $\Omega \times \Omega$ to reach a bounded operator on $L^2(\mathbb{R}^n)$. The novelty of our solution lies in this modification procedure where the boundary terms are treated using the reflection.

The aforementioned ingredients combine in the proof of the following $T\chi_\Omega$ theorem for restricted operators,

$$T\chi_\Omega, T^*\chi_\Omega \in \dot{f}^{m,2}_\infty(\Omega) \Leftrightarrow \{\partial^\alpha T, \partial^\alpha T^* \,:\, |\alpha| = m\} \subset \mathscr{L}(L^2(\Omega)), \tag{3.1}$$

where $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ and $\Omega \subset \mathbb{R}^n$ is a Whitney coplump domain. Altough the operator T is associated with a globally defined kernel, the conditions in (3.1) are intrinsic to Ω so that they depend only on the associated kernel restricted to the set $\Omega \times \Omega \setminus \{(x,x)\}$.

3.1. **Reduced $T1$ theorem in $\mathbb{R}^n$.**

The fact that convolutions do commute, used in conjunction with the Calderón reproducing formula (2.6), lies behind the proof of Theorem 1.7. Not all of the WSIO's commute with convolutions and the following approach is adapted for the proof of the reduced $T1$ theorem which is a boundedness result for globally defined WSIO's under strong cancellation conditions.

The Calderón reproducing formula can be refined to a so called φ-transform identity [FJ85] which states that

$$f = \sum_{Q\in\mathcal{D}} \langle f, \varphi_Q\rangle \psi_Q,$$

where $f \in \mathcal{S}'/\mathcal{P}$ and φ_Q, ψ_Q are translations and dilatations of a dual pair of Littlewood–Paley functions φ, ψ, so that both φ_Q, ψ_Q are concentrated on the dyadic cube $Q \in \mathcal{D}$. The sequence $\{\langle f, \varphi_Q\rangle : Q \in \mathcal{D}\}$ of coefficients can be used to compute norms of f in the scales $\dot{B}^{\alpha,q}_p(\mathbb{R}^n)$ and $\dot{F}^{\alpha,q}_p(\mathbb{R}^n)$. In particular, if $T : \mathcal{S}_\infty \to \mathcal{S}'/\mathcal{P}$ is continuous, various function space norms $||Tf||$ can be computed using the coefficients

$$\langle Tf, \varphi_Q\rangle = \sum_{P\in\mathcal{D}} \langle f, \varphi_P\rangle\langle T\psi_P, \varphi_Q\rangle, \quad Q \in \mathcal{D}.$$

As a consequence,

$$||Tf|| \le C_T||f||$$

given that the matrix $\{\langle T\psi_P, \varphi_Q\rangle : P, Q \in \mathcal{D}\}$ is a so called almost diagonal matrix. Hereby the boundedness of T on various function spaces reduces to an almost diagonality condition. An important example is that, if T is a Calderón–Zygmund type operator, this almost diagonality condition is implied by the integral conditions

$$\int_{\mathbb{R}^n} T^t\varphi_Q(x)dx = 0 = \int_{\mathbb{R}^n} T\psi_Q(x)dx, \quad Q \in \mathcal{D},$$

which correspond to the weak formulation of the familiar cancellation conditions $T1 = 0 = T^t1$. This Frazier–Han–Jawerth–Weiss approach to reduced $T1$ theorem for Calderón–Zygmund type integral operators is well established [FHJW89, FJ90, Wan99] and there are also results for potential operators resembling globally defined WSIO's [GT99, Tor91, Väh08].

The described approach does not support localization since the functions φ_Q, ψ_Q are not compactly supported. We conform to later requirements of locality by adapting the treatment of Meyer and Coifman, originally involving Calderón–Zygmund type operators [MC97, pp. 51–55]. In particular, we use compactly supported wavelets $\{\psi^\varepsilon_Q\}$ and establish almost diagonality estimates for the operator matrices $\{\langle T\psi^\rho_P \mid \psi^\varepsilon_Q\rangle\}$. There is also a price to pay from this wavelet transform point-of-view: small technicalities arise from that compactly supported wavelets are not of class $C^\infty(\mathbb{R}^n)$.

Standing definitions and notation. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a domain. If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, $1/p + 1/p' = 1$, then denote

$$\langle f \mid g\rangle = \int_\Omega f(x)\overline{g(x)}dx. \tag{3.2}$$

This extends the notion of an inner product on $L^2(\Omega)$. For the sake of weak derivatives we need another extension, furnished by the duality. A typical approach via distributions excludes compactly supported wavelets as test functions since these are not smooth. To include these wavelets as test functions we make use of the following modifications. Let $\kappa \in \mathbb{N}_0$ and denote $C_0^\kappa(\Omega) = \{\varphi \in C_0^\kappa(\mathbb{R}^n) : \operatorname{supp}\varphi \subset \Omega\}$, $C_0(\Omega) = C_0^0(\Omega)$. By $(C_0^\kappa(\Omega))^*$ we denote the algebraic dual, consisting of conjugate-linear functionals

$$\Lambda : C_0^\kappa(\Omega) \to \mathbb{C} : \varphi \mapsto \Lambda(\varphi) = \langle \Lambda \mid \varphi \rangle.$$

In the sense of (3.2), $L^p(\Omega) \subset (C_0^\kappa(\Omega))^*$. If $\alpha \in \mathbb{N}_0^n$ and $\kappa = |\alpha|$ then the *weak partial differential operator* ∂^α is the linear operator $(C_0(\Omega))^* \to (C_0^\kappa(\Omega))^*$, defined by

$$\langle \partial^\alpha \Lambda \mid \varphi \rangle = (-1)^{|\alpha|} \langle \Lambda \mid \partial^\alpha \varphi \rangle, \quad \varphi \in C_0^\kappa(\Omega).$$

Here ∂^α on the right-hand side denotes the *pointwise partial differential operator* $C_0^\kappa(\Omega) \to C_0(\Omega)$. Integration by parts shows that these weak and pointwise partial derivatives coincide if $\Lambda \in C_0^\kappa(\Omega) \subset (C_0(\Omega))^*$. Let $\kappa \in \mathbb{N}_0$ and $T : C_0(\Omega) \to (C_0^\kappa(\Omega))^*$ be a linear operator. Then we denote $T \in \mathscr{L}(L^p(\Omega), L^q(\Omega))$ if, given $1 < p, q < \infty$ and $1/q + 1/q' = 1$, it satisfies

$$|\langle Tf \mid g \rangle| \leq C_T ||f||_{L^p(\Omega)} ||g||_{L^{q'}(\Omega)} \tag{3.3}$$

for every $f \in C_0(\Omega)$ and $g \in C_0^\kappa(\Omega)$. In the special case $p = q$ we denote $T \in \mathscr{L}(L^p(\Omega))$ and say that T has a *bounded extension to* $L^p(\Omega)$. This terminology is justified in what follows. Assume that $T \in \mathscr{L}(L^p(\Omega), L^q(\Omega))$. Fix $f \in C_0(\Omega)$. Then the estimate (3.3) implies that Tf is a bounded conjugate-linear functional on the normed vector space

$$\big(C_0^\kappa(\Omega), ||\cdot||_{L^{q'}(\Omega)}\big).$$

Since $C_0^\kappa(\Omega) \subset L^{q'}(\Omega)$ is dense we see that $Tf \in (C_0^\kappa(\Omega))^*$ extends uniquely to a conjugate-linear functional on $L^{q'}(\Omega)$, with norm bounded by $C_T||f||_{L^p(\Omega)}$. Now use the identification $(L^{q'}(\Omega))' = L^q(\Omega)$ [Rud87, p. 127] to conclude that this extension, denoted also by Tf, belongs to $L^q(\Omega)$ and satisfies the norm-estimate

$$||Tf||_{L^q(\Omega)} \leq C_T ||f||_{L^p(\Omega)}. \tag{3.4}$$

Then fix $f \in L^p(\Omega)$. The estimate (3.4) allows us to define Tf as the limit of a Cauchy sequence $(Tf_j)_{j\in\mathbb{N}} \subset L^q(\Omega)$ in $L^q(\Omega)$, where $(f_j)_{j\in\mathbb{N}} \subset C_0(\Omega)$ satisfies $\lim_{j\to\infty} f_j = f$ in $L^p(\Omega)$. This definition of Tf is independent of $(f_j)_{j\in\mathbb{N}}$ and it provides an extension of $T : C_0(\Omega) \to (C_0^\kappa(\Omega))^*$ to a bounded linear operator $T : L^p(\Omega) \to L^q(\Omega)$. Indeed, due to norm-estimate (3.4), we have

$$||Tf||_{L^q(\Omega)} = \lim_{j\to\infty} ||Tf_j||_{L^q(\Omega)} \leq C_T \lim_{j\to\infty} ||f_j||_{L^p(\Omega)} = C_T ||f||_{L^p(\Omega)}.$$

Foundations for WSIO's. Here we establish a fundamental size estimate for WSIO's and extend their domain of definition to include, for instance, the space $\mathrm{BMO}(\mathbb{R}^n)$.

Atoms in connection with function spaces appear in various forms and here they are understood in the sense of $(m+1)$-regular wavelets $\{\psi_Q^\varepsilon : (Q,\varepsilon) \in \mathcal{D} \times \mathcal{E}\} \subset C_0^{m+1}(\mathbb{R}^n)$ which are defined in Appendix B. These possess vanishing moments, up to order $m+1$, and a compact support that is concentrated on the dyadic cube $Q \in \mathcal{D}$. Let $T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$ satisfy certain cancellation condition $T1 = 0 = T^*1$. Then T maps atoms to molecule-like functions that share properties with so called smooth molecules [FJ90, p. 56]. These are, in a vague sense, dispersed atoms and their properties include some smoothness, decay at infinity, and vanishing moments.

These molecule-like size estimates for $T\psi_Q^\varepsilon$'s is the first topic here. A powerful tool for these purposes is the following *Whitney approximation theorem*.

Theorem 3.5. *Let $Q \subset \mathbb{R}^n$ be a cube and $f \in L^1(Q)$. Then we have*

$$\inf_{P \in \mathcal{P}_m(\mathbb{R}^n)} ||f - P||_{L^1(Q)} \le C_{n,m} \sup_{|h| \le \mathrm{diam}(Q)} ||\Delta_h^{m+1}(f, Q, \cdot)||_{L^1(Q)}.$$

A related result is originally due to H. Whitney in dimension $n = 1$ [Whi57]. Whitney's result is further generalized in [Bru70], where the proof of Theorem 3.5 can also be found as a special case. Another proof of Theorem 3.5 via interpolation theory is in [JS77].

We begin establishing the molecule-estimates. Fix $T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$ that it is associated with kernel $K \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$. Let $1 \le p < n/m$. Then using the kernel size-estimate (1.16) and a well known inequality [Gra04, p. 416], yields

$$\begin{aligned} |Tf(x)| &\le \int_{\mathbb{R}^n} |K(x,y)||f(y)|dy \le C_{n,T}\mathcal{I}^m(|f|)(x) \\ &\le C_{n,m,p,T} M(f)(x)^{1-mp/n} ||f||_p^{mp/n} \end{aligned} \tag{3.6}$$

if $x \in \mathbb{R}^n$ and $f \in C_0(\mathbb{R}^n)$. Here $\mathcal{I}^m$ is the Riesz potential operator as in (2.11). As a consequence, T induces a linear operator $C_0(\mathbb{R}^n) \to (C_0(\mathbb{R}^n))^*$, satisfying

$$T \in \mathscr{L}(L^p(\mathbb{R}^n), L^{np/(n-mp)}(\mathbb{R}^n)), \quad \text{if } 1 < p < n/m. \tag{3.7}$$

Estimating as in (3.6) and applying the Fubini's theorem, we also obtain the identity $\langle Tf \mid g \rangle = \langle f \mid T^* g \rangle$ if $f, g \in C_0(\mathbb{R}^n)$. Here $T^* \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$ is the adjoint operator, associated with the adjoint kernel

$$(x,y) \mapsto \overline{K(y,x)}.$$

In particular, $(S+T)^* = S^* + T^*$ if $S, T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$. Next we prove a quantitative size-estimate about $T\psi_Q^\varepsilon$'s. Such an estimate is one of the requirements of molecules.

Lemma 3.8. *Let $T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$. Then for every $Q \in \mathcal{D}$, $\varepsilon \in \mathcal{E}$, and $x \in \mathbb{R}^n$, we have*

$$|T\psi_Q^\varepsilon(x)| \le C|Q|^{-1/2+m/n}(1 + \ell(Q)^{-1}|x - x_Q|)^{-n-\delta}.$$

Here the constant C depends at most on n, m, C_{m+1}, T.

Proof. Recall that x_Q denotes the lower left-corner of Q and x^Q denotes the midpoint of Q. In particular, $|x^Q - x_Q| \le \sqrt{n}\ell(Q)$ and therefore it suffices to verify a modified estimate where $|x-x_Q|$ is replaced with $|x-x^Q|$. Denote by $K \in \mathrm{K}^{-m}_{\mathbb{R}^n}(\delta)$ the kernel associated with operator T. Using the inequality (3.6) with $p=1$ and properties B4)–B5) of wavelets, see Appendix B, we get

$$|T\psi^\varepsilon_Q(x)| \le CM(\psi^\varepsilon_Q)(x)^{1-m/n}||\psi^\varepsilon_Q||_1^{m/n} \le C|Q|^{-1/2+m/n}.$$

As a consequence, it suffices to verify the following inequality

$$|T\psi^\varepsilon_Q(x)| \le C|Q|^{-1/2+m/n}(\ell(Q)^{-1}|x-x^Q|)^{-n-\delta} \tag{3.9}$$

for $|x-x^Q| \ge C_K \operatorname{diam}(C_{m+1}Q)$, where C_K denotes the constant in assumption (1.17). Using B2)–B5) from Appendix[3] B, we get

$$\begin{aligned} |T\psi^\varepsilon_Q(x)| &= \left| \int_{\mathbb{R}^n} K(x,y)\psi^\varepsilon_Q(y)dy \right| \\ &= \inf_{P\in\mathcal{P}_m(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} (K(x,y)-P(y))\psi^\varepsilon_Q(y)dy \right| \\ &\le C_{m+1}|Q|^{-1/2} \inf_{P\in\mathcal{P}_m(\mathbb{R}^n)} \int_{C_{m+1}Q} |K(x,y)-P(y)|dy. \end{aligned} \tag{3.10}$$

Using Theorem 3.5 and the integral estimate (1.17), we obtain the following upper bounds for the right-hand side of (3.10)

$$\begin{aligned} &C|Q|^{-1/2} \sup_{|h|\le \operatorname{diam}(C_{m+1}Q)} \int_{C_{m+1}Q} |\Delta^{m+1}_h(K(x,\cdot),C_{m+1}Q,y)|dy \\ &\le C|Q|^{-1/2+1+(m+\delta)/n}|x-x^Q|^{-n-\delta} = C|Q|^{-1/2+m/n}(\ell(Q)^{-1}|x-x^Q|)^{-n-\delta}. \end{aligned}$$

Here the constant C may vary from one occurence to another but it depends at most on n, m, C_{m+1}, T. Combining these estimates we obtain (3.9). □

This lemma is a powerful tool, used in various occasions. Here we collect some of its implications that turn out to be useful. Lemma 3.8 combined with the Hölder's inequality imply the following L^2-estimates: assuming that $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$, $|\alpha| = m$ and $\varepsilon, \rho \in \mathcal{E}$, we have

$$\sup_{Q\in\mathcal{D}} |\langle T\psi^\varepsilon_Q \mid \partial^\alpha \psi^\rho_Q\rangle| \le \sup_{Q\in\mathcal{D}} ||T\psi^\varepsilon_Q||_{L^2}||\partial^\alpha\psi^\rho_Q||_{L^2} < \infty.$$

An L^1-estimate is recorded in the following corollary of Lemma 3.8. This estimate is used later on when normalizing weakly singular integral operators.

Corollary 3.11. *Let $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ and $(Q,\varepsilon) \in \mathcal{D}\times\mathcal{E}$. Then*

$$||T\psi^\varepsilon_Q||_{L^1} \le C|Q|^{1/2+m/n},$$

where C depends at most on n, m, δ, C_{m+1}, T.

[3]All of the references to B1)–B5) in the sequel will be to Appendix B

Next we extend the domain of definition of $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ to the vector space

$$\mathscr{D}(\delta) = \left\{ b \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \int_{\mathbb{R}^n} |b(x)|(1+\ell(Q)^{-1}|x-x_Q|)^{-n-\delta} dx < \infty \text{ if } Q \in \mathcal{D} \right\}.$$

The quantity Tb, $b \in \mathscr{D}(\delta)$, need not exist as an absolutely convergent integral. Therefore we modify the range of T to reach this extension. Define $Tb : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ for $b \in \mathscr{D}(\delta)$ by

$$Tb(Q,\varepsilon) = \int_{\mathbb{R}^n} b(x)\overline{T^*\psi^\varepsilon_Q(x)}dx, \quad (Q,\varepsilon) \in \mathcal{D} \times \mathcal{E}, \tag{3.12}$$

where ψ^ε_Q's are $(m+1)$-regular wavelets. Because of $b \in \mathscr{D}(\delta)$, Lemma 3.8 shows that the integrals in (3.12) do converge absolutely. This definition induces a linear operator $b \mapsto Tb$, defined in $\mathscr{D}(\delta)$ and valued in the vector space $\{\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}\}$.

The main reason for this extension is that cancellation phenomena occuring in the $T\chi_\Omega$ theorem are best quantified in terms of certain BMO-type sequence spaces, formally $T\chi_\Omega, T^*\chi_\Omega \in \dot{f}^{m,2}_\infty(\Omega)$. The following related definition is used in connection with the reduced $T1$ theorem.

Definition 3.13. We say that $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ satisfies a *strong cancellation condition* if $T1 = 0 \in \{\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}\}$, that is,

$$T1(Q,\varepsilon) = T\chi_{\mathbb{R}^n}(Q,\varepsilon) = \int_{\mathbb{R}^n} \overline{T^*\psi^\varepsilon_Q(x)}dx = 0, \quad (Q,\varepsilon) \in \mathcal{D} \times \mathcal{E}. \tag{3.14}$$

Here the wavelets $\{\psi^\varepsilon_Q\}$ are $(m+1)$-regular.

The sequential approach, which is initiated above, makes it possible to study WSIO's when their domain of definition is $L^\infty(\mathbb{R}^n)$ or $\mathrm{BMO}(\mathbb{R}^n)$.

Example 3.15. Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and denote

$$||b||_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_{Q \text{ cube in } \mathbb{R}^n} \left\{ \frac{1}{|Q|}\int_Q |b(x) - b_Q| dx \right\}, \quad b_Q = \frac{1}{|Q|}\int_Q b(x)dx.$$

We show that if $||b||_{\mathrm{BMO}(\mathbb{R}^n)} < \infty$ and $\delta > 0$, then $b \in \mathscr{D}(\delta)$. For this purpose we invoke the following estimate which is an easy implication of [Gra04, pp. 521–522],

$$\int_{\mathbb{R}^n} |h(x) - h_{[0,1)^n}|(1+|x|)^{-n-\delta}dx \le C_{n,\delta}||h||_{\mathrm{BMO}(\mathbb{R}^n)}, \quad h \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Next, if $Q \in \mathcal{D}$, then by a change of variables $h(x) = b(\ell(Q)x + x_Q)$ for $x \in \mathbb{R}^n$ we are reduced to this estimate,

$$\begin{aligned} &\int_{\mathbb{R}^n} |b(x) - b_Q|(1+\ell(Q)^{-1}|x-x_Q|)^{-n-\delta}dx \\ &\qquad = |Q|\int_{\mathbb{R}^n} |h(x) - h_{[0,1)^n}|(1+|x|)^{-n-\delta}dx \\ &\qquad \le C_{n,\delta}|Q|||h||_{\mathrm{BMO}(\mathbb{R}^n)} = C_{n,\delta}|Q|||b||_{\mathrm{BMO}(\mathbb{R}^n)} < \infty. \end{aligned} \tag{3.16}$$

It follows that $b \in \mathscr{D}(\delta)$.

Almost diagonality estimates for WSIO's. The size estimates combined with strong cancellation condition (3.14) for T and T^* imply almost diagonality estimates for these WSIO's. It is precisely these estimates that ultimately give rise to the reduced $T1$ theorem. We continue with a useful lemma towards the almost diagonality. Here one should keep the primary application in mind:

$$G = |P|^{-m/n} T\psi_P^{\rho}, \quad F = \psi_Q^{\varepsilon}$$

for some $T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$ satisfying $T^*1 = 0$. However, there are also other applications and we present a general result whose proof is modelled after [MC97, p. 53].

Lemma 3.17. *Let $\varrho > 0$ and $P, Q \in \mathcal{D}$ be such that $\ell(P) \leq \ell(Q)$. Let $G \in L^1(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} G(x)dx = 0$. Assume further that, for some $\delta \in (0,1)$, we have*

$$|G(x)| \leq \varrho |P|^{-1/2}(1 + \ell(P)^{-1}|x - x_P|)^{-n-\delta}, \quad \text{if } x \in \mathbb{R}^n. \tag{3.18}$$

Let $F \in C^1(\mathbb{R}^n)$ satisfy $\operatorname{supp} F \subset \varrho Q$ *and*

$$||\partial^\alpha F||_{L^\infty(\mathbb{R}^n)} \leq \varrho |Q|^{-1/2-|\alpha|/n}, \quad \text{if } |\alpha| \leq 1. \tag{3.19}$$

Then we have the estimate

$$|\langle G \mid F\rangle| \leq C\left(\frac{\ell(P)}{\ell(Q)}\right)^{n/2+\delta}\left(1 + \frac{|x_P - x_Q|}{\ell(Q)}\right)^{-(n+\delta)}, \tag{3.20}$$

where the constant C depends on the parameters ϱ, δ, n.

Proof. Let $P \in \mathcal{D}_\mu$ and $Q \in \mathcal{D}_\nu$ be such that $\ell(P) = 2^{-\mu} \leq 2^{-\nu} = \ell(Q)$. Without loss of generality we can further assume that $\varrho > 1$. First we change the variables

$$\begin{aligned} \langle G \mid F\rangle &= 2^{-\mu n}\int_{\mathbb{R}^n} G(2^{-\mu}x + x_P)\overline{F(2^{-\mu}x + x_P)}dx \\ &= 2^{-\mu n/2 + \nu n/2}\int_{\mathbb{R}^n} g(x) f\big(R^{-1}(x - x_0)\big)dx, \end{aligned} \tag{3.21}$$

where $R = 2^{\mu-\nu}$, $x_0 = 2^\mu(x_Q - x_P)$,

$$g(x) = 2^{-\mu n/2}G(2^{-\mu}x + x_P), \quad f(x) = 2^{-\nu n/2}\overline{F(2^{-\nu}x + x_Q)}.$$

Using assumption (3.18), we get the estimate $|g(x)| \leq \varrho(1 + |x|)^{-n-\delta}$. Using (3.19), we have $\operatorname{supp} f \subset \bar{B}(0, \sqrt{n}\varrho)$ and $||\partial^\alpha f||_{L^\infty(\mathbb{R}^n)} \leq \varrho$ if $|\alpha| \leq 1$. As a consequence, we have $\operatorname{supp} f(R^{-1}(\cdot - x_0)) \subset \bar{B}(x_0, R\sqrt{n}\rho)$

First assume that $|x_0| \geq 2R\sqrt{n}\varrho$, that is, $|x_P - x_Q| \geq 2\sqrt{n}\varrho\ell(Q)$. In particular,

$$\left|\int_{\mathbb{R}^n} g(x) f\big(R^{-1}(x - x_0)\big)dx\right| \leq C|x_0|^{-n-\delta}R^n = C2^{\delta(\nu-\mu)}\left(\frac{|x_P - x_Q|}{2^{-\nu}}\right)^{-n-\delta}.$$

This combined with (3.21) yields the desired estimate (3.20) in the present case.

Then assume that $|x_0| < 2R\sqrt{n}\varrho$, that is, $|x_P - x_Q| < 2\sqrt{n}\varrho\ell(Q)$. In this case the functions $f(R^{-1}(\cdot - x_0))$ and g are concentrated roughly on the origin and g is well localized when compared to $f(R^{-1}(\cdot - x_0))$. What saves us here is that the integral of g vanishes and f has smoothness. These facts allow us to utilize cancellation effects as follows. First of all, define

$$g_j = (x_j/|x|^n) \star g, \quad j \in \{1,2,\dots,n\}.$$

The divergence theorem shows that, for every $y \in \mathbb{R}^n$, we have

$$\frac{1}{\omega_{n-1}} \sum_{j=1}^n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^n} \partial_j(f(R^{-1}(x-x_0)))dx = f(R^{-1}(y-x_0)). \tag{3.22}$$

An elementary proof of the identity (3.22) is [EE87, p. 238]. Multiply the identity (3.22) by $g(y)$ and then integrate with respect to the y-variable. Then using the Fubini's theorem, justified by (3.6), we get

$$\frac{1}{\omega_{n-1}} \sum_{j=1}^n R^{-1} \int_{\mathbb{R}^n} g_j(x)(\partial_j f)\big(R^{-1}(x-x_0)\big)dx = \int_{\mathbb{R}^n} g(y) f\big(R^{-1}(y-x_0)\big)dy. \tag{3.23}$$

According to the definition of g and the assumption $\int G = 0$, we have $\int g = 0$. Hence, if $j \in \{1,2,\dots,n\}$ and $x \in \mathbb{R}^n \setminus \{0\}$, we can write

$$g_j(x) = \int_{\mathbb{R}^n} g(y)\bigg(\frac{x_j-y_j}{|x-y|^n} - \frac{x_j}{|x|^n}\bigg)dy. \tag{3.24}$$

Define the sets $A(x) = \{y \in \mathbb{R}^n : |y| < |x|/2\}$, $B(x) = \{y \in \mathbb{R}^n : |x|/2 \le |y| \le 2|x|\}$, and $C(x) = \mathbb{R}^n \setminus (A \cup B)$. In what follows we also use the assumption $0 < \delta < 1$ and denote by C any constant depending at most on n, ϱ. Using the estimate $|g(x)| \le \varrho(1+|x|)^{-n-\delta}$ and the mean value theorem on the difference, we get

$$\int_{A(x)} \bigg| g(y)\bigg(\frac{x_j-y_j}{|x-y|^n} - \frac{x_j}{|x|^n}\bigg)\bigg| dy \le C|x|^{-n} \int_{A(x)} (1+|y|)^{-n-\delta}|y|dy \le \frac{C|x|^{-n+1-\delta}}{1-\delta}.$$

The integrals with respect to $B(x)$ and $C(x)$ are easier to estimate by using again the size estimate about $|g|$, resulting to the inequality

$$\bigg(\int_{B(x)} + \int_{C(x)}\bigg)\bigg| g(y)\bigg(\frac{x_j-y_j}{|x-y|^n} - \frac{x_j}{|x|^n}\bigg)\bigg| dy \le \frac{C|x|^{-n+1-\delta}}{\delta}.$$

Combining the estimates above with the identity (3.24), we have

$$|g_j(x)| \le C_{n,\delta,\varrho}|x|^{-n+1-\delta}, \quad j \in \{1,2,\dots,n\}. \tag{3.25}$$

Using the identity (3.23) and then the estimate (3.25) about g_j's, we get

$$\begin{aligned}
&\bigg| \int_{\mathbb{R}^n} g(x) f\big(R^{-1}(x-x_0)\big)dx \bigg| \\
&\le C_n \sum_{j=1}^n R^{-1} \bigg| \int_{B(0,3R\sqrt{n}\varrho)} g_j(x)(\partial_j f)\big(R^{-1}(x-x_0)\big)dx \bigg| \le C_{n,\delta,\varrho} R^{-\delta} = C_{n,\delta,\varrho} 2^{\delta(\nu-\mu)}.
\end{aligned}$$

Together with (3.21) this shows the desired estimate (3.20) in the present case. □

As a consequence, we obtain the following almost diagonality estimate for WSIO's. It can be interpreted as a technical formulation of the reduced $T1$ theorem.

Proposition 3.26. *Let $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ satisfy $T1 = T^*1 = 0 \in \{\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}\}$. Then for every $P, Q \in \mathcal{D}$ and $\varepsilon, \rho \in \mathcal{E}$ we have the weighted almost diagonality estimate*

$$|\langle T\psi_P^\rho \mid \psi_Q^\varepsilon\rangle| \le C \frac{(\ell(P)\wedge \ell(Q))^{n/2+\delta+m}}{(\ell(P)\vee \ell(Q))^{n/2+\delta}} \left(1 + \frac{|x_P - x_Q|}{\ell(P)\vee \ell(Q)}\right)^{-(n+\delta)},$$

where the constant C depends at most on n, m, δ, C_{m+1}, T.

Proof. Assume first that $\ell(P) \le \ell(Q)$. Let $G = |P|^{-m/n} T\psi_P^\rho$ and $F = \psi_Q^\varepsilon \in C_0^{m+1}(\mathbb{R}^n)$. Using the assumption $T^*1 = 0$ and the identity $(T^*)^* = T$, we have $\int_{\mathbb{R}^n} G(x)dx = \int_{\mathbb{R}^n} T\psi_P^\rho(x)dx = 0$. Also, because $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$, Lemma 3.8 implies that G satisfies the estimate

$$|G(x)| = |P|^{-m/n}|T\psi_P^\rho(x)| \le C_{n,m,C_{m+1},T}|P|^{-1/2}(1+\ell(P)^{-1}|x-x_P|)^{-n-\delta}, \quad x \in \mathbb{R}^n.$$

On the other hand, the localization property B4) for $(m+1)$ regular wavelets implies that $\operatorname{supp} F \subset C_{m+1}Q$. Also, the regularity property B5) implies that

$$||\partial^\alpha F||_{L^\infty(\mathbb{R}^n)}|| = ||\partial^\alpha \psi_Q^\varepsilon||_{L^\infty(\mathbb{R}^n)} \le C_{m+1}|Q|^{-1/2-|\alpha|/n}, \quad |\alpha| \le 1.$$

We have verified the assumptions of Lemma 3.17 with constant ϱ depending at most on the parameters n, m, C_{m+1}, T. Accordingly we obtain the estimate

$$|\langle T\psi_P^\rho \mid \psi_Q^\varepsilon\rangle| = |P|^{m/n}|\langle G \mid F\rangle \le C\frac{\ell(P)^{n/2+\delta+m}}{\ell(Q)^{n/2+\delta}}\left(1+\frac{|x_P-x_Q|}{\ell(Q)}\right)^{-(n+\delta)}.$$

This is the required estimate in the present case $\ell(P) \le \ell(Q)$. The other case, $\ell(P) > \ell(Q)$, reduces to the estimates above. Indeed, we have

$$\langle T\psi_P^\rho \mid \psi_Q^\varepsilon\rangle = \langle \psi_P^\rho \mid T^*\psi_Q^\varepsilon\rangle = \overline{\langle T^*\psi_Q^\varepsilon \mid \psi_P^\rho\rangle},$$

where $T^* \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ and the assumption $T1 = 0$ implies that $\int T^*\psi_Q^\varepsilon = 0$. By setting $G = |Q|^{-m/n}T^*\psi_Q^\varepsilon$ and $F = \psi_P^\rho$ we can proceed as above. □

Reduced $T1$ theorem for WSIO's. Here we finish the proof of reduced $T1$ theorem. The main work is done – culminating in Proposition 3.26. It remains to deal with certain technicalities. Fix $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ satisfying $T1 = 0 = T^*1$. We need to show that $\partial^\alpha T \in \mathscr{L}(L^2(\mathbb{R}^n))$ if $|\alpha| = m$. This is here established by constructing an operator $[\partial^\alpha T] \in \mathscr{L}(L^2(\mathbb{R}^n))$ that satisfies

$$\langle [\partial^\alpha T]f \mid g\rangle = \langle \partial^\alpha T f \mid g\rangle \tag{3.27}$$

if $f \in C_0(\mathbb{R}^n)$ and $g \in C_0^m(\mathbb{R}^n)$. Definition (3.38) for $[\partial^\alpha T] : C_0(\mathbb{R}^n) \to (C_0(\mathbb{R}^n))^*$ is given by using the wavelet transform and the operator matrix

(3.28)

$$M_{\partial^\alpha T} = \{M(P,\rho;Q,\varepsilon) = \langle \partial^\alpha T\psi_P^\rho \mid \psi_Q^\varepsilon\rangle \,:\, Q, P \in \mathcal{D} \text{ and } \varepsilon, \rho \in \mathcal{E}\}, \quad |\alpha| = m.$$

Using Proposition 3.26 and a factorization

$$M_{\partial^\alpha T} = M_{\partial^m T} M_{\partial^\alpha/\partial^m},$$

where $M_{\partial^\alpha/\partial^m}$ is a Riesz-transform type almost diagonal matrix and $M_{\partial^m T}$ is a discrete derivative matrix of T, we will see that $M_{\partial^\alpha T}$ is an almost diagonal matrix, hence bounded on $\ell^2(\mathcal{D}\times\mathcal{E})$. Now the boundedness property $[\partial^\alpha T] \in \mathscr{L}(L^2(\mathbb{R}^n))$ follows because this operator is defined in terms of a bounded matrix operator and the wavelet transform, which is an isometry between $L^2(\mathbb{R}^n)$ and $\ell^2(\mathcal{D}\times\mathcal{E})$.

To complete this sketch we begin with some preparations. Let $P, Q \in \mathcal{D}$ and recall that the lower-left corners of these dyadic cubes are $x_P \in P$ and $x_Q \in Q$, respectively. Let also $\beta, \gamma > 0$. For these parameters, we denote

$$\omega_{P,Q}(\beta,\gamma) = \left(\frac{\ell(P)\wedge\ell(Q)}{\ell(P)\vee\ell(Q)}\right)^{n/2+\gamma}\left(1+\frac{|x_P - x_Q|}{\ell(P)\vee\ell(Q)}\right)^{-(n+\beta)}. \tag{3.29}$$

Notice that $\omega_{P,Q}(\beta,\gamma) = \omega_{Q,P}(\beta,\gamma)$ and $\omega_{P,Q}(\tilde\beta,\tilde\gamma) \geq \omega_{P,Q}(\beta,\gamma)$ if $\tilde\beta \leq \beta$ and $\tilde\gamma \leq \gamma$. A matrix $M : (\mathcal{D}\times\mathcal{E})\times(\mathcal{D}\times\mathcal{E}) \to \mathbb{C}$, denoted by

$$M = \{M(P,\rho;Q,\varepsilon) \in \mathbb{C} \,:\, P,Q \in \mathcal{D} \text{ and } \rho,\varepsilon \in \mathcal{E}\},$$

is *almost diagonal* if there exists $\delta > 0$ such that

$$\sup\left\{\frac{|M(P,\rho;Q,\varepsilon)|}{\omega_{P,Q}(\delta,\delta)} \,:\, P,Q \in \mathcal{D} \text{ and } \rho,\varepsilon \in \mathcal{E}\right\} < \infty. \tag{3.30}$$

This condition is symmetric so that the adjoint matrix M^*, defined by

$$M^*(P,\rho;Q,\varepsilon) = \overline{M(Q,\varepsilon;P,\rho)},$$

is almost diagonal if, and only if, M is. The same holds true for the transpose matrix defined by $M^t = \overline{M^*}$.

Example 3.31. Let $0 < m < n$ and fix $(m+1)$-regular wavelets $\{\psi_Q^\varepsilon\}$ as described in Appendix B. Here are certain almost diagonal matrices:

- Let $T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$ satisfy $T1 = T^*1 = 0 \in \{\tau : \mathcal{D}\times\mathcal{E}\}$. Denote the order m discrete derivative matrix of T by

$$M_{\partial^m T}(P,\rho;Q,\varepsilon) = |Q|^{-m/n}\langle T\psi_P^\rho \mid \psi_Q^\varepsilon\rangle$$

  if $P,Q \in \mathcal{D}$ and $\rho,\varepsilon \in \mathcal{E}$. Then $M_{\partial^m T}$ is almost diagonal by Proposition 3.26.
- Let $\alpha \in \mathbb{N}_0^n$ satisfy $|\alpha| = m$ and consider the Riesz-transform type matrix

$$M_{\partial^\alpha/\partial^m}(P,\rho;Q,\varepsilon) = |P|^{m/n}\langle \partial^\alpha\psi_P^\rho \mid \psi_Q^\varepsilon\rangle$$

  if $P,Q \in \mathcal{D}$ and $\rho,\varepsilon \in \mathcal{E}$. Then $M_{\partial^\alpha/\partial^m}$ is almost diagonal and it satisfies (3.30) for every $\delta \in (0,1)$. Indeed, we can do the case studies $\ell(P) \leq \ell(Q)$ and $\ell(P) > \ell(Q)$ and choose the functions $G = |P|^{m/n}\partial^\alpha\psi_P^\rho$ and $F = \psi_Q^\varepsilon$ in the former case and vice versa in the latter. The assumptions of Lemma 3.17 are satisfied in any case by B4) and B5).

- If $\rho \in \mathcal{E}$ then there exists a canonical multi-index $\alpha = \alpha(\rho, m) \in \mathbb{N}_0^n$ with $|\alpha| = m$ and a function $\psi^{\rho,m} : \mathbb{R}^n \to \mathbb{C}$ as in Lemma B.2. This satisfies $\psi^\rho = \partial^\alpha \psi^{\rho,m}$. Define
$$M_m^{\mathrm{lift}}(P, \rho; Q, \varepsilon) = \langle \psi_P^{\rho,m} \mid \psi_Q^\varepsilon \rangle$$
for $P, Q \in \mathcal{D}$ and $\rho, \varepsilon \in \mathcal{E}$. Then the lift matrix M_m^{lift} is almost diagonal. In the proof we proceed in a case study as above and choose $F, G \in \{\psi_P^{\rho,m}, \psi_Q^\varepsilon\}$.

The matrix multiplication MN corresponds to the composition of the matrix operators M and N on $\ell^2(\mathcal{D} \times \mathcal{E})$. The following lemma shows that almost diagonal matrices are closed under this matrix multiplication. A proof is in [FJ90, p. 158–159] and, in comparison therein, we only changed the variable γ.

Lemma 3.32. *Let $P, Q \in \mathcal{D}$. Let $\beta, \gamma_1, \gamma_2 > 0$, $\gamma_1 \neq \gamma_2$, and $\gamma_1 + \gamma_2 > \beta$. Then*
$$\sum_{R \in \mathcal{D}} \omega_{P,R}(\beta, \gamma_1) \omega_{R,Q}(\beta, \gamma_2) \leq C \omega_{P,Q}(\beta, \gamma_1 \wedge \gamma_2),$$
where the constant C depends at most on $n, \beta, \gamma_1, \gamma_2$.

Now we can show the almost diagonality of the matrix $M_{\partial^\alpha T}$ defined in (3.28).

Lemma 3.33. *Let $T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$ satisfy $T1 = T^*1 = 0 \in \{\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}\}$. Then, if $|\alpha| = m$, we have the factorization $M_{\partial^\alpha T} = M_{\partial^m T} M_{\partial^\alpha / \partial^m}$. As a consequence, the matrix $M_{\partial^\alpha T}$ is almost diagonal if $|\alpha| = m$.*

Proof. First of all, the relation (3.7) implies that $T \in \mathscr{L}(L^p(\mathbb{R}^n), L^{np/(n-mp)}(\mathbb{R}^n))$ if $1 < p < n/m$. Also, according to Proposition 3.26, the matrix $M_{\partial^m T}$ defined by
$$M_{\partial^m T}(P, \rho; Q, \varepsilon) = |Q|^{-m/n} \langle T \psi_P^\rho \mid \psi_Q^\varepsilon \rangle$$
is almost diagonal. Fix $p \in (1, n/m)$ and denote $q = np/(n - mp) \in (p, \infty)$.

First we prove that the matrix $M_{\partial^\alpha T}$ in (3.28) is almost diagonal. Fix $P, Q \in \mathcal{D}$ and $\rho, \varepsilon \in \mathcal{E}$. Now, according to the property B2), the wavelet approximation of $T\psi_P^\rho \in L^q(\mathbb{R}^n)$ converges uncoditionally. That is, the series
$$T\psi_P^\rho = \sum_{R \in \mathcal{D}} \sum_{\sigma \in \mathcal{E}} \langle T \psi_P^\rho \mid \psi_R^\sigma \rangle \psi_R^\sigma$$
converges unconditionally in $L^q(\mathbb{R}^n)$. Because $\partial^\alpha \psi_Q^\varepsilon \in L^{q'}(\mathbb{R}^n)$ and $1 = |R|^{-m/n} |R|^{m/n}$, $R \in \mathcal{D}$, we have
$$\begin{aligned} M_{\partial^\alpha T}(P, \rho; Q, \varepsilon) &= \langle \partial^\alpha T \psi_P^\rho \mid \psi_Q^\varepsilon \rangle = (-1)^{|\alpha|} \langle T \psi_P^\rho \mid \partial^\alpha \psi_Q^\varepsilon \rangle \\ &= \sum_{R \in \mathcal{D}} \sum_{\sigma \in \mathcal{E}} |R|^{-m/n} \langle T \psi_P^\rho \mid \psi_R^\sigma \rangle |R|^{m/n} \langle \partial^\alpha \psi_R^\sigma \mid \psi_Q^\varepsilon \rangle \\ &= M_{\partial^m T} M_{\partial^\alpha / \partial^m}(P, \rho; Q, \varepsilon). \end{aligned} \tag{3.34}$$
As a consequence, we have $M_{\partial^\alpha T} = M_{\partial^m T} M_{\partial^\alpha / \partial^m}$. Using Lemma 3.32 and Example 3.31, we see that $M_{\partial^\alpha T}$ is almost diagonal, being a product of almost diagonal

matrices. To be more precise, there exists δ, δ' such that $0 < \delta < \delta' < 1$ and

$$|M_{\partial^\alpha T}(P,\rho;Q,\varepsilon)| \leq C_{T,M_{\partial^\alpha/\partial^m}} \sum_{R\in\mathcal{D}} \omega_{P,R}(\delta,\delta)\omega_{R,Q}(\delta,\delta') \leq C\omega_{P,Q}(\delta,\delta). \tag{3.35}$$

We refer to this estimate later and it serves also for those purposes. □

We need yet another fact: Almost diagonal matrices can be interpreted as bounded operators on $\ell^2(\mathcal{D}\times\mathcal{E})$. This is formally stated in the following result whose proof is based on Schur's lemma and can be found in [MC97, pp. 54–55].

Lemma 3.36. *Let M be an almost diagonal matrix, therefore satisfying the estimate*

$$|M(P,\rho;Q,\varepsilon)| \leq C_M \omega_{P,Q}(\delta,\delta)$$

if $P, Q \in \mathcal{D}$ and $\rho, \varepsilon \in \mathcal{E}$. Then M is a bounded matrix operator on $\ell^2(\mathcal{D}\times\mathcal{E})$. That is, assuming $x \in \ell^2(\mathcal{D}\times\mathcal{E})$ we define

$$y(P,\rho) = Mx(P,\rho) = \sum_{Q\in\mathcal{D},\varepsilon\in\mathcal{E}} M(P,\rho;Q,\varepsilon)x(Q,\varepsilon), \quad (P,\rho)\in\mathcal{D}\times\mathcal{E}.$$

Then $y = Mx \in \ell^2(\mathcal{D}\times\mathcal{E})$ and it satisfies the norm-estimate

$$\sum_{P\in\mathcal{D},\rho\in\mathcal{E}} |y(P,\rho)|^2 \leq C \sum_{Q\in\mathcal{D},\varepsilon\in\mathcal{E}} |x(Q,\varepsilon)|^2,$$

where the constant C depends only on the matrix M.

Next we combine all the pieces together for the reduced $T1$ theorem.

Theorem 3.37. *Let $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ be such that $T1 = T^*1 = 0 \in \{\tau : \mathcal{D}\times\mathcal{E} \to \mathbb{C}\}$. Then, if $|\alpha| = m$, we have $\partial^\alpha T, \partial^\alpha T^* \in \mathscr{L}(L^2(\mathbb{R}^n))$.*

Proof. Because of the symmetry it suffices to verify the boundedness of the operator $\partial^\alpha T$ for $|\alpha| = m$. According to Lemma 3.33, the matrix $M_{\partial^\alpha T}$ is almost diagonal. We define a linear operator $[\partial^\alpha T] : C_0(\mathbb{R}^n) \to (C_0(\mathbb{R}^n))^*$. If $f, g \in C_0(\mathbb{R}^n)$, we set

$$\langle [\partial^\alpha T] f \mid g\rangle = \sum_{Q\in\mathcal{D},\varepsilon\in\mathcal{E}} \sum_{P\in\mathcal{D},\rho\in\mathcal{E}} \langle f \mid \psi^\rho_P\rangle M_{\partial^\alpha T}(P,\rho;Q,\varepsilon)\langle \psi^\varepsilon_Q \mid g\rangle. \tag{3.38}$$

With the aid of property B1) and Lemma 3.36, we can first change the order of summation and then use the Hölder's inequality for that $[\partial^\alpha T] \in \mathscr{L}(L^2(\mathbb{R}^n))$ in the sense of (3.3). In particular, to reach the desired conclusion $\partial^\alpha T \in \mathscr{L}(L^2(\mathbb{R}^n))$ it suffices to verify that

$$\langle [\partial^\alpha T] f, g\rangle = \langle \partial^\alpha T f, g\rangle \tag{3.39}$$

if $f \in C_0(\mathbb{R}^n)$ and $g \in C^m_0(\mathbb{R}^n)$.

To do this, we invoke definition (3.38) and expand the double series therein by applying the identity (3.34). This results in

$$\langle [\partial^\alpha T] f \mid g\rangle = \sum_{Q\in\mathcal{D},\varepsilon\in\mathcal{E}} \sum_{P\in\mathcal{D},\rho\in\mathcal{E}} \sum_{R\in\mathcal{D},\sigma\in\mathcal{E}} \langle f \mid \psi^\rho_P\rangle\langle T\psi^\rho_P \mid \psi^\sigma_R\rangle\langle \partial^\alpha \psi^\sigma_R \mid \psi^\varepsilon_Q\rangle\langle \psi^\varepsilon_Q \mid g\rangle.$$

This sum converges absolutely. This is seen by estimating as in (3.34)–(3.35) and using Lemma 3.36. Hence we can organize the summation to the order R, Q, P. By (3.7) we have $T \in \mathscr{L}(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$ if $p \in (1, n/m)$ and $q = np/(n - mp)$. Using this relation with B2), and the definition of weak derivative, we reach the identity

$$\langle [\partial^\alpha T]f \mid g\rangle = (-1)^{|\alpha|}\langle Tf \mid \partial^\alpha g\rangle = \langle \partial^\alpha Tf \mid g\rangle.$$

This is the required representation formula (3.39). □

Application to convolution operators. The assumptions of the reduced $T1$ theorem hold true if $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ is of convolution type. Notice that the assumptions here are stronger than those in Remark 2.31 (i).

Theorem 3.40. *Let $0 < m < n$ and $k \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a kernel satisfying the size condition $|k(x)| \le c_k|x|^{-n+m}$ and the regularity condition*

$$|\Delta_h^{m+1}(k, x)| \le c_k|h|^{m+\delta}|x|^{-n-\delta}, \qquad \textit{if } 2(m+1)|h| \le |x|.$$

Then the operator $T = f \mapsto k \star f \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ satisfies the cancellation conditions $T1 = 0 = T^*1$ and the boundedness property $\partial^\alpha T \in \mathscr{L}(L^2(\mathbb{R}^n))$ if $|\alpha| = m$.

Proof. To begin with, it is simple to verify that the convolution operator T associated with the kernel $K(x, y) = k(x - y)$ satisfies $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ and we have a generalization of the integral condition (1.8), which occured in Remark 2.31,

$$\sup_{h\neq 0}\left\{|h|^{-m}\int_{\mathbb{R}^n} |\Delta_h^{m+1}(k, x)|dx\right\} < \infty. \tag{3.41}$$

According to Theorem 3.37 it suffices to verify that $T1 = 0 = T^*1$. Due to symmetry it suffices to prove that $T^*1 = 0$ and for this purpose we fix $(Q, \varepsilon) \in \mathcal{D}\times\mathcal{E}$. According to Definition 3.13 it suffices to show that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(x - y)\psi_Q^\varepsilon(y)dydx = \int_{\mathbb{R}^n} T\psi_Q^\varepsilon(x)dx = 0, \tag{3.42}$$

where ψ_Q^ε is an $(m+1)$-regular compactly supported wavelet.

Notice that $T\psi_Q^\varepsilon$ is integrable due to Lemma 3.8 and, in the formal level, the identity (3.42) is trivial since $\int \psi_Q^\varepsilon = 0$; one would change the variables and then the order of integration. However, Fubini's theorem does not apply because the kernel may not be integrable at infinity. Instead the cancellation condition (3.41) is utilized by using the Fourier transform as follows

$$\begin{aligned}\int_{\mathbb{R}^n} T\psi_Q^\varepsilon(x)dx &= \lim_{\xi\to 0,\xi\neq 0}\int_{\mathbb{R}^n} T\psi_Q^\varepsilon(x)e^{-2\pi i x\cdot\xi}dx \\ &= (-2^{-1})^{m+1}\lim_{\xi\to 0,\xi\neq 0}\int_{\mathbb{R}^n} \Delta_{g(\xi)}^{m+1}(T\psi_Q^\varepsilon, x)e^{-2\pi i x\cdot\xi}dx,\end{aligned}$$

where $g(\xi) = 2^{-1}\xi/|\xi|^2$ if $\xi \neq 0$. The last identity follows by iterating the corresponding identity with $m = 0$ and in this special case it follows by a simple change of variables with the aid of the Euler's identity $e^{\pi i} = -1$. Using linearity we get

$$\int_{\mathbb{R}^n} T\psi_Q^\varepsilon(x)dx = (-2^{-1})^{m+1} \lim_{\xi\to 0, \xi\neq 0} \int_{\mathbb{R}^n} (\Delta_{g(\xi)}^{m+1}(k,\cdot) \star \psi_Q^\varepsilon)(x) e^{-2\pi i x\cdot\xi} dx.$$

If $\xi \neq 0$ both of the functions $\Delta_{g(\xi)}^{m+1}(k,\cdot)$ and ψ_Q^ε are integrable and applying the integral estimate (3.41) we have

$$\begin{aligned}&\left| \int_{\mathbb{R}^n} (\Delta_{g(\xi)}^{m+1}(k,\cdot) \star \psi_Q^\varepsilon)(x) e^{-2\pi i x\cdot\xi} dx \right| = |\mathcal{F}(\Delta_{g(\xi)}^{m+1}(k,\cdot))(\xi)||\mathcal{F}(\psi_Q^\varepsilon)(\xi)| \\ &\le ||\Delta_{g(\xi)}^{m+1}(k,\cdot)||_{L^1} |\mathcal{F}(\psi_Q^\varepsilon)(\xi)| \le c_k |g(\xi)|^m |\mathcal{F}(\psi_Q^\varepsilon)(\xi)|.\end{aligned}$$

We also have $|g(\xi)|^m < |\xi|^{-m}$. The Fourier transform of ψ_Q^ε is smooth and all of its partial derivatives of order $\le m+1$ vanish in the origin because of the corresponding vanishing moments for the $(m+1)$-regular wavelets. Estimating $\mathcal{F}(\psi_Q^\varepsilon)$ near the origin by using the Taylor expansion, we get

$$\limsup_{\xi\to 0} |\xi|^{-m} |\mathcal{F}(\psi_Q^\varepsilon)(\xi)| = 0.$$

Combining this with previous estimates leads to the desired identity (3.42). □

3.2. BMO**-type spaces on domains.** To advance beyond the reduced $T1$ theorem we need certain BMO-type sequence and function spaces. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain and $\alpha > 0$. We utilize the following partition of the dyadic cubes into interior (I), boundary (B), and exterior (E) cubes:

- $\mathcal{D}_I(\alpha,\Omega) = \{Q \in \mathcal{D} : Q \subset \Omega \text{ and } \operatorname{dist}(Q,\partial\Omega) \ge \alpha \operatorname{diam}(Q)\}$,
- $\mathcal{D}_B(\alpha,\Omega) = \{Q \in \mathcal{D} : Q \cap \overline{\Omega} \neq \emptyset\} \setminus \mathcal{D}_I(\alpha,\Omega)$,
- $\mathcal{D}_E(\Omega) = \mathcal{D} \setminus (\mathcal{D}_I(\alpha,\Omega) \cup \mathcal{D}_B(\alpha,\Omega)) = \{Q \in \mathcal{D} : Q \cap \overline{\Omega} = \emptyset\}$.

The BMO-type spaces $\dot{f}_\infty^{m,2}(\Omega)$ and $\dot{F}_\infty^{m,2}(\Omega)$ depend on the interior cubes $\mathcal{D}_I(\alpha,\Omega)$ with

$$\alpha = C_{m+1} > 0 \tag{3.43}$$

being the constant in Appendix B for which B4)–B5) hold true in case of $(m+1)$-regular wavelets. A function $f \in L^1_{\text{loc}}(\Omega)$ belongs to $\dot{F}_\infty^{m,2}(\Omega)$ if its wavelet coefficients

$$\{\langle f \mid \psi_Q^\varepsilon\rangle : (Q,\varepsilon) \in \mathcal{D}_I(C_{m+1},\Omega) \times \mathcal{E}\} \tag{3.44}$$

belong to certain sequence space $\dot{f}_\infty^{m,2}(\Omega)$ which, in turn, is defined in terms of a Carleson's condition. The novelty of this definition is that the supports of the wavelets ψ_Q^ε in (3.44) are contained in the domain because of (3.43). These BMO-type spaces turn out to be useful in the difficult direction of the $T\chi_\Omega$ theorem but only if we restrict to the class of Whitney coplump domains where there is a reflection

$$Q \mapsto Q^s : \mathcal{D}_B(C_{m+1},\Omega) \to \mathcal{D}_E(\Omega),$$

satisfying $\operatorname{diam}(Q) = \operatorname{diam}(Q^s)$ and $\operatorname{dist}(Q,Q^s) \le \beta_m \operatorname{diam}(Q)$ for some $\beta_m > 0$.

Spaces $\dot{f}^{m,2}_{\infty}(\Omega)$ and $\dot{F}^{m,2}_{\infty}(\Omega)$ on general domains. We define the sequence spaces $\dot{f}^{m,2}_{\infty}(\Omega)$ and the corresponding function spaces $\dot{F}^{m,2}_{\infty}(\Omega)$ on general domains. These spaces depend on the α-interior cubes for a suitable $\alpha = \alpha_m$. The boundary and exterior cubes are not used here.

Definition 3.45. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $m \in \mathbb{N}_0$. Denote

$$\mathcal{D}^m_I(\Omega) = \mathcal{D}_I(C_{m+1}, \Omega),$$

where $C_{m+1} > 0$ is the constant defined in Appendix B for which B4)–B5) holds true in the case of $(m+1)$-regular wavelets.

Remark 3.46. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $m \in \mathbb{N}_0$. Let $\{\psi^{\varepsilon}_Q\}$ denote the set of $(m+1)$-regular wavelets. Then, according to B4) in Appendix B and Definition 3.45, we have $\operatorname{supp} \psi^{\varepsilon}_Q \subset C_{m+1}Q \subset\subset \Omega$ if $(Q, \varepsilon) \in \mathcal{D}^m_I(\Omega) \times \mathcal{E}$.

Definition 3.47. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $m \in \mathbb{N}_0$. Assume that $\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ is such that

$$||\tau||^2_{\dot{f}^{m,2}_{\infty}(\Omega)} = \sup_{P \in \mathcal{D}^m_I(\Omega)} \frac{1}{|P|} \sum_{Q \subset P} \sum_{\varepsilon \in \mathcal{E}} |Q|^{-2m/n} |\tau(Q, \varepsilon)|^2 < \infty,$$

where the outer summation is over all $Q \in \mathcal{D}$ satisfying $Q \subset P$. Then denote $\tau \in \dot{f}^{m,2}_{\infty}(\Omega)$. Assuming that $f \in L^1_{\mathrm{loc}}(\Omega)$, we denote $f(Q, \varepsilon) = \langle f \mid \psi^{\varepsilon}_Q \rangle$, if $(Q, \varepsilon) \in \mathcal{D}^m_I(\Omega) \times \mathcal{E}$, and $f(Q, \varepsilon) = 0$, if $Q \in \mathcal{D} \setminus \mathcal{D}^m_I(\Omega)$ and $\varepsilon \in \mathcal{E}$. The wavelets $\{\psi^{\varepsilon}_Q\}$ here are $(m+1)$-regular. Furthermore, we denote $f \in \dot{F}^{m,2}_{\infty}(\Omega)$ if $||f||_{\dot{F}^{m,2}_{\infty}(\Omega)} = ||f : \mathcal{D} \times \mathcal{E} \to \mathbb{C}||_{\dot{f}^{m,2}_{\infty}(\Omega)} < \infty$.

We also need the space of bounded mean oscillation on domains.

Definition 3.48. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a domain. Then $\mathrm{BMO}(\Omega)$ is the seminormed vector space of $f \in L^1_{\mathrm{loc}}(\Omega)$ satisfying

$$||f||_{\mathrm{BMO}(\Omega)} = \sup_{Q \subset\subset \Omega} \left\{ \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \right\} < \infty, \quad f_Q = \frac{1}{|Q|} \int_Q f(x) dx.$$

The supremum is taken over all of the cubes compactly contained in the domain.

The function space $\dot{F}^{0,2}_{\infty}(\mathbb{R}^n)$ gives a characterization of $\mathrm{BMO}(\mathbb{R}^n)$ as follows. Assuming that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is bounded at infinity, then

$$f \in \mathrm{BMO}(\mathbb{R}^n) \Leftrightarrow f \in \dot{F}^{0,2}_{\infty}(\mathbb{R}^n)$$

and the corresponding norms are equivalent [Mey92, p. 154]. The analogous identification does not hold true on proper domains $\Omega \subsetneq \mathbb{R}^n$ because the boundary behaviour of functions in the space $\dot{F}^{0,2}_{\infty}(\Omega)$ is less restricted than in the space $\mathrm{BMO}(\Omega)$.

The function spaces $\dot{F}^{m,2}_{\infty}(\mathbb{R}^n)$ for $m \geq 1$ are related to certain Triebel–Lizorkin spaces that are also denoted by $\dot{F}^{m,2}_{\infty}(\mathbb{R}^n)$, see [FJ90, p. 70]. These spaces satisfy

$$\dot{F}^{m,2}_{\infty}(\mathbb{R}^n) = \mathcal{I}^m(\mathrm{BMO}(\mathbb{R}^n)) = \{f \in \mathcal{S}'/\mathcal{P} \,:\, \partial^{\alpha} f \in \mathrm{BMO}(\mathbb{R}^n) \text{ for } |\alpha| = m\}.$$

The first identity follows from the φ-independence and the second can be found in [Str80, Proposition 3.1.(b)]. Also, a function $f : \mathbb{R}^n \to \mathbb{C}$ is in $\mathcal{I}^1(\mathrm{BMO}(\mathbb{R}^n))$ if, and only if, the difference quotients $x \mapsto |h|^{-1}(f(x+h) - f(x))$ are in $\mathrm{BMO}(\mathbb{R}^n)$ uniformly with respect to $h \in \mathbb{R}^n$ [Str80, Theorem 3.2.].

The $T\chi_\Omega$ theorem motivates our definition for the sequence spaces $\dot{f}^{m,2}_\infty(\Omega)$ on domains. These spaces furnish the correct means to quantify certain cancellation effects therein. The corresponding function spaces $\dot{F}^{m,2}_\infty(\Omega)$ are large spaces because we ignore the boundary behaviour in a certain sense. A smaller space in the sense of bounded inclusion is the restriction of $\dot{F}^{m,2}_\infty(\mathbb{R}^n)$ on the domain Ω. This restriction space is more natural in many applications. However, our definition for the sequence spaces $\dot{f}^{m,2}_\infty(\Omega)$ has the advantage that the projection

$$\tau \mapsto \tau\chi_{\mathcal{D}^m_I(\Omega)\times\mathcal{E}} : \{\mathcal{D}\times\mathcal{E} \to \mathbb{C}\} \to \{\mathcal{D}\times\mathcal{E} \to \mathbb{C}\}$$

induces a bounded linear operator $\dot{f}^{m,2}_\infty(\Omega) \to \dot{f}^{m,2}_\infty(\mathbb{R}^n)$ on general domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$. This important auxiliary result is verified next.

Lemma 3.49. *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $m \in \mathbb{N}_0$. Let $\tau \in \dot{f}^{m,2}_\infty(\Omega)$ and $\sigma : \mathcal{D}\times\mathcal{E} \to \mathbb{C}$ be such that $\sigma(Q,\varepsilon) = \tau(Q,\varepsilon)$ if $Q \in \mathcal{D}^m_I(\Omega)$ and $\sigma(Q,\varepsilon) = 0$ if $Q \in \mathcal{D}\setminus\mathcal{D}^m_I(\Omega)$. Then $\sigma \in \dot{f}^{m,2}_\infty(\mathbb{R}^n)$ and $||\sigma||_{\dot{f}^{m,2}_\infty(\mathbb{R}^n)} \leq ||\tau||_{\dot{f}^{m,2}_\infty(\Omega)}$.*

Proof. Fix $P \in \mathcal{D}$. It suffices to prove that

$$\frac{1}{|P|}\sum_{Q\subset P}\sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n}|\sigma(Q,\varepsilon)|^2 \leq ||\tau||^2_{\dot{f}^{m,2}_\infty(\Omega)}.$$

Denote $\beta(Q) = \sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n}|\sigma(Q,\varepsilon)|^2$. This vanishes if $Q \notin \mathcal{D}^m_I(\Omega)$. If $Q \in \mathcal{D}^m_I(\Omega)$ and $Q \subset P$ then there exists a maximal cube $Q^m \in \mathcal{D}^m_I(\Omega)$ with the property $Q \subset Q^m \subset P$. Denote

$$\mathcal{M} = \{Q^m : Q \subset P \text{ and } Q \in \mathcal{D}^m_I(\Omega)\}.$$

Fix $Q, R \in \mathcal{M}$. Then $Q \cup R \subset P$ and $Q \cap R = \emptyset$ because of maximality and properties of dyadic cubes. Combining these facts, we get

$$\frac{1}{|P|}\sum_{Q\subset P}\beta(Q) = \frac{1}{|P|}\sum_{Q\in\mathcal{M}}|Q||Q|^{-1}\sum_{R\subset Q}\beta(R) \leq \frac{||\tau||^2_{\dot{f}^{m,2}_\infty(\Omega)}}{|P|}\sum_{Q\in\mathcal{M}}|Q| \leq ||\tau||^2_{\dot{f}^{m,2}_\infty(\Omega)},$$

which is as required. □

Reflection of dyadic cubes on Whitney coplump domains. We provide a geometric characterization of Whitney coplump domains in terms of so called (α,β)-coplump domains. This gives us a certain reflection of dyadic cubes.

Definition 3.50. Let $\alpha, \beta > 0$. A domain $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, is (α,β)-coplump if for every $Q \in \mathcal{D}_B(\alpha,\Omega)$ there is a *reflected cube* $Q^s \in \mathcal{D}_E(\Omega)$ satisfying $\mathrm{diam}(Q) = \mathrm{diam}(Q^s)$ and $\mathrm{dist}(Q,Q^s) \leq \beta\,\mathrm{diam}(Q)$. If Ω is (α,β)-coplump domain then we extend the *reflection* $Q \mapsto Q^s$ to all of the dyadic cubes by setting $Q^s = Q$ if $Q \in \mathcal{D}\setminus\mathcal{D}_B(\alpha,\Omega)$.

Remark 3.51. Regarding an (α,β)-coplump domain $\Omega\subset\mathbb{R}^n$,

- If $\alpha\geq\tilde\alpha$ and $\beta\leq\tilde\beta$ then Ω is $(\tilde\alpha,\tilde\beta)$-coplump. We will show that for every $\tilde\alpha>0$ there exists $\tilde\beta>0$, depending on the parameters $\tilde\alpha, n$, so that Ω is $(\tilde\alpha,\tilde\beta)$-coplump.
- Ω is *minimally regular*, that is, $\Omega=\operatorname{int}(\overline{\Omega})$. To prove the nontrivial inclusion, let $x\in\operatorname{int}(\overline{\Omega})$. Then $B(x,r)\subset\overline{\Omega}$ for some $r>0$. We assume, aiming at a contradiction, that $x\notin\Omega$. Then $x\in\overline{\Omega}\setminus\Omega=\partial\Omega$. Choose a cube $Q\in\mathcal{D}_B(\alpha,\Omega)$ such that $x\in Q$ and $\operatorname{diam}(Q)<r/(2+\beta)$. Then there exists a cube $Q^s\in\mathcal{D}_E(\Omega)$ so that $Q^s\cap\overline{\Omega}=\emptyset$ but also $Q^s\subset B(x,r)\subset\overline{\Omega}$ because, for $y\in Q^s$, we have

  $$|x-y|\leq\operatorname{diam}(Q)+\operatorname{dist}(Q,Q^s)+\operatorname{diam}(Q^s)\leq(2+\beta)\operatorname{diam}(Q)<r.$$
- Assume that $\emptyset\neq\Omega\neq\mathbb{R}^n$. Then there exists $x\in\partial\Omega$ and therefore also arbitrarily large cubes $Q\in\mathcal{D}_B(\alpha,\Omega)$. In particular, there are arbitrarily large cubes $Q^s\subset\mathbb{R}^n\setminus\Omega$ and therefore $\mathbb{R}^n\setminus\Omega$ is unbounded.
- Assume that $\Omega\neq\mathbb{R}^n$. For every interior cube $Q\in\mathcal{D}_I(\alpha,\Omega)$ there exists a unique maximal cube $Q^{\max}(Q)\in\mathcal{D}_I(\alpha,\Omega)$ such that $P\subset Q^{\max}(Q)$ if $P\in\mathcal{D}_I(\alpha,\Omega)$ and $Q\subset P$. Furthermore, we have

  $$\mathcal{D}_I(\alpha,\Omega)=\bigcup_{P\in\mathcal{D}_I(\alpha,\Omega)}\{Q\in\mathcal{D}\,:\,Q\subset Q^{\max}(P)\}.$$

  and the family $\{Q^{\max}(Q)\,:\,Q\in\mathcal{D}_I(\alpha,\Omega)\}$ is a partition of the domain Ω.

The following result is due to J. Väisälä [Väi08] but the proof is ours. For the definition of c-coplump domains, see Definition 1.14.

Theorem 3.52. *Let $\emptyset\neq\Omega\subset\mathbb{R}^n$, $n\geq 2$, be a domain such that* $\operatorname{diam}(\mathbb{R}^n\setminus\Omega)=\infty$. *If Ω is c-coplump then it is $(c,3c)$-coplump. Conversely, if Ω is (α,β)-coplump then it is c-coplump with $c=\sqrt{n}(12+4\beta)$.*

Proof. First assume that Ω is c-coplump and fix $Q\in\mathcal{D}_B(c,\Omega)$. Then $Q\cap\overline{\Omega}\neq\emptyset$ and $\operatorname{dist}(Q,\partial\Omega)<c\operatorname{diam}(Q)$. Fix a point $x\in\partial\Omega$ such that $\operatorname{dist}(x,Q)\leq c\operatorname{diam}(Q)$. Using the assumptions we find a point $z\in\bar B(x,2c\operatorname{diam}(Q))$ such that $B(z,2\operatorname{diam}(Q))\subset\mathbb{R}^n\setminus\Omega$. Let Q^s be the unique dyadic cube such that $z\in Q^s$ and $\ell(Q^s)=\ell(Q)$. Then $Q^s\subset B(z,2\operatorname{diam}(Q))\subset\mathbb{R}^n\setminus\Omega$; that is, $Q^s\in\mathcal{D}_E(\Omega)$. Furthermore, we have

$$\operatorname{dist}(Q^s,Q)\leq 2c\operatorname{diam}(Q)+\operatorname{dist}(x,Q)\leq 3c\operatorname{diam}(Q).$$

We conclude that Ω is $(c,3c)$-coplump.

Conversely, assume that Ω is (α,β)-coplump. Consider any $x\in\mathbb{R}^n\setminus\Omega$ and $r>0$ so that

$$B(x,r/(3+\beta))\not\subset\mathbb{R}^n\setminus\Omega.$$

Then there is $w\in\partial\Omega$ satisfying $|x-w|<r/(3+\beta)$. Let $Q\in\mathcal{D}$ be a cube such that $w\in Q$ and $r/(6+2\beta)\leq\operatorname{diam}(Q)<r/(3+\beta)$. Clearly $Q\in\mathcal{D}_B(\alpha,\Omega)$ and

therefore there exists a cube $Q^s \in \mathcal{D}_E(\Omega)$ such that $\ell(Q^s) = \ell(Q)$ and $\operatorname{dist}(Q^s, Q) \leq \beta \operatorname{diam}(Q)$. In particular, if $y \in Q^s$, then

$$|x - y| \leq |x - w| + \operatorname{diam}(Q) + \operatorname{dist}(Q, Q^s) + \operatorname{diam}(Q^s) < r$$

proving that $Q^s \subset B(x, r)$. As a consequence, the centerpoint z of Q^s belongs to $B(x, r)$. Furthermore, we have

$$B\left(z, \frac{r}{4\sqrt{n}(3+\beta)}\right) \subset B(z, \ell(Q^s)/2) \subset Q^s \subset \mathbb{R}^n \setminus \Omega,$$

proving the claim. □

We have the following characterization of Whitney coplump domains.

Theorem 3.53. *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain. Then*

- *Ω is Whitney coplump if, and only if, it is (α, β)-coplump for some $\alpha, \beta > 0$.*
- *If Ω is (α, β)-coplump and $\tilde{\alpha} > 0$ then there exists $\tilde{\beta} > 0$, depending at most on $n, \tilde{\alpha}, \beta$, so that Ω is $(\tilde{\alpha}, \tilde{\beta})$-coplump.*

Proof. The first claim follow from Theorem 3.52 and Remark 3.51 about the unboundedness of $\mathbb{R}^n \setminus \Omega$ if $\Omega \neq \mathbb{R}^n$. Then we prove the second claim. If $\Omega = \mathbb{R}^n$ then we are done because $\mathbb{R}^n$ is $(\tilde{\alpha}, \tilde{\beta})$-coplump for every $\beta > 0$. Assume then that $\Omega \neq \mathbb{R}^n$. Then $\operatorname{diam}(\mathbb{R}^n \setminus \Omega) = \infty$, see Remark 3.51, and we are in the position to apply Theorem 3.52. But first note that Ω is $(\alpha, \tilde{\alpha} + \beta)$-coplump because $\tilde{\alpha} + \beta > \beta$. In particular, Ω is c-coplump where $c = \sqrt{n}(12 + 4(\tilde{\alpha} + \beta)) > \tilde{\alpha}$. Applying Theorem 3.52 again we see that Ω is $(c, 3c)$-coplump and, because $c > \tilde{\alpha}$, it is also $(\tilde{\alpha}, 3c)$-coplump. □

This characterization allows us to choose the α-parameter such that $\operatorname{supp} \psi_Q^\varepsilon \subset\subset \Omega$ if Q is an α-interior cube and ψ_Q^ε is an $(m+1)$-regular wavelet. This is recorded next.

Definition 3.54. Let $\Omega \subset \mathbb{R}^n$ be a Whitney coplump domain and $m \in \mathbb{N}_0$. Denote by $C_{m+1} > 0$ the constant for the $(m+1)$-regular wavelets, see B4)–B5) in Appendix B, and denote also $\beta_m = 1 + \inf\{\beta > 0 : \Omega \text{ is } (C_{m+1}, \beta)\text{-coplump}\} < \infty$.

Remark 3.55. Let $\Omega \subset \mathbb{R}^n$ be a Whitney coplump domain and let $m \in \mathbb{N}_0$. Then the domain Ω is (C_{m+1}, β_m)-coplump. In particular, if ψ_Q^ε's are $(m+1)$-regular, we have

- $\operatorname{supp} \psi_Q^\varepsilon \subset C_{m+1} Q \subset\subset \Omega$ if $(Q, \varepsilon) \in \mathcal{D}_I^m(\Omega) \times \mathcal{E}$,
- $\ell(Q) = \ell(Q^s)$ and $Q^s \subset B(x_Q, (2 + \beta_m) \operatorname{diam}(Q))$ if $Q \in \mathcal{D}$,
- $Q^s \subset \mathbb{R}^n \setminus \Omega$ if $Q \in \mathcal{D} \setminus \mathcal{D}_I^m(\Omega)$.

In the sequel $m \in \mathbb{N}_0$ is explicitly given but the parameters C_{m+1}, β_m are often omitted. The convention is that the given index m (implicitly) determines the parameters C_{m+1} and β_m as in Definition 3.54.

Remark 3.56. Later in Theorem 6.6 we prove that Whitney coplump domains are invariant under K-quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n$. This result combined with Theorem 3.53 shows that (α, β)-coplump domains are also invariant: if $\Omega \subset \mathbb{R}^n$ is (α, β)-coplump, then the image $f\Omega$ is $(\tilde{\alpha}, \tilde{\beta})$-coplump for some $\tilde{\alpha}, \tilde{\beta}$ that depend at most on $\alpha, \beta, n, K, \Omega$.

3.3. **Endpoint estimates for restricted operators.** Let $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ be associated with a global kernel and $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a domain. The ultimate goal in this section is to characterize when

$$\{\partial^\alpha T \,:\, |\alpha| = m\} \subset \mathscr{L}(L^2(\Omega)) \tag{3.57}$$

and the same with T replaced by T^*. This is the $T\chi_\Omega$ theorem for restricted operators. The present object of study is the boundedness on certain endpoint spaces and the following results give the easy direction of the $T\chi_\Omega$ theorem. Our results here read as follows

- Assume that $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, is any domain and the condition (3.57) holds. Then $T \in \mathscr{L}(L^\infty(\Omega), \dot{f}^{m,2}_\infty(\Omega))$.
- Assume that $\Omega \subset \mathbb{R}^n$ is a uniform domain and that the condition (3.57) holds. Then $T\chi_\Omega = 0$ in $\dot{f}^{m,2}_\infty(\Omega)$ if, and only if, $T \in \mathscr{L}(\mathrm{BMO}(\Omega), \dot{f}^{m,2}_\infty(\Omega))$.

The downside is that the target $\dot{f}^{m,2}_\infty(\Omega)$ is a large space. We enhance the target space in connection with the $T\chi_\Omega$ theorem, see later Corollary 3.117, but there we need to restrict to Whitney coplump domains.

Standing assumptions and notation. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain and $0 < m < n$. Let $\{\psi^\varepsilon_Q\} \subset C^{m+1}_0(\mathbb{R}^n)$ be the family of $(m+1)$-regular wavelets. Fix the family $\mathcal{D}^m_I(\Omega)$ in Definition 3.45. Fix $S, T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ and denote $S \sim T$ if the associated kernels $K_S, K_T \in \mathrm{K}^{-m}_{\mathbb{R}^n}(\delta)$ satisfy

$$K_S|\Omega \times \Omega \setminus \{(x,x)\} = K_T|\Omega \times \Omega \setminus \{(x,x)\}. \tag{3.58}$$

Notice that $\sim$ defines an equivalence relation in $\mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$. Let $\alpha \in \mathbb{N}^n_0$ satisfy $|\alpha| = m$. We denote $\partial^\alpha T \in \mathscr{L}(L^p(\Omega))$ if

$$\partial^\alpha \circ \mathrm{id}^* \circ T \circ \mathrm{id} : C_0(\Omega) \to (C^m_0(\Omega))^* \in \mathscr{L}(L^p(\Omega)) \tag{3.59}$$

in the sense of (3.3). Here $\mathrm{id} : C_0(\Omega) \hookrightarrow C_0(\mathbb{R}^n)$ and $\mathrm{id}^* : (C_0(\mathbb{R}^n))^* \hookrightarrow (C_0(\Omega))^*$ are canonical inclusions. If $S \sim T$ and $\partial^\alpha S \in \mathscr{L}(L^p(\mathbb{R}^n))$, then $\partial^\alpha T \in \mathscr{L}(L^p(\Omega))$.

Fix $b \in \mathscr{D}(\delta)$. Then $Tb : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ is defined in (3.12) and we denote $Tb \in \dot{f}^{m,2}_\infty(\Omega)$ if $\{Tb(Q, \varepsilon)\} \in \dot{f}^{m,2}_\infty(\Omega)$. For instance, we have $b\chi_\Omega \in \mathscr{D}(\delta)$ if $b \in L^\infty(\Omega)$ and we denote $T \in \mathscr{L}(L^\infty(\Omega), \dot{f}^{m,2}_\infty(\Omega))$ if

$$||T(b\chi_\Omega)||_{\dot{f}^{m,2}_\infty(\Omega)} \leq C||b||_{L^\infty(\Omega)} \tag{3.60}$$

holds with C independent of $b \in L^\infty(\Omega)$.

Assume that $\Omega \subset \mathbb{R}^n$ is a uniform domain. There exists a bounded and linear extension operator $E : \mathrm{BMO}(\Omega) \to \mathrm{BMO}(\mathbb{R}^n)$, $Eb|\Omega = b$ if $b \in \mathrm{BMO}(\Omega)$ [Jon80].

Example 3.15 shows that $b\chi_\Omega = (Eb)\chi_\Omega \in \mathscr{D}(\delta)$ if $b \in \mathrm{BMO}(\Omega)$. This allows us to define $T(b\chi_\Omega)$ using (3.12) and $T \in \mathscr{L}(\mathrm{BMO}(\Omega), \dot{f}^{m,2}_\infty(\Omega))$ if

$$||T(b\chi_\Omega)||_{\dot{f}^{m,2}_\infty(\Omega)} \le C||b||_{\mathrm{BMO}(\Omega)} \tag{3.61}$$

holds with C independent of $b \in \mathrm{BMO}(\Omega)$. Because of $||\chi_\Omega||_{\mathrm{BMO}(\Omega)} = 0$, the condition $T\chi_\Omega = 0$ in $\dot{f}^{m,2}_\infty(\Omega)$ is necessary for the inequality (3.61) to hold.

Norm estimates of Tb for $b \in L^\infty(\Omega)$. A Calderón–Zygmund operator $T \in \mathrm{CZO}$ maps $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ boundedly [Gra04, p. 580–582]. The counterpart in our setting is formally stated as follows:

$$\begin{aligned} &T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta) \text{ and } \{\partial^\alpha T \,:\, |\alpha| = m\} \subset \mathscr{L}(L^2(\Omega)) \\ &\qquad\qquad\qquad\qquad \Rightarrow T \in \mathscr{L}(L^\infty(\Omega), \dot{f}^{m,2}_\infty(\Omega)). \end{aligned} \tag{3.62}$$

Definition (3.12) and Corollary 3.11 combined yield the estimate

$$\sup\{|Q|^{-1/2-m/n}|Tb(Q,\varepsilon)| \,:\, (Q,\varepsilon) \in \mathcal{D}\times\mathcal{E}\} \le C_{n,m,\delta,C_{m+1},T}||b||_{L^\infty(\mathbb{R}^n)} \tag{3.63}$$

if $b \in L^\infty(\mathbb{R}^n)$. This estimate is sharp: Theorem 3.93 implies that there exists a so called (adjoint) paraproduct operator $\Pi^*_\tau \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ satisfying $\Pi^*_\tau 1(Q,\varepsilon) = |Q|^{1/2+m/n}$ for every $(Q,\varepsilon) \in \mathcal{D}\times\mathcal{E}$. As a consequence, $\Pi^*_\tau 1 \notin \dot{f}^{m,2}_\infty(\mathbb{R}^n)$. In the light of (3.62) we see that

$$\{\partial^\alpha \Pi^*_\tau \,:\, |\alpha| = m\} \not\subset \mathscr{L}(L^2(\mathbb{R}^n)).$$

Thus the boundedness properties for WSIO's are not trivial. The paraproduct operators are defined later and now we turn to the proof of (3.62).

We begin with a tail estimate which arises from the globally defined kernel. This *tail lemma* is useful also later in connection with the so called interior paraproduct operators.

Lemma 3.64. *Assume that $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ and $b \in L^\infty(\mathbb{R}^n)$. Let $P \in \mathcal{D}$ and $B \subset \mathbb{R}^n$ be a measurable set, satisfying $\mathrm{dist}(B,P) \ge \varrho\ell(P)$ for some $\varrho > 0$. Then*

$$\frac{1}{|P|}\sum_{Q\subset P}\sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n}|T(b\chi_B)(Q,\varepsilon)|^2 \le C||b\chi_B||^2_{L^\infty(\mathbb{R}^n)}$$

so that the constant C depends at most on the parameters $\varrho, n, m, \delta, C_{m+1}, T$.

Proof. By scaling we can assume that $||b\chi_B||_{L^\infty(\mathbb{R}^n)} = 1$. Fix a dyadic cube $Q \subset P$ and $\varepsilon \in \mathcal{E}$. Applying Lemma 3.8 we have the estimate

$$\begin{aligned} |T(b\chi_B)(Q,\varepsilon)| &= \left|\int_B b(x)\overline{T^*\psi^\varepsilon_Q(x)}dx\right| \le \int_B |T^*\psi^\varepsilon_Q(x)|dx \\ &\le C|Q|^{-1/2+m/n}\int_B (1+\ell(Q)^{-1}|x-x_Q|)^{-n-\delta}dx. \end{aligned} \tag{3.65}$$

Fix $x \in B$. Due to the assumptions, we have $|x-x_Q| \geq \varrho\ell(P)$ and also $1+\frac{|x-x_Q|}{\ell(Q)} \geq \frac{\varrho\ell(P)}{\ell(Q)}$. Using this and the estimate (3.65), we get

(3.66)

$$\begin{aligned}|T(b\chi_B)(Q,\varepsilon)| &\leq C|Q|^{-1/2+(m+\delta/2)/n}|P|^{-(\delta/2)/n}\int_{\mathbb{R}^n}(1+\ell(Q)^{-1}|x-x_Q|)^{-n-\delta/2}dx\\ &\leq C|Q|^{1/2+(m+\delta/2)/n}|P|^{-(\delta/2)/n}.\end{aligned}$$

Summing the squared estimates with respect to $\varepsilon \in \mathcal{E}$, we get

$$\sum_{\varepsilon\in\mathcal{E}}|T(b\chi_B)(Q,\varepsilon)|^2 \leq C|Q|^{1+(2m+\delta)/n}|P|^{-\delta/n}, \quad \text{if } Q\in\mathcal{D} \text{ and } Q\subset P. \tag{3.67}$$

Denote $\ell(P)=2^{-\mu}$ and $\ell(Q)=2^{-\nu}$. Then, applying the estimate (3.67), we get

$$\begin{aligned}\frac{1}{|P|}\sum_{Q\subset P}\sum_{\varepsilon\in\mathcal{E}}|Q|^{-2m/n}|T(b\chi_B)(Q,\varepsilon)|^2 &\leq |P|^{-\delta/n}\frac{C}{|P|}\sum_{Q\subset P}|Q|^{1+\delta/n}\\ &\leq |P|^{-\delta/n}\frac{C}{|P|}\sum_{\nu=\mu}^{\infty}\frac{|P|}{2^{-\nu n}}2^{-\nu n}2^{-\nu\delta} = C|P|^{-\delta/n}\sum_{\nu=\mu}^{\infty}2^{-\nu\delta} \leq C_{\varrho,n,m,\delta,C_{m+1},T}.\end{aligned}$$

This estimate is as required. □

In order to estimate $||T(b\chi_\Omega)||_{\dot{f}^{m,2}_\infty(\Omega)}$ we apply the tail lemma above which reduces the situation to the boundedness assumption

$$\{\partial^\alpha T : |\alpha|=m\}\subset\mathscr{L}(L^2(\Omega)).$$

The inclusion to the smaller space $\mathscr{L}(L^2(\mathbb{R}^n))$ need not hold true.

Theorem 3.68. *Assume that $\emptyset\neq\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a domain. Let $T\in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ be such that $\partial^\alpha T\in\mathscr{L}(L^2(\Omega))$ if $|\alpha|=m$. Then $T\in\mathscr{L}(L^\infty(\Omega),\dot{f}^{m,2}_\infty(\Omega))$. To put this otherwise, we have $T(b\chi_\Omega)\in\dot{f}^{m,2}_\infty(\Omega)$ for every $b\in L^\infty(\Omega)$ and*

$$||T(b\chi_\Omega)||_{\dot{f}^{m,2}_\infty(\Omega)}\leq C||b||_{L^\infty(\Omega)},$$

where the constant C is independent of b.

Proof. By using the linearity, we can assume that $||b||_{L^\infty(\Omega)}=1$. Let us denote $\beta(Q)=\sum_{\varepsilon\in\mathcal{E}}|T(b\chi_\Omega)(Q,\varepsilon)|^2$ if $Q\in\mathcal{D}^m_I(\Omega)$. According to Definition 3.47 it suffices to prove that

$$||T(b\chi_\Omega)||^2_{\dot{f}^{m,2}_\infty(\Omega)} = \sup_{P\in\mathcal{D}^m_I(\Omega)}\frac{1}{|P|}\sum_{Q\subset P}|Q|^{-2m/n}\beta(Q)\leq C \tag{3.69}$$

with constant C independent of b. For this purpose we fix $P\in\mathcal{D}^m_I(\Omega)$ and denote $\Sigma_P=|P|^{-1}\sum_{Q\subset P}|Q|^{-2m/n}\beta(Q)$. If $(Q,\varepsilon)\in\mathcal{D}^m_I(\Omega)\times\mathcal{E}$ satisfies $Q\subset P$, we write

$$T(b\chi_\Omega)(Q,\varepsilon) = T(b\chi_{\Omega\setminus 4P})(Q,\varepsilon)+T(b\chi_{4P\cap\Omega})(Q,\varepsilon). \tag{3.70}$$

Denote also $\beta_1(Q) = \sum_{\varepsilon\in\mathcal{E}} |T(b\chi_{\Omega\setminus 4P})(Q,\varepsilon)|^2$ and $\beta_2(Q) = \sum_{\varepsilon\in\mathcal{E}} |T(b\chi_{4P\cap\Omega})(Q,\varepsilon)|^2$. Then

$$\Sigma_P \le \frac{3}{|P|}\sum_{Q\subset P} |Q|^{-2m/n}\beta_1(Q) + \frac{3}{P}\sum_{Q\subset P} |Q|^{-2m/n}\beta_2(Q) = \Sigma_{P,1} + \Sigma_{P,2},$$

Notice that $\operatorname{dist}(\Omega\setminus 4P, P) \ge \ell(P)$. Hence, by using Lemma 3.64 with $B = \Omega\setminus 4P$, we get the estimate

$$\Sigma_{P,1} = \frac{1}{|P|}\sum_{Q\subset P} |Q|^{-2m/n}\beta_1(Q) \le C_1. \tag{3.71}$$

Here the finite constant C_1 is independent of P and b.

In order to estimate the sum $\Sigma_{P,2}$ we need some preparations. Fix $\varepsilon \in \mathcal{E}$ and consider the corresponding lifted wavelet in Lemma B.2 – there is a canonical $\alpha = \alpha(\varepsilon, m) \in \mathbb{N}_0^n$, $|\alpha| = m$, and a lifted wavelet $\psi^{\varepsilon,m}$ satisfying $\psi^\varepsilon = \partial^\alpha\psi^{\varepsilon,m}$ and B4)–B5) in Appendix B. Fix a dyadic cube $Q \subset P$. It satisfies $Q \in \mathcal{D}_I^m(\Omega)$ and

$$|Q|^{-m/n}\psi_Q^\varepsilon = \partial^\alpha\psi_Q^{\varepsilon,m}.$$

Here $\operatorname{supp}\psi_Q^{\varepsilon,m} \subset C_{m+1}Q \subset\subset \Omega$, which follows from Lemma B.2 and Remark 3.46. The assumption $\partial^\alpha T \in \mathscr{L}(L^2(\Omega))$ implies that $f^\alpha = \partial^\alpha T(b\chi_{4P\cap\Omega}) \in L^2(\Omega)$ satisfies

$$||f^\alpha||_{L^2(\Omega)} \le C_{T,\alpha}||b\chi_{4P\cap\Omega}||_{L^2(\Omega)}. \tag{3.72}$$

Using these preparations we can now proceed as follows

$$\begin{aligned} |T(b\chi_{4P\cap\Omega})(Q,\varepsilon)| &= |\langle b\chi_{4P\cap\Omega} \mid T^*\psi_Q^\varepsilon\rangle| \\ &= |Q|^{m/n}|\langle\partial^\alpha T(b\chi_{4P\cap\Omega}) \mid \psi_Q^{\varepsilon,m}\rangle| \\ &= |Q|^{m/n}|\langle\psi_Q^{\varepsilon,m} \mid \chi_\Omega f^\alpha\rangle|. \end{aligned} \tag{3.73}$$

Squaring this identity and summing it with respect to the dyadic cubes $Q \subset P$, but still keeping ε fixed, we get

$$\Sigma_{P,2}(\varepsilon) = \frac{1}{|P|}\sum_{Q\subset P} |Q|^{-2m/n}|T(b\chi_{4P\cap\Omega})(Q,\varepsilon)|^2 \le \frac{1}{|P|}\sum_{Q\in\mathcal{D}}\sum_{\sigma\in\mathcal{E}} |\langle\psi_Q^{\sigma,m} \mid \chi_\Omega f^\alpha\rangle|^2. \tag{3.74}$$

Then, by expanding the lifted wavelets $\psi_Q^{\sigma,m} \in L^2(\mathbb{R}^n)$, we obtain

$$|\langle\psi_Q^{\sigma,m} \mid \chi_\Omega f^\alpha\rangle| = \left|\sum_{P\in\mathcal{D}}\sum_{\rho\in\mathcal{E}}\langle\psi_Q^{\sigma,m} \mid \psi_P^\rho\rangle\langle\psi_P^\rho \mid \chi_\Omega f^\alpha\rangle\right|.$$

The matrix $M = M_m^{\mathrm{lift}}$ defined by

$$M_m^{\mathrm{lift}}(Q,\sigma;P,\rho) = \langle\psi_Q^{\sigma,m} \mid \psi_P^\rho\rangle$$

for $Q, P \in \mathcal{D}$ and $\sigma, \rho \in \mathcal{E}$ is almost diagonal, see Example 3.31. Thus, applying Lemma 3.36, we see that M is a bounded matrix operator on $\ell^2(\mathcal{D}\times\mathcal{E})$. Using this result within the right-hand side of the inequality (3.74) and then using (3.72) yields

$$\Sigma_{P,2}(\varepsilon) \leq \frac{C_M}{|P|} ||\chi_\Omega f^\alpha||^2_{L^2(\mathbb{R}^n)} \leq \frac{C_{M,T,\alpha}}{|P|} ||b\chi_{4P\cap\Omega}||^2_{L^2(\Omega)} \leq C_2(\varepsilon). \tag{3.75}$$

Here the finite constant $C_2(\varepsilon)$ is independent of P and b because $||b||_{L^\infty(\Omega)} \leq 1$. Combining this with the inequality (3.71) we see that

$$\Sigma_P \leq 3\Sigma_{P,1} + 3 \sum_{\varepsilon\in\mathcal{E}} \Sigma_{P,2}(\varepsilon) \leq C$$

with upper bound independent of P and b. Taking supremum of Σ_P over all $P \in \mathcal{D}^m_I(\Omega)$ we obtain the estimate (3.69) as required. □

Norm estimates of Tb for $b \in \mathrm{BMO}(\Omega)$. A Calderón–Zygmund operator $T \in \mathrm{CZO}$ is bounded on $\mathrm{BMO}(\mathbb{R}^n)$ if, and only if, $T1 = 0$ [MC97, p. 23]. Here we establish a similar result for WSIO's restricted to uniform domains.

Theorem 3.76. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain. Let $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ be such that $\partial^\alpha T \in \mathscr{L}(L^2(\Omega))$ if $|\alpha| = m$. Then $T\chi_\Omega = 0$ in $\dot{f}^{m,2}_\infty(\Omega)$ if, and only if, $T \in \mathscr{L}(\mathrm{BMO}(\Omega), \dot{f}^{m,2}_\infty(\Omega))$. The latter condition is equivalent to the estimate*

$$||T(b\chi_\Omega)||_{\dot{f}^{m,2}_\infty(\Omega)} \leq C||b||_{\mathrm{BMO}(\Omega)},$$

where C is independent of $b \in \mathrm{BMO}(\Omega)$.

Proof. The sufficiency follows as $||\chi_\Omega||_{\mathrm{BMO}(\Omega)} = 0$. Next we assume that $T\chi_\Omega = 0$ in $\dot{f}^{m,2}_\infty(\Omega)$. Because the extension operator $E : \mathrm{BMO}(\Omega) \to \mathrm{BMO}(\mathbb{R}^n)$ is bounded, it suffices to prove the estimate

$$||T(b\chi_\Omega)||_{\dot{f}^{m,2}_\infty(\Omega)} = ||T((Eb)\chi_\Omega)||_{\dot{f}^{m,2}_\infty(\Omega)} \leq C||Eb||_{\mathrm{BMO}(\mathbb{R}^n)}, \tag{3.77}$$

where C should be independent of Eb. The proof of this is estimate very similar to the proof of Theorem 3.68 and we only indicate the required modifications here.

Fix $P \in \mathcal{D}^m_I(\Omega)$ and $(Q,\varepsilon) \in \mathcal{D}^m_I(\Omega) \times \mathcal{E}$ satisfying $Q \subset P$. Analogous to (3.70), but using also the assumption $T\chi_\Omega = 0$ in $\dot{f}^{m,2}_\infty(\Omega)$, we have

$$\begin{aligned} T((Eb)\chi_\Omega)(Q,\varepsilon) &= T((Eb)\chi_\Omega)(Q,\varepsilon) - (Eb)_{4P} T\chi_\Omega(Q,\varepsilon) \\ &= T((Eb - (Eb)_{4P})\chi_{\Omega\setminus 4P})(Q,\varepsilon) + T((Eb-(Eb)_{4P})\chi_{\Omega\cap 4P})(Q,\varepsilon). \end{aligned}$$

First of all, we need to reach the estimate

$$\frac{1}{|P|} \sum_{Q\subset P} \sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n} |T((Eb-(Eb)_{4P})\chi_{\Omega\setminus 4P})(Q,\varepsilon)|^2 \leq C_1 ||Eb||^2_{\mathrm{BMO}(\mathbb{R}^n)}, \tag{3.78}$$

where C_1 should be independent of P and Eb. Estimating as in Lemma 3.64, we have

(3.79)
$$|T((Eb-(Eb)_{4P})\chi_{\Omega\setminus 4P})(Q,\varepsilon)| \\ \leq C|Q|^{-1/2+m/n}\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\delta/2}\int_{\mathbb{R}^n}|Eb(x)-(Eb)_{4P}|(1+\ell(Q)^{-1}|x-x_Q|)^{-n-\delta/2}dx.$$

We invoke following estimate, valid for functions in the space $\mathrm{BMO}(\mathbb{R}^n)$,

$$|(Eb)_{4P}-(Eb)_Q| = C_n||Eb||_{\mathrm{BMO}(\mathbb{R}^n)}\Big(\log\frac{\ell(P)}{\ell(Q)}+\log 8\Big). \tag{3.80}$$

Combining the estimates (3.80), (3.16), and (3.79), we get

$$|T((Eb-(Eb)_{4P})\chi_{\Omega\setminus 4P})(Q,\varepsilon)| \leq C||Eb||_{\mathrm{BMO}(\mathbb{R}^n)}|Q|^{1/2+m/n}\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\delta/2}\Big(1+\log\frac{\ell(P)}{\ell(Q)}\Big).$$

This corresponds to the estimate (3.66) altough this is slightly worse due to logarithmic factor. However, this factor is easily compensated and we can continue to estimate as in Lemma 3.64 to reach the required estimate (3.78). Second we need the estimate

(3.81)
$$\Sigma_{P,2} = \frac{1}{|P|}\sum_{Q\subset P}\sum_{\varepsilon\in\mathcal{E}}|Q|^{-2m/n}|T((Eb-(Eb)_{4P})\chi_{\Omega\cap 4P})(Q,\varepsilon)|^2 \leq C||Eb||^2_{\mathrm{BMO}(\mathbb{R}^n)},$$

with C independent of P and Eb. First of all, $(Eb-(Eb)_{4P})\chi_{\Omega\cap 4P}\in L^2(\Omega)$ and therefore we can proceed beginning from the estimate (3.73) and until we reach (3.75), which is replaced by the following estimate in the present context

$$\begin{aligned}\Sigma_{P,2} &\leq \frac{C}{|P|}||(Eb-(Eb)_{4P})\chi_{\Omega\cap 4P}||^2_{L^2(\Omega)} \leq \frac{C}{|P|}||(Eb-(Eb)_{4P})\chi_{4P}||^2_{L^2(\mathbb{R}^n)}\\ &= \frac{C}{|P|}\int_{4P}|Eb(x)-(Eb)_{4P}|^2dx \leq C||Eb||^2_{\mathrm{BMO}(\mathbb{R}^n)}.\end{aligned}$$

This is the required estimate (3.81) since the constant C is independent of P and Eb. Finally, combining the estimates (3.78) and (3.81), we reach the estimate (3.77). □

Example 3.82. Here are examples when Theorem 3.76 is applicable.

- The assumptions of Theorem 3.76 hold if $\Omega=\mathbb{R}^n$ and T is of convolution type. This follows from Theorem 3.40.
- Later in Definition 3.91 we define operators $\Pi_\tau\in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ satisfying $\Pi_\tau 1 = 0\in \dot{f}^{m,2}_\infty(\mathbb{R}^n)$ but $\Pi^*_\tau 1 = 0 \in \dot{f}^{m,2}_\infty(\mathbb{R}^n)$ only if $\Pi_\tau\equiv 0$. Hence Theorem 3.76 applies to those operators Π_τ that meet the $L^2(\mathbb{R}^n)$-bundedness criterion.

3.4. $T\chi_\Omega$ **theorem for restricted operators.** Next we treat the $T\chi_\Omega$ theorem for restricted operators. Its difficult direction remains to be proven and, to describe it, we fix $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ that is associated with a globally defined kernel $K \in \mathrm{K}^{-m}_{\mathbb{R}^n}(\delta)$. The difficult direction of the $T\chi_\Omega$ theorem states that

$$T\chi_\Omega, T^*\chi_\Omega \in \dot{f}^{m,2}_\infty(\Omega) \Rightarrow \{\partial^\alpha T, \partial^\alpha T^* \,:\, |\alpha| = m\} \subset \mathscr{L}(L^2(\Omega)), \tag{3.83}$$

where $\Omega \subset \mathbb{R}^n$ is a Whitney coplump domain. The assumption on the left-hand side of (3.83) is rather weak because the space $\dot{f}^{m,2}_\infty(\Omega)$ is defined in terms of the interior cubes and the boundary cubes are omitted.

The main tools here are certain reflected paraproduct operators that are obtained from the usual paraproducts by establishing a geometric modification. This modification is based on the reflection of dyadic cubes that exists in a Whitney coplump domain.

Let us sketch the proof of (3.83). If $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$, there exists reflected paraproduct operators $\pi_T, \pi_{T^*} \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ such that the reduced operator

$$M = T - \pi_{T^*} - \pi_T^* \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta) \tag{3.84}$$

satisfies the all the assumptions of the reduced $T1$ theorem, Theorem 3.37. In particular, it satisfies $M1 = 0 = M^*1$. Applying the reduced $T1$ theorem, we see that

$$\{\partial^\alpha M, \partial^\alpha M^* \,:\, |\alpha| = m\} \subset \mathscr{L}(L^2(\mathbb{R}^n)). \tag{3.85}$$

The proof of (3.83) in the case $\Omega = \mathbb{R}^n$ is an adaptation of the proof of the $T1$ theorem of David and Journé. First we show the implication

$$T1, T^*1 \in \dot{f}^{m,2}_\infty(\mathbb{R}^n) \Rightarrow \{\partial^\alpha \pi_{T^*}, \partial^\alpha \pi_T^* \,:\, |\alpha| = m\} \subset \mathscr{L}(L^2(\mathbb{R}^n)). \tag{3.86}$$

This is a direct computation, using the special properties of the paraproduct operators and the reflection of dyadic cubes. Combining (3.84), (3.85), and (3.86), we obtain the conclusion in (3.83) for this special case $\Omega = \mathbb{R}^n$. To proceed in the case of general Whitney coplump domains $\Omega \subset \mathbb{R}^n$ we decompose the reflected paraproduct operators π_{T^*} and π_T to interior (I) and residual (R) parts $\pi_{T^*} = \mathrm{I}_{T^*} + \mathrm{R}_{T^*}$ and $\pi_T = \mathrm{I}_T + \mathrm{R}_T$. This decomposition and the definition (3.84) yield the following important identity

$$M + \mathrm{I}_{T^*} + \mathrm{I}_T^* = T - \mathrm{R}_{T^*} - \mathrm{R}_T^*. \tag{3.87}$$

The interior and residual parts are defined so that, first of all,

$$\langle \mathrm{R}_{T^*} f \mid g\rangle = 0 = \langle \mathrm{R}_T^* f \mid g\rangle, \qquad \text{if } f, g \in C_0(\Omega), \tag{3.88}$$

but also

$$T\chi_\Omega, T^*\chi_\Omega \in \dot{f}^{m,2}_\infty(\Omega) \Rightarrow \{\partial^\alpha \mathrm{I}_{T^*}, \partial^\alpha \mathrm{I}_T^* \,:\, |\alpha| = m\} \subset \mathscr{L}(L^2(\mathbb{R}^n)). \tag{3.89}$$

Both (3.88) and (3.89) are crucial and follow from the properties of reflection $Q \mapsto Q^s$ and the globally defined kernel. Fix $\alpha \in \mathbb{N}_0^n$ so that $|\alpha| = m$, $f \in C_0(\Omega)$, and

$g \in C_0^m(\Omega)$. Then, using (3.87) and (3.88), we get

$$|\langle \partial^\alpha Tf \mid g\rangle| = |\langle \partial^\alpha (T - \mathrm{R}_{T^*} - \mathrm{R}_T^*)f \mid g\rangle = |\langle \partial^\alpha (M - \mathrm{I}_{T^*} - \mathrm{I}_T^*)f \mid g\rangle|.$$

Finally, applying (3.89) with this identity, we get the required norm-estimate

$$|\langle \partial^\alpha Tf \mid g\rangle| \le C_T ||f||_{L^2(\mathbb{R}^n)} ||g||_{L^2(\mathbb{R}^n)} = C_T ||f||_{L^2(\Omega)} ||g||_{L^2(\Omega)}.$$

As a consequence, we have $\partial^\alpha T \in \mathscr{L}(L^2(\Omega))$. Due to symmetry in the assumptions, we also have $\partial^\alpha T^* \in \mathscr{L}(L^2(\Omega))$. These are as required in (3.83).

We stick to the definitions and notation introduced in the beginning of Section 3.3 with the exception that the domain $\Omega \subset \mathbb{R}^n$ is assumed to be Whitney coplump.

Basic properties of reflected paraproducts. Here we cover the definition and basic properties of reflected paraproducts. We adapt the treatment in [MC97, pp. 57–60] to meet our needs and the geometric modification involved in this adaptation is apparently new. Fix a real-valued function $\Phi \in C_0^\infty(\mathbb{R}^n)$ so that

$$0 \le \Phi \le 1, \quad \operatorname{supp} \Phi \subset [0,1)^n, \quad \int_{\mathbb{R}^n} \Phi(x)dx = 1.$$

Recall that we denote

$$\Phi_Q(x) = |Q|^{-1/2}\Phi(\ell(Q)^{-1}(x - x_Q)) \tag{3.90}$$

if $Q \in \mathcal{D}$ and $x \in \mathbb{R}^n$. Thus $\operatorname{supp} \Phi_Q \subset Q$ and $|Q|^{-1/2} \int_{\mathbb{R}^n} \Phi_Q(x)dx = 1$ if $Q \in \mathcal{D}$. A reflected paraproduct depends on $m \in \{1, 2, \dots, n-1\}$, on the function Φ, on the $(m+1)$-regular wavelets $\{\psi_Q^\varepsilon\}$, and on the reflection

$$Q \mapsto Q^s : \mathcal{D} \to \mathcal{D}$$

associated with the given Whitney coplump domain Ω which is (C_{m+1}, β_m)-coplump; see Remark 3.55. There is also a parameter $\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ involved that is more variable than the fixed quantities before.

Definition 3.91. Let $n \ge 2$, $0 < m < n$, and $\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ satisfy

$$|\tau(Q,\varepsilon)| \le \lambda |Q|^{1/2+m/n}, \quad \text{ if } (Q,\varepsilon) \in \mathcal{D} \times \mathcal{E},$$

where $\lambda > 0$. Let $f \in C_0(\mathbb{R}^n)$. Then the *reflected paraproduct (of τ and f)* is the function $\Pi_\tau f : \mathbb{R}^n \to \mathbb{C}$ that is defined pointwise for $x \in \mathbb{R}^n$ by

$$\Pi_\tau f(x) = \Pi_{\Phi,m,s,\Omega,\tau} f(x) = \sum_{Q\in\mathcal{D}} \sum_{\varepsilon\in\mathcal{E}} \tau(Q,\varepsilon)|Q|^{-1/2}\Phi_{Q^s}(x)\langle f \mid \psi_Q^\varepsilon\rangle. \tag{3.92}$$

Let us then quantify the operator theoretic setting and properties of reflected paraproducts. We show that the reflected paraproduct is well defined and it coincides with a weakly singular integral operator that is associated with a standard kernel of order $-m$. Accordingly we interprete reflected paraproducts as WSIO's. One of the important properties of a reflected paraproduct as a WSIO is that $\Pi_\tau^* 1 = \bar{\tau}$ in $\dot{f}_\infty^{m,2}(\mathbb{R}^n)$ and the dependence of Π_τ on the $(m+1)$-regular wavelets is a reflection of this property.

Theorem 3.93. *Let $\delta \in (0,1)$, $n \geq 2$, and $0 < m < n$. Let $\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ be such that $|\tau(P,\rho)| \leq \lambda |P|^{1/2+m/n}$ if $(P,\rho) \in \mathcal{D} \times \mathcal{E}$. Then the series (3.92) converges absolutely and $\Pi_\tau \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ is associated with a kernel*

$$\kappa = \kappa_\tau \in C^{m+1}(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\})$$

that is defined by an absolutely convergent series

$$\kappa(x,y) = \sum_{Q\in\mathcal{D}} \sum_{\varepsilon\in\mathcal{E}} \tau(Q,\varepsilon)|Q|^{-1/2} \Phi_{Q^s}(x) \overline{\psi_Q^\varepsilon(y)} \tag{3.94}$$

and satisfies the estimates

$$|\partial_x^\alpha \partial_y^\beta \kappa(x,y)| \leq C|x-y|^{m-n-|\alpha|-|\beta|}$$

for every $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq m+1$. Here the constant C depends at most on the parameters $n, m, \beta_m, \lambda, \Phi, C_{m+1}$. We also have $\Pi_\tau^* 1 = \bar{\tau}$ and $\Pi_\tau 1 = 0$ in $\dot{f}^{m,2}_\infty(\mathbb{R}^n)$. To formulate this otherwise, we have

$$\int_{\mathbb{R}^n} \Pi_\tau \psi_P^\rho(x)dx = \tau(P,\rho), \quad \int_{\mathbb{R}^n} \Pi_\tau^* \psi_P^\rho(x)dx = 0,$$

if $(P,\rho) \in \mathcal{D} \times \mathcal{E}$ and ψ_Q^ε's are $(m+1)$-regular wavelets.

Proof. Denote $r = m+1$ and by C we denote any constant that may depend at most on $n, m, \beta_m, \lambda, \Phi, C_{m+1}$. If $\nu \in \mathbb{Z}$ we denote

$$\kappa_\nu(x,y) = \sum_{Q\in\mathcal{D}_\nu} \sum_{\varepsilon\in\mathcal{E}} \tau(Q,\varepsilon)|Q|^{-1/2} \Phi_{Q^s}(x) \overline{\psi_Q^\varepsilon(y)}.$$

Fix $\nu \in \mathbb{Z}$ and $(Q,\varepsilon) \in \mathcal{D}_\nu \times \mathcal{E}$. Using (3.90) and Remark 3.55, we see that

$$\operatorname{supp} \Phi_{Q^s} \subset Q^s \subset B(x_Q, (2+\beta_m)\operatorname{diam}(Q)).$$

Applying this relation and the property B4) of $(m+1)$-regular wavelets yields

$$\operatorname{supp} \Phi_{Q^s} \cup \operatorname{supp} \psi_Q^\varepsilon \subset B(x_Q, \gamma \operatorname{diam}(Q)), \quad \gamma = 2 + \beta_m + C_{m+1}. \tag{3.95}$$

Fix $x \in \mathbb{R}^n$. Assuming that $x \in \operatorname{supp} \Phi_{Q^s}$, we have $Q \subset B(x, (1+\gamma)\operatorname{diam}(Q))$. Since the cubes in the family $\mathcal{D}_\nu$ are disjoint and $\operatorname{diam}(P) = \sqrt{n}\ell(P) = \sqrt{n}2^{-\nu}$ if $P \in \mathcal{D}_\nu$, there are at most

$$\frac{|B(x, (1+\gamma)\sqrt{n}2^{-\nu})|}{2^{-\nu n}} = C_n(1+\gamma)^n = \varrho$$

cubes in the family

$$\mathcal{D}_\nu(x) = \{P \in \mathcal{D}_\nu \,:\, x \in \operatorname{supp} \Phi_{P^s}\}.$$

This upper bound is independent of $\nu \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Similar reasoning shows that the number of cubes in the family $\mathcal{D}_\nu(K) = \{P \in \mathcal{D}_\nu \,:\, K \cap \operatorname{supp} \Phi_{P^s} \neq \emptyset\}$, where $K \subset \mathbb{R}^n$ is a compact set, is finite depending essentially on $\operatorname{diam}(K)$ and ν

but also on other ambient parameters. This implies that $\kappa_\nu \in C^{m+1}(\mathbb{R}^n \times \mathbb{R}^n)$ and, if $x, y \in \mathbb{R}^n$ and $|\alpha| + |\beta| \le m+1$, we have the estimate

$$\begin{aligned} |\partial_x^\alpha \partial_y^\beta \kappa_\nu(x,y)| &\le \sum_{Q \in \mathcal{D}_\nu(x)} \sum_{\varepsilon \in \mathcal{E}} |\tau(Q,\varepsilon)|Q|^{-1/2} \partial_x^\alpha \Phi_{Q^s}(x) \overline{\partial_y^\beta \psi_Q^\varepsilon(y)}| \\ &\le \lambda 2^n ||\partial^\alpha \Phi||_{L^\infty(\mathbb{R}^n)} C_{m+1} \sum_{Q \in \mathcal{D}_\nu(x)} |Q|^{m/n-1-|\alpha|/n-|\beta|/n} \\ &\le \lambda 2^n ||\partial^\alpha \Phi||_{L^\infty(\mathbb{R}^n)} C_{m+1} \varrho 2^{\nu(n+|\alpha|+|\beta|-m)} = C 2^{\nu(n+|\alpha|+|\beta|-m)}. \end{aligned} \tag{3.96}$$

Furthermore, if $\partial_x^\alpha \partial_x^\beta \kappa_\nu(x,y) \neq 0$, then there exists $(Q,\varepsilon) \in \mathcal{D}_\nu \times \mathcal{E}$ such that $x \in \operatorname{supp} \Phi_{Q^s}$ and $y \in \operatorname{supp} \psi_Q^\varepsilon$. Using (3.95) we see that

$$\frac{|x-y|}{\gamma\sqrt{n}} < \ell(Q) = 2^{-\nu}. \tag{3.97}$$

Let $d > 0$ and denote $\Omega_d = \{(x,y) : |x-y| > d\} \subset \mathbb{R}^n \times \mathbb{R}^n$. Fix $\nu_d \in \mathbb{Z}$ so that that

$$2^{-\nu_d} \le d/(\gamma\sqrt{n}) < 2^{-\nu_d+1}.$$

Assuming $|\alpha| + |\beta| \le m+1$, $(x,y) \in \Omega_d$ and using the (3.96) and (3.97), we get

$$\sum_{\nu=-\infty}^{\infty} |\partial_x^\alpha \partial_y^\beta \kappa_\nu(x,y)| \le C \sum_{\nu=-\infty}^{\nu_d} 2^{\nu(n+|\alpha|+|\beta|-m)} \le C d^{m-n-|\alpha|-|\beta|}.$$

The Weierstrass M–test shows that the left-hand side converges uniformly in Ω_d and, since $\cup_{d>0} \Omega_d = \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\}$, we have $\kappa = \sum_{\nu \in \mathbb{Z}} \kappa_\nu \in C^{m+1}(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\})$ and this series can be differentiated termwise. Assume that $|\alpha| + |\beta| \le m+1$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\}$, and $\nu_{x,y} \in \mathbb{Z}$ satisfies $2^{-\nu_{x,y}} \le |x-y|/(\gamma\sqrt{n}) < 2^{-\nu_{x,y}+1}$. Then (3.96) and (3.97) yields

$$\begin{aligned} \sum_{\nu \in \mathbb{Z}} |\partial_x^\alpha \partial_y^\beta \kappa_\nu(x,y)| &\le \sum_{\nu=-\infty}^{\nu_{x,y}} \sum_{Q \in \mathcal{D}_\nu(x)} \sum_{\varepsilon \in \mathcal{E}} |\tau(Q,\varepsilon)|Q|^{-1/2} \partial_x^\alpha \Phi_{Q^s}(x) \overline{\partial_y^\beta \psi_Q^\varepsilon(y)}| \\ &\le C|x-y|^{m-n-|\alpha|-|\beta|}. \end{aligned} \tag{3.98}$$

A simple modification of later Proposition 4.6 shows that $\kappa \in \mathrm{K}_{\mathbb{R}^n}^{-m}(\delta)$ if $0 < \delta < 1$. Next we prove that the paraproduct is associated with kernel κ. Indeed, if $f \in C_0(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, then estimate (3.98) and local integrability of $y \mapsto |x-y|^{m-n}$ yields

$$\int_{\mathbb{R}^n} \sum_{\nu \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_\nu(x)} \sum_{\varepsilon \in \mathcal{E}} |\tau(Q,\varepsilon)|Q|^{-1/2} \Phi_{Q^s}(x) \overline{\psi_Q^\varepsilon(y)}||f(y)|dy < \infty. \tag{3.99}$$

The dominated convergence theorem implies $\Pi_\tau f(x) = \int_{\mathbb{R}^n} \kappa(x,y) f(y) dy$. Furthermore, the series (3.92) converges absolutely.

Fix $(P,\rho) \in \mathcal{D} \times \mathcal{E}$. Applying the property B1) of wavelets to the definition of Π_τ we see that $\int_{\mathbb{R}^n} \Pi_\tau \psi_P^\rho = \tau(P,\rho)$.

Next we prove that $\int_{\mathbb{R}^n} \Pi_\tau^* \psi_P^\rho = 0$. First notice that the family $\mathcal{D}_\nu(P) = \mathcal{D}_\nu(\operatorname{supp} \psi_P^\rho)$ is finite for every $\nu \in \mathbb{Z}$. Therefore we can apply the Fubini's theorem and the identity $\int \psi_Q^\varepsilon = 0$ for

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{\kappa_\nu(y,x)} \psi_P^\rho(y) dy dx = 0. \tag{3.100}$$

We claim that it suffices to verify that the function $F : \mathbb{R}^n \to [0, \infty)$,

$$F(x) = \sum_{\nu \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \overline{\kappa_\nu(y,x)} \psi_P^\rho(y) dy \right|, \tag{3.101}$$

is integrable. Indeed, assuming that $F \in L^1(\mathbb{R}^n)$, we apply the dominated convergence theorem twice – first justified by (3.98) and then by $F \in L^1(\mathbb{R}^n)$ – thereby reaching the identity

$$\int_{\mathbb{R}^n} \Pi_\tau^* \psi_P^\rho(x) dx = \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{\kappa_\nu(y,x)} \psi_P^\rho(y) dy dx = 0.$$

Notice that we also used (3.100). It remains to prove that $F \in L^1(\mathbb{R}^n)$. Using (3.98) and (3.6) we see that $F \in L^\infty(\mathbb{R}^n)$. Assume then that $|x - x_P| \geq 2\gamma \operatorname{diam}(P)$. Lemma B.2 shows that there exists $\alpha = \alpha(\rho, m+1)$ such that $|\alpha| = m+1$ and a function $\psi^{\rho,m+1} : \mathbb{R}^n \to \mathbb{C}$ such that $\psi_P^{\rho,m+1}$ satisfies B4) and B5) in Appendix B and

$$\psi_P^\rho = |P|^{(m+1)/n} \partial^\alpha \psi_P^{\rho,m+1}.$$

Using these properties and integrating by parts, we have

$$\begin{aligned} F(x) &= |P|^{(m+1)/n} \sum_{\nu \in \mathbb{Z}} \left| (-1)^{m+1} \int_{\mathbb{R}^n} \overline{\partial_y^\alpha \kappa_\nu(y,x)} \psi_P^{\rho,m+1}(y) dy \right| \\ &\leq C_{m+1} |P|^{(m+1)/n - 1/2} \int_{B(x_P, \gamma \operatorname{diam}(P))} \sum_{\nu \in \mathbb{Z}} |\partial_y^\alpha \kappa(y,x)| dy \end{aligned}$$

Using (3.98) with $\beta = 0$ and the estimate $|y - x| \geq |x - x_P|/2$ for $y \in B(x_P, \gamma \operatorname{diam}(P))$ yields $F(x) \leq C|P|^{1/2 + (m+1)/n} |x - x_P|^{-n-1}$. Combining this with the boundedness of F we see that $F \in L^1(\mathbb{R}^n)$ as required. □

Boundedness of reflected paraproducts. From Definition 3.47 it follows that, if $\tau \in \dot{f}_\infty^{m,2}(\mathbb{R}^n)$, then

$$|\tau(Q, \varepsilon)| \leq ||\tau||_{\dot{f}_\infty^{m,2}(\mathbb{R}^n)} |Q|^{1/2 + m/n}, \qquad \text{if } (Q, \varepsilon) \in \mathcal{D} \times \mathcal{E}. \tag{3.102}$$

Hence Definition 3.91 applies and the reflected paraproduct operator $\Pi_\tau = \Pi_{\Phi,m,s,\Omega,\tau}$ exists and it satisfies all the properties described in Theorem 3.93. Here we verify the following additional boundedness property

$$\tau \in \dot{f}_\infty^{m,2}(\mathbb{R}^n) \Rightarrow \{\partial^\alpha \Pi_\tau, \partial^\alpha \Pi_\tau^* : |\alpha| = m\} \subset \mathscr{L}(L^2(\mathbb{R}^n)).$$

The main tool here is the following variant of the Carleson's lemma, which is an adaptation from [MC97, p. 59].

Lemma 3.103. *Let* $\tau \in \dot{f}^{m,2}_\infty(\mathbb{R}^n)$ *and* $\omega : \mathcal{D}\times\mathcal{E} \to [0,\infty)$ *be a positive sequence. Define* $\omega(x) = \sup\{\omega(Q,\varepsilon) : x \in Q \in \mathcal{D}$ *and* $\varepsilon \in \mathcal{E}\}$ *if* $x \in \mathbb{R}^n$*. Then*

$$\sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n}|\tau(Q,\varepsilon)|^2\omega(Q,\varepsilon) \le ||\tau||^2_{\dot{f}^{m,2}_\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} \omega(x)dx.$$

Proof. Denote $p(Q,\varepsilon) = |Q|^{-2m/n}|\tau(Q,\varepsilon)|^2$ if $(Q,\varepsilon) \in \mathcal{D}\times\mathcal{E}$. Define $\chi : \mathcal{D}\times\mathcal{E}\times[0,\infty) \to \mathbb{R}$ by $\chi(Q,\varepsilon,t) = 1$, if $0 \le t < \omega(Q,\varepsilon)$, and $\chi(Q,\varepsilon,t) = 0$ otherwise. Then

$$\sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} p(Q,\varepsilon)\omega(Q,\varepsilon) = \int_0^\infty \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} p(Q,\varepsilon)\chi(Q,\varepsilon,t)dt. \tag{3.104}$$

Fix $t > 0$. Denote $D_t = \{x \in \mathbb{R}^n : \omega(x) > t\}$. Then using the Chebyshev's inequality, we have

$$|D_t| \le t^{-1}\int_{\mathbb{R}^n} \omega(x)dx.$$

If $\int_{\mathbb{R}^n}\omega(x)dx = \infty$ then we are done. Hence we can assume that $|D_t| < \infty$. Assume that $(Q,\varepsilon) \subset \mathcal{D}\times\mathcal{E}$ satisfies $\chi(Q,\varepsilon,t) \ne 0$. Then $\omega(Q,\varepsilon) > t$ and therefore $Q \subset D_t$. Applying the estimate $|D_t| < \infty$ we see that there exists a unique maximal cube $Q^m(Q) \in \mathcal{D}$ satisfying $Q \subset Q^m(Q) \subset D_t$. The family $\mathcal{M}_t$ of these maximal cubes is disjoint and $Q^m \subset D_t$ if $Q^m \in \mathcal{M}_t$. Taking the discussion above into consideration and using Definition 3.47, we have

$$\begin{aligned}\sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} p(Q,\varepsilon)\chi(Q,\varepsilon,t) &\le \sum_{Q^m\in\mathcal{M}_t}\sum_{Q\in\mathcal{D}\,:\,Q\subset Q^m}\sum_{\varepsilon\in\mathcal{E}} p(Q,\varepsilon)\\ &\le ||\tau||^2_{\dot{f}^{m,2}_\infty(\mathbb{R}^n)}\sum_{Q^m\in\mathcal{M}_t}|Q^m| \le ||\tau||^2_{\dot{f}^{m,2}_\infty(\mathbb{R}^n)}|D_t|.\end{aligned}$$

Combining this estimate with the identities $\int_0^\infty |D_t|dt = \int_{\mathbb{R}^n}\omega(x)dx$ and (3.104) we reach the required estimate. □

We prove a boundedness result for the reflected paraproduct operators. Analogous treatment in the limiting case $m = 0$ is in [MC97, p. 58–59].

Theorem 3.105. *Let* $n \ge 2$ *and* $0 < m < n$*. Assume that* $\tau \in \dot{f}^{m,2}_\infty(\mathbb{R}^n)$*. Then*

$$\partial^\alpha\Pi_\tau, \partial^\alpha\Pi^*_\tau \in \mathscr{L}(L^2(\mathbb{R}^n))$$

for every $\alpha \in \mathbb{N}_0^n$ *satisfying* $|\alpha| = m$*.*

Proof. First of all, the reflected paraproduct Π_τ is well defined because of the estimate (3.102). Theorem 3.93 implies that $\Pi_\tau \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ and therefore the basic estimates (3.6) and (3.7), along with other properties related to weakly singular integral operators, are at our disposal. Fix $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| = m$. First we prove that

$$\partial^\alpha\Pi_\tau \in \mathscr{L}(L^2(\mathbb{R}^n)).$$

For this purpose we fix $f \in C_0(\mathbb{R}^n)$ and $g \in C_0^m(\mathbb{R}^n)$. Using the definition on weak derivative and properties of weakly singular integral operators, we have

$$\langle \partial^\alpha \Pi_\tau f \mid g\rangle = (-1)^m \langle f \mid \Pi_\tau^* \partial^\alpha g\rangle. \tag{3.106}$$

Applying (3.7) we see that $\Pi_\tau^*\partial^\alpha g \in L^q(\mathbb{R}^n)$ for some $1 < q < \infty$. Since the wavelet approximation converges unconditionally in $L^{q'}(\mathbb{R}^n)$, see B2), we have

$$\begin{aligned} |\langle f \mid \Pi_\tau^*\partial^\alpha g\rangle| &= \left| \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} \langle f \mid \psi_Q^\varepsilon\rangle\langle \psi_Q^\varepsilon \mid \Pi_\tau^*\partial^\alpha g\rangle\right| \\ &\le \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} |\langle f \mid \psi_Q^\varepsilon\rangle||\langle \Pi_\tau\psi_Q^\varepsilon \mid \partial^\alpha g\rangle| \le ||f||_{L^2(\mathbb{R}^n)} G. \end{aligned} \tag{3.107}$$

In the last inequality we used the Cauchy–Schwarz inequality and the term G is quantified below. Indeed, using Definition (3.91) and Lemma 3.103, we have

$$\begin{aligned} G^2 &= \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} |\tau(Q,\varepsilon)|^2|Q|^{-1}|\langle \Phi_{Q^s} \mid \partial^\alpha g\rangle|^2 \\ &= \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n}|\tau(Q,\varepsilon)|^2|Q|^{-1}|\langle (\partial^\alpha\Phi)_{Q^s} \mid g\rangle|^2 \le C_\tau \int_{\mathbb{R}^n} \omega(x)dx. \end{aligned}$$

Here $\omega(x) = \sup\{\omega(Q,\varepsilon) \,:\, x \in Q \in \mathcal{D} \text{ and } \varepsilon \in \mathcal{E}\}$ if $x \in \mathbb{R}^n$ and

$$\omega(Q,\varepsilon) = |Q|^{-1}|\langle(\partial^\alpha\Phi)_{Q^s} \mid g\rangle|^2, \quad \text{if } (Q,\varepsilon) \in \mathcal{D}\times\mathcal{E}.$$

Fix $x \in \mathbb{R}^n$ and $(Q,\varepsilon) \in \mathcal{D}\times\mathcal{E}$ such that $x \in Q$. According to Remark 3.55 we have

$$x \in \operatorname{supp}(\partial^\alpha\Phi)_{Q^s} \cup Q \subset Q^s \cup Q \subset B(x_Q, (2+\beta_m)\operatorname{diam}(Q)).$$

Using this we have the estimate

$$\sqrt{\omega(Q,\varepsilon)} \le ||\partial^\alpha\Phi||_\infty |Q|^{-1} \int_{B(x_Q,(2+\beta_m)\operatorname{diam}(Q))} |g(y)|dy \le C_{\Phi,n,m,\Omega} Mg(x),$$

where Mg is the (non-centered) Hardy–Littlewood maximal function of g. Hence $\omega(x) \le C^2_{\Phi,n,m,\Omega} Mg(x)^2$ and, combining the estimates above, we get

$$G^2 \le C_{\tau,\Phi,n,m,\Omega}||Mg||_2^2 \le C_{M,\tau,\Phi,n,m,\Omega}||g||_2^2$$

This combined with the identity (3.106) and the estimate (3.107) shows that $\partial^\alpha\Pi_\tau \in \mathscr{L}(L^2(\mathbb{R}^n))$.

Then we study the operator $\partial^\alpha\Pi_\tau^*$. If f, g are as above then reasoning as in connection with (3.107), we have

(3.108)

$$\begin{aligned} (-1)^m\langle \partial^\alpha\Pi_\tau^* f \mid g\rangle &= \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} \langle \Pi_\tau^* f \mid \psi_Q^\varepsilon\rangle\langle\psi_Q^\varepsilon \mid \partial^\alpha g\rangle \\ &= \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} |Q|^{-m/n}\overline{\tau(Q,\varepsilon)}|Q|^{-1/2}\langle f \mid \Phi_{Q^s}\rangle |Q|^{m/n}\langle \psi_Q^\varepsilon \mid \partial^\alpha g\rangle. \end{aligned}$$

Applying the Cauchy–Schwarz inequality and estimating as in the first part of this proof, we get the estimate

$$|\langle \partial^\alpha \Pi_\tau^* f \mid g\rangle| \le C_{M,\tau,n,\Omega}||f||_2 \bigg(\sum_{Q\in\mathcal{D}} \sum_{\varepsilon\in\mathcal{E}} |Q|^{2m/n} |\langle \psi_Q^\varepsilon \mid \partial^\alpha g\rangle|^2 \bigg)^{1/2}. \tag{3.109}$$

Because $g, (\partial^\alpha \psi^\varepsilon)_Q \in L^2(\mathbb{R}^n)$ we have

$$\begin{aligned} |Q|^{m/n} \langle \psi_Q^\varepsilon \mid \partial^\alpha g\rangle &= (-1)^m |Q|^{m/n} \langle \partial^\alpha \psi_Q^\varepsilon \mid g\rangle \\ &= (-1)^m \sum_{P\in\mathcal{D}} \sum_{\rho\in\mathcal{E}} |Q|^{m/n} \langle \partial^\alpha \psi_Q^\varepsilon \mid \psi_P^\rho \rangle \langle \psi_P^\rho \mid g\rangle. \end{aligned}$$

Example 3.31 and Lemma 3.36 show that

$$M = M_{\partial^\alpha/\partial^m} = \{ |Q|^{m/n} \langle \partial^\alpha \psi_Q^\varepsilon \mid \psi_P^\rho \rangle \}$$

is bounded on $\ell^2(\mathcal{D}\times\mathcal{E})$. Combining this fact with the inequality (3.109), we get the desired estimate

$$|\langle \partial^\alpha \Pi_\tau^* f, g\rangle| \le C_{M,\tau,n,m,\Omega} ||f||_2 ||g||_2.$$

Hence we have $\partial^\alpha \Pi_\tau^* \in \mathscr{L}(L^2(\mathbb{R}^n))$ as required. □

Interior and residual paraproducts. Here we split the reflected paraproduct into interior and residual parts and collect their important properties. We keep the setting described above, that is, reflected paraproducts depend on the sequence τ, on the function Φ, on the $(m+1)$-regular wavelets, and on the reflection $Q \mapsto Q^s$ associated with the Whitney coplump domain Ω. Also the family $\mathcal{D}_I^m(\Omega)$ of dyadic cubes, defined in 3.45, plays a significant role.

Definition 3.110. Let $T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$. We pose the following definitions

- If $\varepsilon \in \mathcal{E}$ we define $\tau(Q,\varepsilon) = \int_{\mathbb{R}^n} T^*\psi_Q^\varepsilon(x)dx$, if $Q \in \mathcal{D}_I^m(\Omega)$, and $\tau(Q,\varepsilon) = 0$, if $Q \in \mathcal{D}\setminus\mathcal{D}_I^m(\Omega)$. The *interior paraproduct associated with* T is defined by $\mathrm{I}_T = \Pi_\tau$.
- If $\varepsilon \in \mathcal{E}$ we define $\sigma(Q,\varepsilon) = \int_{\mathbb{R}^n} T^*\psi_Q^\varepsilon(x)dx$, if $Q \in \mathcal{D}\setminus\mathcal{D}_I^m(\Omega)$, and $\sigma(Q,\varepsilon) = 0$, if $Q \in \mathcal{D}_I^m(\Omega)$. The *residual paraproduct associated with* T is defined by $\mathrm{R}_T = \Pi_\sigma$.
- The *full paraproduct associated with* T is defined by $\pi_T = \mathrm{I}_T + \mathrm{R}_T$.

Remark 3.111. Let $T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$. Combining Corollary 3.11 and Theorem 3.93 we see that $\mathrm{I}_T, \mathrm{R}_T \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$. The residual paraproduct R_T is associated with a kernel satisfying

$$\kappa_\sigma | \Omega\times\Omega \setminus \{(x,x)\} \equiv 0.$$

To see this apply the presentation (3.94) and use Remark 3.55 for $\operatorname{supp}\Phi_{Q^s} \subset Q^s \subset \mathbb{R}^n\setminus\Omega$ if $Q \in \mathcal{D}\setminus\mathcal{D}_I^m(\Omega)$. In particular, we have the important equivalence

$$T \sim T - \mathrm{R}_M - \mathrm{R}_N^*$$

if $M, N \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta)$. This allows us to normalize operators by using residual paraproducts.

The following boundedness result provides a step towards the $T\chi_\Omega$ theorem. The earlier tail lemma for globally defined kernels turns out to be useful here.

Lemma 3.112. *Assume that* $\emptyset \neq \Omega \subset \mathbb{R}^n$ *is a Whitney coplump domain and* $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ *satisfies* $T\chi_\Omega \in \dot{f}^{m,2}_\infty(\Omega)$. *Then* $\partial^\alpha \mathrm{I}_T, \partial^\alpha \mathrm{I}_T^* \in \mathscr{L}(L^2(\mathbb{R}^n))$ *if* $|\alpha| = m$.

Proof. In this proof C is a generic constant whose value may depend at most on the parameters n, m, C_{m+1}, T. Applying Corollary 3.11 and Theorem 3.105 we see that it suffices to prove that $\tau \in \dot{f}^{m,2}_\infty(\mathbb{R}^n)$ where τ is as in Definition 3.110, so that $\mathrm{I}_T = \Pi_\tau$. By Lemma 3.49 we are reduced to showing that

$$||\tau||^2_{\dot{f}^{m,2}_\infty(\Omega)} = \sup_{P \in \mathcal{D}^m_I(\Omega)} \frac{1}{|P|} \sum_{Q \subset P} \sum_{\varepsilon \in \mathcal{E}} |Q|^{-2m/n} |\tau(Q,\varepsilon)|^2 < \infty. \tag{3.113}$$

For this purpose, if $Q \in \mathcal{D}^m_I(\Omega)$, we denote $\beta(Q) = \sum_{\varepsilon\in\mathcal{E}} |\tau(Q,\varepsilon)|^2$ and

$$\tau_\Omega(Q,\varepsilon) = \overline{T\chi_\Omega(Q,\varepsilon)} = \int_\Omega T^*\psi^\varepsilon_Q, \quad \tau_{\mathbb{R}^n\setminus\Omega}(Q,\varepsilon) = \tau(Q,\varepsilon) - \tau_\Omega(Q,\varepsilon) = \overline{T(\chi_{\mathbb{R}^n\setminus\Omega})(Q,\varepsilon)},$$

$$\beta_\Omega(Q) = \sum_{\varepsilon\in\mathcal{E}} |\tau_\Omega(Q,\varepsilon)|^2, \quad \beta_{\mathbb{R}^n\setminus\Omega}(Q) = \sum_{\varepsilon\in\mathcal{E}} |\tau_{\mathbb{R}^n\setminus\Omega}(Q,\varepsilon)|^2.$$

Fix an interior cube $P \in \mathcal{D}^m_I(\Omega)$ and denote $\Sigma_P = |P|^{-1}\sum_{Q\subset P} |Q|^{-2m/n}\beta(Q)$. This is well defined since, if $Q \in \mathcal{D}$ is such that $Q \subset P$, then we have $Q \in \mathcal{D}^m_I(\Omega)$. Applying the triangle-inequality, we get

$$\begin{aligned} \Sigma_P &\leq \frac{3}{|P|}\sum_{Q\subset P} |Q|^{-2m/n}\beta_\Omega(Q) + \frac{3}{|P|}\sum_{Q\subset P}|Q|^{-2m/n}\beta_{\mathbb{R}^n\setminus\Omega}(Q) \\ &\leq 3||T\chi_\Omega||^2_{\dot{f}^{m,2}_\infty(\Omega)} + \frac{3}{|P|}\sum_{Q\subset P}|Q|^{-2m/n}\beta_{\mathbb{R}^n\setminus\Omega}(Q). \end{aligned} \tag{3.114}$$

In order to estimate the tail series we fix a dyadic cube $Q \subset P$ and index $\varepsilon \in \mathcal{E}$. Recall that $P \in \mathcal{D}^m_I(\Omega)$ and, using Definition 3.45, we have $\mathrm{dist}(\mathbb{R}^n \setminus \Omega, P) \geq C_{m+1}\ell(P)$. Invoking Lemma 3.64 with $B = \mathbb{R}^n \setminus \Omega$, we get the estimate

$$\frac{1}{|P|}\sum_{Q\subset P}|Q|^{-2m/n}\beta_{\mathbb{R}^n\setminus\Omega}(Q) = \frac{1}{|P|}\sum_{Q\subset P}\sum_{\varepsilon\in\mathcal{E}}|Q|^{-2m/n}|T(\chi_{\mathbb{R}^n\setminus\Omega})(Q,\varepsilon)|^2 \leq C_1. \tag{3.115}$$

Here the constant $C_1 < \infty$ is independent of the cube $P \in \mathcal{D}^m_I(\Omega)$. Now combining the estimates (3.115) and (3.114) above, we obtain

$$\sup_{P\in\mathcal{D}^m_I(\Omega)} \Sigma_P \leq 3||T\chi_\Omega||^2_{\dot{f}^{m,2}_\infty(\Omega)} + 3C_1 < \infty.$$

This is the required estimate (3.113). □

Difficult direction of $T\chi_\Omega$ theorem. Now we are able to prove the difficult direction of the $T\chi_\Omega$ theorem. Ultimately we rely on the reduced $T1$ theorem but also on the reflected paraproduct operators which enable us to extend the restricted operator to a bounded operator on the whole space.

Theorem 3.116. *Assume that $\emptyset \neq \Omega \subset \mathbb{R}^n$ is a Whitney coplump domain and $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ satisfies $T\chi_\Omega, T^*\chi_\Omega \in \dot{f}^{m,2}_\infty(\Omega)$. Then there exists $S \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ such that $S \sim T$ and*

$$\{\partial^\alpha S, \partial^\alpha S^* \,:\, |\alpha| = m\} \subset \mathscr{L}(L^2(\mathbb{R}^n)).$$

In particular, $\partial^\alpha T, \partial^\alpha T^* \in \mathscr{L}(L^2(\Omega))$ if $|\alpha| = m$.

Proof. Fix $\alpha \in \mathbb{N}^n_0$ such that $|\alpha| = m$. Theorem 3.93 and Corollary 3.11 combined show that $M = T - \pi_{T^*} - \pi^*_T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ satisfies $M1 = M^*1 = 0 \in \dot{f}^{m,2}_\infty(\mathbb{R}^n)$. Theorem 3.37 shows that $\partial^\alpha M, \partial^\alpha M^* \in \mathscr{L}(L^2(\mathbb{R}^n))$. Define

$$S = M + \mathrm{I}_{T^*} + \mathrm{I}^*_T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta).$$

Using the assumptions and Theorem 3.112 yields $\partial^\alpha S, \partial^\alpha S^* \in \mathscr{L}(L^2(\mathbb{R}^n))$. On the other hand, we have the identity

$$S = M + \mathrm{I}_{T^*} + \mathrm{I}^*_T = T - \mathrm{R}_{T^*} - \mathrm{R}^*_T.$$

Taking also the Remark 3.111 into account we see that $S \sim T$ and $S^* \sim T^*$. Based on the discussion in connection with (3.59), we see that $\partial^\alpha T, \partial^\alpha T^* \in \mathscr{L}(L^2(\Omega))$. □

Combining Theorem 3.116 and Theorem 3.68, we obtain the following boundedness result complementing Theorem 3.68.

Corollary 3.117. *Assume that $\emptyset \neq \Omega \subset \mathbb{R}^n$ is a Whitney coplump domain and $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ satisfies $T\chi_\Omega, T^*\chi_\Omega \in \dot{f}^{m,2}_\infty(\Omega)$. Then there exists $S \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ such that $S \sim T$ and*

$$S \in \mathscr{L}(L^\infty(\mathbb{R}^n), \dot{f}^{m,2}_\infty(\mathbb{R}^n)).$$

In particular, if $b \in C_0(\Omega)$, there exists $Sb \in \dot{F}^{m,2}_\infty(\mathbb{R}^n)$ such that $Sb|\Omega = Tb|\Omega$ pointwise and $||Sb||_{\dot{F}^{m,2}_\infty(\mathbb{R}^n)} \leq C||b||_{L^\infty(\Omega)}$ with C independent of b.

Formulation of the $T\chi_\Omega$ theorem. Finally we obtain the $T\chi_\Omega$ theorem. Its proof follows by combining Theorem 3.116 and Theorem 3.68.

Theorem 3.118. *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a Whitney coplump domain and $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$, where $0 < m < n$ and $0 < \delta < 1$. Then the following two conditions are equivalent*

- $T\chi_\Omega, T^*\chi_\Omega \in \dot{f}^{m,2}_\infty(\Omega)$,
- $\partial^\alpha T, \partial^\alpha T^* \in \mathscr{L}(L^2(\Omega))$ *if* $|\alpha| = m$.

Furthermore, if these conditions hold true, then there exists $S \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ such that $S \sim T$ and the operator S satisfies the conditions above with $\Omega = \mathbb{R}^n$.

Remark 3.119.
- We strengthen the $T\chi_\Omega$ theorem in later Theorem 6.12. Therein we show that the condition

$$\{\partial^\alpha T, \partial^\alpha T^* \,:\, |\alpha| = m\} \subset \mathscr{L}(L^p(\Omega)), \quad \text{if } 1 < p < \infty, \tag{3.120}$$

is equivalent with the two conditions occuring in the $T\chi_\Omega$ theorem.

- The characterizing conditions in the $T\chi_\Omega$ theorem are intrinsic to the domain $\Omega \subset \mathbb{R}^n$ and they are invariant under the equivalence relation $\sim$. This applies also to the condition (3.120).

  The proof of our main result, Theorem 6.19, relies on these intrincity or invariance properties. In this so called $T1$ theorem for WSIO's on admissible domains $\Omega \subset \mathbb{R}^n$ we begin with an operator $T \in \mathrm{SK}_\Omega^{-m}(\delta)$ and in the proof we extend the associated kernel $K \in \mathrm{K}_\Omega^{-m}(\delta)$ to a global kernel $\hat{K} \in \mathrm{K}_{\mathbb{R}^n}^{-m}(\delta')$. This defines an extension of the original operator T and we apply the $T\chi_\Omega$ theorem to this extension. However, as a consequence of the intrincity, the required conditions now depend only on T.
- Here are some results that are related to the $T\chi_\Omega$ theorem:

  In the formal limiting case $\Omega = \mathbb{R}^n$ and $m = 0$ the $T\chi_\Omega$ theorem coincides with the $T1$ theorem of David and Journé, formulated in Theorem 1.12. This limiting case is not included in our treatment. There are further results for Calderón–Zygmund type operators on more general spaces. As an example, F. Nazarov, S. Treil, and A. Volberg have proved a Tb theorem on non-homogeneous spaces [NTV03]. A theory of Calderón–Zygmund operators on Euclidean domains $\Omega \subset \mathbb{R}^n$ follows as a special case if we consider the space $(\mathbb{R}^n, \mu)$, where the Borel measure μ is defined in terms of the Lebesgue measure by $\mu(A) = m_n(\Omega \cap A)$ for every Borel set $A \subset \mathbb{R}^n$.

  The treatment [Tor91] of R. H. Torres deals with Calderón–Zygmund type singular integral operators but also with integral operators associated with kernels of different order. These include operators resembling the global WSIO's [Tor91, Theorem 4.3.12.]. The function spaces involved in Torres' work are the global Triebel–Lizorkin spaces. In our work the emphasis is in the boundedness properties of WSIO's on domains.

4. Regularity of standard kernels

Finding a solution to certain kernel extension problem in a uniform domain $\Omega \subset \mathbb{R}^n$ is our last major topic. We will prove that a standard kernel $K \in \mathrm{K}_\Omega^{-m}(\delta)$ has an extension to a globally defined kernel of the class $\mathrm{K}_{\mathbb{R}^n}^{-m}(\delta')$ if $\delta' < \delta$. This result complements the $T\chi_\Omega$ theorem, where the kernels need to be globally defined. Our solution to this kernel extension problem is divided in two parts as follows:

- The first part consists of a regularity result for standard kernels, formally

$$\mathcal{K}_\Omega^{-m}(\delta) \subset \mathrm{K}_\Omega^{-m}(\delta) \subset \mathcal{K}_\Omega^{-m}(\delta'), \tag{4.1}$$

  where the kernel space $\mathcal{K}_\Omega^{-m}$ consists of certain smooth kernels of the class C^m and whose order m derivatives are Calderón–Zygmund standard kernels if $\Omega = \mathbb{R}^n$. The second inclusion in (4.1) is the main result in this section. It follows from certain almost diagonality estimates combined with a so called dyadic resolution of unity that we will construct by utilizing the geometry of uniform domains.
- The second part consists of an extension result for the smooth kernels $\mathcal{K}_\Omega^{-m}(\delta)$ and this is the main result in the following Section 5.

4.1. **Kernel spaces.** We define various kernel spaces and collect those inclusions between these spaces that are somewhat easy to verify. The standard kernel space $\mathrm{K}_\Omega^{-m}(\delta)$ is defined in the Introduction. Next we define the Hölder–Zygmund kernels $k_{\mathrm{loc}}^{m+\delta}(\Omega)$ and the smooth kernels $\mathcal{K}_\Omega^{-m}(\delta)$.

Definition 4.2. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain. Assume that $m \in \mathbb{N}$, $m < n$, and $0 < \delta < 1$. The space of *Hölder–Zygmund kernels*, denoted by $k_{\mathrm{loc}}^{m+\delta}(\Omega)$, consists of complex-valued functions $K \in C(\Omega \times \Omega \setminus \{(x,x)\})$ satisfying

- size-estimate $|K(x,y)| \leq C_K |x-y|^{m-n}$, if $x, y \in \Omega$,
- $(m+\delta)$-Hölder–Zygmund condition

$$|\Delta_h^{m+1}(K(x,\cdot), Q, y)| \leq C_K |h|^{m+\delta} |x-y|^{-n-\delta}$$

  if $x, y \in \Omega$, $Q \subset\subset \Omega$ is a cube, and $2(m+1)|h| \leq |x-y|$. We also assume the same estimate but with $K(x,\cdot)$ replaced by $K(\cdot,x)$.

Definition 4.3. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain. Assume that $m \in \mathbb{N}$, $m < n$, and $0 < \delta < 1$. The space of *smooth kernels*, denoted by $\mathcal{K}_\Omega^{-m}(\delta)$, consists of complex-valued functions $K \in C^m(\Omega \times \Omega \setminus \{(x,x)\})$ satisfying

- size-estimate, given $\alpha, \beta \in \mathbb{N}_0^n$ so that $|\alpha| + |\beta| \leq m$,

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C_K |x-y|^{m-n-|\alpha|-|\beta|}$$

  if $x, y \in \Omega$,
- Hölder-regularity estimate, given $\alpha, \beta \in \mathbb{N}_0^n$ so that $|\alpha| + |\beta| = m$,

$$|\partial_x^\alpha \partial_y^\beta K(x+h,y) - \partial_x^\alpha \partial_y^\beta K(x,y)| \leq C_K |h|^\delta |x-y|^{-n-\delta}$$

if $x, y, x+h \in \Omega$ satisfy $2|h| \le |x-y|$. We also assume the same estimate with h-difference placed to the y-variable and $x, y, y+h \in \Omega$ satisfying $2|h| \le |x-y|$.

Remark 4.4.
- Let $K \in \mathcal{K}^{-m}_{\mathbb{R}^n}(\delta)$, $|\alpha| + |\beta| = m$. Then $\partial^\alpha_x \partial^\beta_y K$ is a Calderón–Zygmund standard kernel as defined in connection with (1.9). Hence WSIO's with smooth kernels and Calderón–Zygmund type operators are related to each other.
- There are analogous function spaces. The kernel space $\mathrm{K}^{-m}_\Omega(\delta)$ corresponds to so called local smoothness space $\mathscr{C}^{m+\delta}_\infty(\Omega)$. Smooth kernel space $\mathcal{K}^{-m}_\Omega(\delta)$ corresponds to a Hölder space $C^{m,\delta}(\overline{\Omega})$. Both of these function spaces are defined in Section 5.

First inclusions between kernel spaces. In this section we will show the following inclusions

$$\mathcal{K}^{-m}_\Omega(\delta) \subset k^{m+\delta}_{\mathrm{loc}}(\Omega) \subset \mathrm{K}^{-m}_\Omega(\delta) \subset \mathcal{K}^{-m}_\Omega(\delta'), \tag{4.5}$$

where $\Omega \subset \mathbb{R}^n$ is a uniform domain. We begin with the first two inclusions because these are easier to prove. The difficulties lie in the later verification of the last inclusion, where the uniformity is utilized.

Proposition 4.6. *Let $\emptyset \ne \Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain. Assume that $m \in \mathbb{N}$, $0 < m < n$, and $0 < \delta < 1$. Then we have the inclusions*

$$\mathcal{K}^{-m}_\Omega(\delta) \subset k^{m+\delta}_{\mathrm{loc}}(\Omega) \subset \mathrm{K}^{-m}_\Omega(\delta).$$

Proof. The size-estimates are valid. Due to symmetry it suffices to prove the regularity estimates with respect to the y-variable only. First we verify that $\mathcal{K}^{-m}_\Omega(\delta) \subset k^{m+\delta}_{\mathrm{loc}}(\Omega)$ and, for this purpose, let $K \in \mathcal{K}^{-m}_\Omega(\delta)$, $Q \subset\subset \Omega$ be a cube, $x, y \in \Omega$, and $h \in \mathbb{R}^n$ be so that $2(m+1)|h| \le |x-y|$ and $\{y, y+h, \dots, y+(m+1)h\} \subset Q$. It suffices to show that

$$|\Delta^{m+1}_h(K(x,\cdot), Q, y)| \le C_{n,K}|h|^{m+\delta}|x-y|^{-n-\delta}. \tag{4.7}$$

Denoting $f = K(x,\cdot)$ and using the integral presentation (A.1) for the first m differences, we are reduced to proving the estimate

$$|\Delta^1_h(\partial^\alpha_y K(x,\cdot), y + (\theta_1 + \dots + \theta_m)h)| \le C_{n,K}|h|^\delta|x-y|^{-n-\delta} \tag{4.8}$$

if $|\alpha| = m$ and $\theta \in [0,1]^m$. Notice that

$$2|h| \le 2(m+1)|h| - m|h| \le |x-y| - m|h| \le |x - (y + (\theta_1 + \dots + \theta_m)h)|.$$

Hence we can apply the Hölder-regularity estimate, satisfied by the smooth kernel K. After doing so we use the estimate $2|x - (y + (\theta_1 + \dots + \theta_m)h)| \ge |x-y|$. This gives us

$$|\Delta^1_h(\partial^\alpha_y K(x,\cdot), y + (\theta_1 + \dots + \theta_m)h)| \le 2^{n+\delta}C_K|h|^\delta|x-y|^{-n-\delta},$$

and this is the estimate (4.8). Hence (4.7) holds and therefore $K \in k^{m+\delta}_{\mathrm{loc}}(\Omega)$.

Next we prove that $k^{m+\delta}_{\mathrm{loc}}(\Omega) \subset \mathrm{K}^{-m}_{\Omega}(\delta)$. Let $K \in k^{m+\delta}_{\mathrm{loc}}(\Omega)$ and denote by C_K the associated constant. Let $x \in \Omega$ and fix a cube $Q = Q(x^Q, r) \subset\subset \Omega$, $r \le |x - x^Q|/8\sqrt{n}$. Let $y \in Q$ and $h \in \mathbb{R}^n$. If $(m+1)|h| > \mathrm{diam}(Q)$, then $\{y, y+h, \ldots, y+(m+1)h\} \not\subset Q$ and therefore $\Delta^{m+1}_h(K(x,\cdot), Q, y) = 0$. Next assume that $(m+1)|h| \le \mathrm{diam}(Q)$. Then, since $r \le |x - x^Q|/8\sqrt{n}$ and $|x - x^Q| < 2|x-y|$, we have

$$(m+1)|h| \le \mathrm{diam}(Q) = 2\sqrt{n}r \le |x - x^Q|/4 < |x-y|/2.$$

Thus $2(m+1)|h| < |x-y|$ and, using the assumptions, we get

$$|\Delta^{m+1}_h(K(x,\cdot), Q, y)| \le C_K |h|^{m+\delta}|x-y|^{-n-\delta} \le C_{n,K}|Q|^{(m+\delta)/n}|x - x^Q|^{-n-\delta}.$$

Combining the estimates above, we obtain

$$\sup_{|h|\le \mathrm{diam}(Q)} \frac{1}{|Q|^{1+(m+\delta)/n}} \int_Q |\Delta^{m+1}_h(K(x,\cdot), Q, y)| dy \le C_{n,K}|x - x^Q|^{-n-\delta}.$$

We assumed that $Q(x^Q, r) \subset\subset \Omega$ satisfies $4\,\mathrm{diam}(Q) = 8\sqrt{n}r \le |x - x^Q|$. As a consquence, $K \in \mathrm{K}^{-m}_{\Omega}(\delta)$ with constant $\max\{C_K, C_{n,K}, 4\}$. □

4.2. **Dyadic resolution of unity.** In order to prove the last inclusion in (4.5) we first construct a so called dyadic resolution of unity in uniform domains. This is a generalization of a similar construction on special Lipschitz domains [Ryc99].

Let us explain what we mean by a dyadic resolution. If $f : \Omega \to \mathbb{C}$ is continuous, then $f(x) = \langle f, \delta_x \rangle$, where δ_x is the Dirac's delta located at the point $x \in \Omega$. This identity gives rise to a dyadic resolution of f as follows. First we approximate the Dirac's delta with a bump function $\varphi_{x,M}$. Then we expand this bump function as a sum of a fixed coarse scale bump function and a telescoping series of differences of two consecutive bump functions. That is,

$$f(x) \approx \int_\Omega f(y)\varphi_{x,\ell}(y)dy + \sum_{j=\ell+1}^{M} \int_\Omega f(y)(\varphi_{x,j} - \varphi_{x,j-1})(y)dy, \tag{4.9}$$

where the integrands are smooth for every fixed y, as functions of x. For instance, if $\Omega = \mathbb{R}^n$, then we can fix one bump function $\varphi \in C^\infty_0(\mathbb{R}^n)$ so that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$ and define $\varphi_{j,x} = 2^{jn}\varphi(2^j(\cdot - x))$ for $j \in \mathbb{Z}$. Decompositions like (4.9) allow us to connect to cancellation properties of f as we will see.

In proper domains the difficulties lie in ensuring that we can do this construction so that the supports of the bump functions are included in the domain. To indicate some of the difficulties, one expects vanishing moments from the difference of two consecutive bump functions in order to induce cancellation. There are also certain geometric properties that the construction should possess. We show that these difficulties can be overcome in the case of uniform domains.

Bump functions. Our construction requires special bump functions which we first describe. Here we follow Triebel [Tri92, p. 173–174] until we reach the bump functions supported in rotated cones, see Lemma 4.13 below. We begin with one-dimensional bump functions.

Lemma 4.10. *Let $m \in \mathbb{N}$, $\varepsilon > 0$, and $\rho \in \mathbb{R}$. There exists a function $g = g_{\rho,\varepsilon} \in C_0^\infty(\mathbb{R})$ such that* $\operatorname{supp} g \subset B(\rho, \varepsilon)$ *and*

$$\int_{\mathbb{R}} g(t)dt = 1, \quad \int_{\mathbb{R}} t^k g(t)dt = 0, \quad k = 1, 2, \dots, m.$$

Proof. Choose $m+1$ points $\rho_0 < \rho_1 < \cdots < \rho_m \in \mathbb{R}$ such that $\rho - \varepsilon/2 < \rho_0$ and $\rho_m < \rho + \varepsilon/2$. Fix $\delta > 0$ and let $g_0^\delta, g_1^\delta, \dots, g_m^\delta \in C_0^\infty(\mathbb{R})$ be such that, if $j = 0, 2, \dots, m$, we have $\int_{\mathbb{R}} g_j^\delta(t)dt = 1$ and $\operatorname{supp} g_j^\delta \subset B(\rho_j, \delta)$. Then consider the linear system

$$\sum_{j=0}^{m} \mu_j \int_{\mathbb{R}} g_j^\delta(t)dt = 1; \quad \sum_{j=0}^{m} \mu_j \int_{\mathbb{R}} t^k g_j^\delta(t)dt = 0, \quad k = 1, 2, \dots, m \tag{4.11}$$

of the variable $\mu = (\mu_0, \mu_1, \dots, \mu_m) \in \mathbb{R}^{m+1}$. This linear system has the equivalent matrix form

$$A_\delta \mu = (1, 0, 0, \dots, 0) \in \mathbb{R}^{m+1}, \quad A_\delta = \left\{ \int_{\mathbb{R}} t^j g_k^\delta(t)dt \right\}_{j,k=0}^{m} \in \mathbb{R}^{(m+1)\times(m+1)}.$$

Due to properties of Vandermonde determinants,

$$\det A_\delta = \det A_\delta^T \xrightarrow{\delta \to 0} \det\{\rho_j^k\}_{j,k=0}^m = \Pi_{j>k}(\rho_j - \rho_k) \neq 0.$$

In particular, for some $\delta = \delta_0 < \varepsilon/2$, the linear system of equations (4.11) has a solution $\mu \in \mathbb{R}^{m+1}$. Then we can choose $g(t) = \sum_{j=0}^m \mu_j g_j^{\delta_0}(t)$ if $t \in \mathbb{R}$. □

Multivariate bump functions are then obtained as tensor products of corresponding one-dimensional bump functions. For later purposes we need the supports to be contained in a small neighborhood of the point $e_n \in \mathbb{R}^n$.

Lemma 4.12. *Let $m \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $\varphi \in C_0^\infty(B(e_n, \varepsilon))$, $e_n \in \mathbb{R}^n$ being the n'th base vector, such that*

$$\int_{\mathbb{R}^n} \varphi(x)dx = 1; \quad \int_{\mathbb{R}^n} x^\alpha \varphi(x)dx = 0, \quad 0 < |\alpha| \leq m.$$

Proof. Fix $g, h \in C_0^\infty(\mathbb{R})$ provided by Lemma 4.10 with $\operatorname{supp} g \subset B(0, \varepsilon/2n)$ and $\operatorname{supp} h \subset B(1, \varepsilon/2n)$. Define $\varphi(x) = g(x_1)g(x_2)\cdots g(x_{n-1})h(x_n)$, $x \in \mathbb{R}^n$. Assume that $x \in \mathbb{R}^n$ is such that $\varphi(x) \neq 0$. Then $x_1, \dots, x_{n-1} \in B(0, \varepsilon/2n)$ and $x_n \in B(1, \varepsilon/2n)$. Thus $|x - e_n| < \varepsilon/2$ and we have $x \in B(e_n, \varepsilon/2)$. In particular, $\operatorname{supp} \varphi \subset C_0^\infty(B(e_n, \varepsilon))$. The Fubini's theorem yields

$$\int_{\mathbb{R}^n} \varphi(x)dx = \int_{\mathbb{R}} g(x_1)dx_1 \cdots \int_{\mathbb{R}} g(x_{n-1})dx_{n-1} \int_{\mathbb{R}} h(x_n)dx_n = 1.$$

Assume that $0 < |\alpha| \leq m$. Then $0 < \alpha_j \leq m$ for some j and the Fubini's theorem with the identities

$$\int_{\mathbb{R}} x^{\alpha_j} g(x)dx = 0 = \int_{\mathbb{R}} x^{\alpha_j} h(x)dx$$

yields $\int_{\mathbb{R}^n} x^\alpha \varphi(x)dx = 0$ in any case. □

Then we define multivariate bump functions that are supported in rotated cones. For this purpose we need notation regarding rotation and dilatation.

Let $n \geq 2$ and φ be as in Lemma 4.12 with parameters m and ε. Let $\sigma \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. We denote by $T_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ arbitrary but fixed rotation about the origin such that $T_\sigma(e_n) = \sigma$; in the special case $\sigma = e_n$, we choose $T_{e_n} = \mathrm{id}$. Denote $\varphi_\sigma = \varphi \circ T_\sigma^{-1}$. If $\sigma \in \mathbb{R}^n \setminus \{0\}$ then we denote $\varphi_\sigma = |\sigma|^{-n} \varphi_{\sigma/|\sigma|}(|\sigma|^{-1} \cdot)$.

Lemma 4.13. *Functions φ_σ, as defined above, satisfy the following*

1) $\operatorname{supp} \varphi_\sigma \subset B(\sigma, |\sigma|\varepsilon)$, *if* $\sigma \in \mathbb{R}^n \setminus \{0\}$,
2) $\int_{\mathbb{R}^n} \varphi_\sigma(x) dx = 1$, *if* $\sigma \in \mathbb{R}^n \setminus \{0\}$,
3) $\int_{\mathbb{R}^n} x^\alpha \varphi_\sigma(x) dx = 0$, *if* $0 < |\alpha| \leq m$ *and* $\sigma \in \mathbb{R}^n \setminus \{0\}$,
4) *Let* $\alpha \in \mathbb{N}_0^n$ *and* $\sigma \in \mathbb{S}^{n-1}$. *Then*

$$|\partial^\alpha \varphi_\sigma(x)| \leq \sum_{|\beta|=|\alpha|} c_{\alpha,\beta} |(\partial^\beta \varphi)(T_\sigma^{-1} x)|, \quad x \in \mathbb{R}^n.$$

In particular, if $\rho \in \mathbb{R}^n \setminus \{0\}$, we have $||\partial^\alpha \varphi_\rho||_{L^\infty} \leq C_{\alpha,\varphi} |\rho|^{-n-|\alpha|}$.

Proof. We prove 1)–3) under the assumption that $\sigma \in \mathbb{S}^{n-1}$. General cases follow.

1) We have $\operatorname{supp} \varphi_\sigma = \operatorname{supp} \varphi \circ T_\sigma^{-1} \subset T_\sigma \operatorname{supp} \varphi \subset T_\sigma B(e_n, \varepsilon) = B(\sigma, \varepsilon)$.
2) Recall that T_σ is a rotation and change the variables.
3) $T_\sigma(y^\alpha) = \sum_{|\beta|=|\alpha|} c_\beta y^\beta$ and therefore

$$\int_{\mathbb{R}^n} x^\alpha \varphi_\sigma(x) dx = \int_{\mathbb{R}^n} (T_\sigma y)^\alpha \varphi(y) dy = \sum_{|\beta|=|\alpha|} c_{\alpha,\beta} \int_{\mathbb{R}^n} y^\beta \varphi(y) dy = 0.$$

4) Denote the matrix of T_σ^{-1} by $\{c_{j,k}\}_{j,k=1}^n$. Apply the chain rule for

$$\partial_j(\varphi \circ T_\sigma^{-1})(x) = \sum_{k=1}^n c_{k,j} (\partial_k \varphi)(T_\sigma^{-1} x), \quad j = 1, 2, \ldots, n. \tag{4.14}$$

Note that $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$. Denoting $a \in \{n\}^{\alpha_n} \times \{n-1\}^{\alpha_{n-1}} \times \cdots \times \{1\}^{\alpha_1}$ and iterating (4.14), we have

$$\partial^\alpha(\varphi \circ T_\sigma^{-1})(x) = \sum_{k_1=1}^n c_{k_1,a_1} \cdots \sum_{k_{|\alpha|}=1}^n c_{k_{|\alpha|},a_{|\alpha|}} (\partial_{k_{|\alpha|}} \cdots \partial_{k_1} \varphi)(T_\sigma^{-1} x),$$

if $x \in \mathbb{R}^n$. Observe that $|c_{jk}| = |e_j \cdot T_\sigma^{-1} e_k| \leq 1$ for every $1 \leq j, k \leq n$ and

$$|\partial^\alpha \varphi_\sigma(x)| = |\partial^\alpha(\varphi \circ T_\sigma^{-1})(x)| \leq \sum_{|\beta|=|\alpha|} c_{\alpha,\beta} |(\partial^\beta \varphi)(T_\sigma^{-1} x)|, \quad x \in \mathbb{R}^n.$$

If $\rho \in \mathbb{R}^n \setminus \{0\}$, use the chain rule. □

Special Lipschitz domains. Having the bump functions at our disposal, we are ready to illustrate the ideas behind the construction of a dyadic resolution of unity on domains. The relevant ideas are best demonstrated in the context of so called special Lipschitz domains.

Definition 4.15. Let $n \geq 2$. A *special Lipschitz domain* is a domain $\Omega \subset \mathbb{R}^n$ lying above the graph of a Lipschitz function $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\omega(0) = 0$. More precisely,

$$\Omega = \{(x', x_n) \in \mathbb{R}^n \,:\, x_n > \omega(x')\}$$

and there exists a constant $L > 0$ such that $|\omega(x') - \omega(y')| \leq L|x' - y'|$ if $x', y' \in \mathbb{R}^{n-1}$.

It is straightforward to show that any special Lipschitz domain is an image of the half space $\{(x', x_n) \,:\, x_n > 0\} \subset \mathbb{R}^n$ under a bi-Lipschitz mapping of $\mathbb{R}^n$ onto itself. Thus, according to later Theorem 6.4, special Lipschitz domains are uniform (even admissible).

The following construction of a dyadic resolution of unity is due to Rychkov [Ryc99]. It also serves as a starting point for our construction. Let $\Omega \subset \mathbb{R}^n$ be a special Lipschitz domain with constant L. Consider the following *cone*

$$K = K_L = \{(x', x_n) \,:\, x_n > L|x'|\} \subset \Omega.$$

This cone is convex and positively homogeneous so that $\lambda K = K$ if $\lambda > 0$. Its translations satisfy $x + K \subset \Omega$ for every $x \in \Omega$. The *core* of the translated cone $x + K$, $x \in \Omega$, is the image $\sigma_x[0, \infty) = \{\sigma_x(t) \,:\, t \geq 0\}$ of the path

$$\sigma_x : [0, \infty) \to x + K, \quad \sigma_x(t) = x + te_n.$$

We call σ_x the *arc length parametrization of the core.*

Fix $m \in \mathbb{N}$ and $\varepsilon > 0$ so that $B(e_n, \varepsilon) \subset K$. Let $\varphi \in C_0^\infty(B(e_n, \varepsilon))$ denote the bump function associated with the parameters m and ε as in Lemma 4.12. Using the homogeneity and translation properties of K, we have

$$\operatorname{supp}(\varphi(\lambda^{-1}(\cdot - x))) \subset x + \lambda B(e_n, \varepsilon) \subset x + \lambda K = x + K \subset \Omega$$

for every $\lambda > 0$ and $x \in \Omega$. Notice that the center $\sigma_x(\lambda) = x + \lambda e_n$ of the ball $x + \lambda B(e_n, \varepsilon)$ lies in the core $\sigma_x[0, \infty)$ of the translated cone $x + K$ and $\varphi = \varphi_{e_n} = \varphi_{(\sigma_x(1) - x)}$.

If $j \in \mathbb{N}_0$ and $x \in \Omega$, define

$$\psi_{x,j}(y) = \begin{cases} \varphi(y - x), & j = 0, \\ 2^{jn}\varphi(2^j(y - x)) - 2^{(j-1)n}\varphi(2^{j-1}(y - x)), & j > 0. \end{cases}$$

Notice that, if $j \in \mathbb{N}$ and $x \in \Omega$, then $\operatorname{supp} \psi_{x,j} \subset \Omega$ and its diameter is roughly 2^{-j}. If $M \in \mathbb{N}$, then $\sum_{j=0}^M \psi_{x,j} = 2^{Mn}\varphi(2^M(\cdot - x))$ approximates the Dirac's delta at $x \in \Omega$. Furthermore, if $x \in \Omega$, then we have $\int_\Omega \psi_{x,0}(y)dy = 1$ and $\int_\Omega x^\alpha \psi_{x,j}(y)dy = 0$ in the case $|\alpha| \leq m$ and $j > 0$.

Quasihyperbolic geodesics in uniform domains. The previous construction of a dyadic resolution of unity in special Lipschitz domains admits a generalization to uniform domains. The translation $x+K$ of a convex cone is, in this general case, replaced with a cone like object that is implicitly defined via its core. Recall that the core of $x+K$ is the geodesic $\sigma_x[0,\infty) = \{x+te_n : t \geq 0\}$. In uniform domains a core is given by a quasihyperbolic geodesic joining two given points $x, y \in \Omega$.

We invoke the quasihyperbolic geodesics from [GO79]. Assume that $\Omega \subset \mathbb{R}^n$ is a uniform domain and $x, y \in \Omega$. There exists a path $\sigma : [0, \ell(\sigma)] \to \Omega$ that is parametrized by the arc length and satisfies the following:

- $\sigma(0) = x$, $\sigma(\ell(\sigma)) = y$,
- if $0 \leq s < t \leq \ell(\sigma)$, then

$$\int_s^t \operatorname{dist}(\sigma(x), \partial\Omega)^{-1} dx = \inf_\gamma \int_0^{\ell(\gamma)} \operatorname{dist}(\gamma(x), \partial\Omega)^{-1} dx,$$

  where the infimum is taken over all of the rectifiable paths $\gamma : [0, \ell(\gamma)] \to \Omega$ that are parametrized by the arc length and satisfy $\gamma(0) = \sigma(s)$ and $\gamma(\ell(\gamma)) = \sigma(t)$. This infimum is the *quasihyperbolic distance* between $\sigma(s)$ and $\sigma(t)$ in Ω.

The path σ *is (the arc length parametrization of) a quasihyperbolic geodesic* joining the two points $x, y \in \Omega$. Such paths are denoted by $\sigma : x \curvearrowright y$. The following useful result is proven by Gehring and Osgood.

Lemma 4.16. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with uniformity constant $a \geq 1$. There is a constant b, depending on Ω and satisfying $b \geq a$, so that the following holds. Let $x, y \in \Omega$ and $\sigma : x \curvearrowright y$ be a quasihyperbolic geodesic joining the two points x and y. Then we have a)–b) below*

a) $|s-t| \leq b|\sigma(s) - \sigma(t)|$ *if* $0 \leq s, t \leq \ell(\sigma)$,

b) $\min(t, \ell(\sigma) - t) \leq b \operatorname{dist}(\sigma(t), \partial\Omega)$ *if* $t \in [0, \ell(\sigma)]$.

As a consequence of b) above and the arc length parametrization of σ, we have for every $t \in [0, |x-y|/2] \subset [0, \ell(\sigma)/2]$,

c) $t < 2b \operatorname{dist}(\sigma(t) + (z-x), \partial\Omega)$ *if* $z \in B(x, \operatorname{dist}(x, \partial\Omega)/4b)$.

Proof. Both a) and b) are stated in [GO79, Corollary 2] and the proofs can be found therein. It remains to verify c). Denote $R(x) = \operatorname{dist}(x, \partial\Omega)$.

Assume first that $t \in [0, R(x)/2)$. Then using the arc length parametrization of $\sigma : x \curvearrowright y$, we have

$$|(\sigma(t) + z - x) - x| \leq |\sigma(t) - x| + |z - x| < R(x)/2 + R(x)/4b \leq 3R(x)/4.$$

Therefore $\bar{B}(\sigma(t) + z - x, t/2b) \subset B(\sigma(t) + z - x, R(x)/4) \subset \Omega$.

Next we assume that $t \in [R(x)/2, |x-y|/2]$. Applying b), we get $B(\sigma(t), t/b) \subset \Omega$. We also have

$$|(\sigma(t) + z - x) - \sigma(t)| = |z - x| < R(x)/4b \leq t/2b.$$

When combined, these estimates imply that $\bar{B}(\sigma(t) + z - x, t/2b) \subset \Omega$. □

Dyadic resolution of unity in uniform domains. Let $m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a uniform domain. Let $b \geq 1$ be a constant depending on the domain Ω as in Lemma 4.16. Let

$$\varphi \in C_0^\infty(B(e_n, 1/8b\sqrt{n}))$$

be a bump function as in Lemma 4.12 with vanishing moments for α, $0 < |\alpha| \leq m$. Let $p \in \mathbb{N}$ be such that

$$8b\sqrt{n} + 1 \geq p > 8b\sqrt{n}.$$

Given $j \in \mathbb{N}$ we divide the interval $[2^{-j}, 2^{-j+1}]$ into p similar pieces and denote the division points by

$$2^{-j} = t(j,1) < t(j,2) < \cdots < t(j,p) = 2^{-j+1}.$$

The aforementioned parameters depend at most on Ω and m (and possibly on j but we only worry about the number p of division points and the norm bounds for the derivatives $\partial^\alpha \varphi$ because these quantities occur as constants in later estimates).

If $x \in \Omega$ we denote $R(x) = \operatorname{dist}(x, \partial\Omega)$ and $r(x) = R(x)/4b$. Let $x_0, y_0 \in \Omega$ be distinct points and $\ell = \ell(x_0, y_0)$ be determined by $2^{-\ell} < |x_0 - y_0|/16 \leq 2^{-\ell+1}$. Let $\sigma : x_0 \curvearrowright y_0$ be a quasihyperbolic geodesic satisfying a)–c) in Lemma 4.16. Denote

$$\varphi_{\sigma,t} = \varphi_{(\sigma(t)-x_0)}, \quad t \in (0, |x_0 - y_0|/2],$$

so that the function $y \mapsto \varphi_{(\sigma(t)-x_0)}(y - x_0)$ is supported in the ball $B(\sigma(t), t/8b\sqrt{n})$ whose center $\sigma(t)$ lies in the *core* $\sigma[0, \ell(\sigma)]$. Finally denote

$$\psi_{\sigma,j} = \begin{cases} 0, & j < \ell = \ell(x_0, y_0), \\ \varphi_{\sigma,2^{-j}}, & j = \ell, \\ \sum_{q=1}^{p-1} \big(\varphi_{\sigma,t(j,q)} - \varphi_{\sigma,t(j,q+1)}\big), & j > \ell. \end{cases}$$

Then $\{\psi_{\sigma,j}\}_{j \geq \ell}$ is an *m-regular dyadic resolution of unity along the quasihyperbolic geodesic* $\sigma : x_0 \curvearrowright y_0$.

The expansion related to the indexing scheme $\{t(j,q)\}$, $j > \ell$, is used to ensure that there is a cube $\subset\subset \Omega$ containing the support of $\varphi_{\sigma,t(j,q)} - \varphi_{\sigma,t(j,q+1)}$, $j > \ell$. This property is later required by the Whitney approximation theorem.

In what follows we collect properties that a dyadic resolution of unity satisfies. Denote by $Q(x,r) \subset \mathbb{R}^n$ the unique open cube whose sides are parallel to the coordinate axes and which satisfies $B(x,r) \subset Q(x,r) \subset B(x, \sqrt{n}r)$. Hence $Q(x,r)$ is centered at the point x and has side-length $2r$.

Lemma 4.17. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and $x_0, y_0 \in \Omega$. Let $\{\psi_{\sigma,j}\}_{j\geq\ell}$ be the m-regular dyadic resolution of unity along the quasihyperbolic geodesic $\sigma : x_0 \curvearrowright y_0$. Let $x \in B(x_0, r(x_0))$, $\alpha \in \mathbb{N}_0^n$, $t \in (0, |x_0 - y_0|/2]$, $j > \ell$, and $q \in \{1, 2, \ldots, p-1\}$. Then we have 1)–4) below*

1) $\operatorname{supp} \varphi_{\sigma,t}(\cdot - x) \subset Q(\sigma(t) + x - x_0, t/8b\sqrt{n}) \subset\subset \Omega$,
2) $||\partial_x^\alpha(\varphi_{\sigma,t}(\cdot - x))||_{L^\infty} \leq C_{\alpha,m,\Omega} t^{-n-|\alpha|}$,
3) $\operatorname{supp}(\varphi_{\sigma,t(j,q)}(\cdot - x) - \varphi_{\sigma,t(j,q+1)}(\cdot - x)) \subset Q(\sigma(t(j,q)) + x - x_0, 2^{-j}/2b\sqrt{n}) \subset\subset \Omega$,
4) $||\partial_x^\alpha(\varphi_{\sigma,t(j,q)}(\cdot - x) - \varphi_{\sigma,t(j,q+1)}(\cdot - x))||_{L^\infty} \leq C_{\alpha,m,\Omega} 2^{j(n+|\alpha|)}$.

Proof. 1) Using 4.13.1), the arc length parametrization of σ, and 4.16.c), we have

$$\begin{aligned}\operatorname{supp}\varphi_{\sigma,t}(\cdot - x) &= \operatorname{supp}\varphi_{(\sigma(t)-x_0)}(\cdot - x) \subset B(\sigma(t)+x-x_0, t/8b\sqrt{n})\\ &\subset Q(\sigma(t)+x-x_0, t/8b\sqrt{n}) \subset B(\sigma(t)+x-x_0, t/8b) \subset\subset \Omega.\end{aligned}$$

2) Applying 4.13.4) and 4.16.a) with $s=0$, we get

$$\begin{aligned}||\partial_x^\alpha(\varphi_{\sigma,t}(\cdot - x))||_{L^\infty} &= ||(\partial^\alpha\varphi_{(\sigma(t)-x_0)})(\cdot - x)||_{L^\infty}\\ &\le C_{\alpha,\varphi}|\sigma(t)-x_0|^{-n-|\alpha|} \le C_{\alpha,m,\Omega}t^{-n-|\alpha|}.\end{aligned}$$

3) The inequalities $2^{-j} \le t(j,q) < t(j,q+1) \le 2^{-j+1} < |x_0-y_0|/2$ show that the difference is well defined and 1) is applicable to the individual terms. These estimates are also used below. We continue with the estimate

$$\begin{aligned}|\sigma(t(j,q+1)) - \sigma(t(j,q))| &\le |t(j,q+1)-t(j,q)|\\ &= p^{-1}(2^{-j+1}-2^{-j}) = p^{-1}2^{-j} < 2^{-j+1}/8b\sqrt{n}.\end{aligned}$$

Using this estimate, we get

$$\sigma(t(j,q+1)) + x - x_0 \in B(\sigma(t(j,q)) + x - x_0, 2^{-j+1}/8b\sqrt{n}).$$

Applying also the proof of 1) above and using 4.16.c), we get

$$\begin{aligned}&\operatorname{supp}(\varphi_{\sigma,t(j,q)}(\cdot - x) - \varphi_{\sigma,t(j,q+1)}(\cdot - x))\\ &\quad\subset Q(\sigma(t(j,q)) + x - x_0, 2^{-j}/2b\sqrt{n}) \subset \bar{B}(\sigma(t(j,q)) + x - x_0, 2^{-j}/2b) \subset\subset \Omega.\end{aligned}$$

4) We have $t(j,q) < t(j,q+1) \le 2^{-j+1} \le |x_0-y_0|/16$. Therefore, applying 2) above, we have the following estimate for the left-hand side of 4) from above

$$\begin{aligned}&||\partial_x^\alpha(\varphi_{\sigma,t(j,q)}(\cdot - x))||_{L^\infty} + ||\partial_x^\alpha(\varphi_{\sigma,t(j,q+1)}(\cdot - x))||_{L^\infty}\\ &\qquad\le C_{\alpha,m,\Omega}t(j,q)^{-n-|\alpha|} + C_{\alpha,m,\Omega}t(j,q+1)^{-n-|\alpha|}.\end{aligned}$$

Taking also the inequality $2^{-j} \le t(j,q) \wedge t(j,q+1)$ into account, we are able to estimate the right-hand side by $C_{\alpha,m,\Omega}2^{j(n+|\alpha|)}$ from above. □

Lemma 4.18. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and $x_0, y_0 \in \Omega$. Let $\{\psi_{\sigma,j}\}_{j\ge\ell}$ be the m-regular dyadic resolution of unity along the quasihyperbolic geodesic $\sigma : x_0 \curvearrowright y_0$. Let $x \in B(x_0, r(x_0))$, $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| \le m$, $j > \ell$, and $q \in \{1, 2, \dots, p-1\}$. Then*

$$\int_\Omega y^\alpha \partial_x^\beta\big(\varphi_{\sigma,t(j,q)}(y-x) - \varphi_{\sigma,t(j,q+1)}(y-x)\big)dy = 0.$$

Proof. Applying 4.17.3) and integrating by parts in $\mathbb{R}^n$, we get

$$\begin{aligned}\int_\Omega &= (-1)^{|\beta|}\int_{\mathbb{R}^n} y^\alpha\big((\partial^\beta\varphi_{\sigma,t(j,q)})(y-x) - (\partial^\beta\varphi_{\sigma,t(j,q+1)})(y-x)\big)dy\\ &= (-1)^{|\beta|}\int_{\mathbb{R}^n}(y+x)^\alpha\partial^\beta(\varphi_{\sigma,t(j,q)} - \varphi_{\sigma,t(j,q+1)})(y)dy\\ &= \int_{\mathbb{R}^n}\partial_y^\beta(y+x)^\alpha(\varphi_{\sigma,t(j,q)} - \varphi_{\sigma,t(j,q+1)})(y)dy.\end{aligned}$$

Using Lemma 4.13, 2) and 3) therein, and that $\partial_y^\beta (y+x)^\alpha \in \mathcal{P}_m(\mathbb{R}^n)$ (x is treated as a constant here) we see that the last integral vanishes. □

Remark 4.19. Let $\Omega \subset \mathbb{R}^n$ be a uniform domain, $x_0, y_0 \in \Omega$ be distinct, and $\{\psi_{\sigma,j}\}_{j\geq\ell}$ be an m-regular dyadic resolution of unity along a quasihyperbolic geodesic $\sigma : x_0 \curvearrowright y_0$. Due to cancellation of terms, we have $\psi_{\sigma,j} = \varphi_{\sigma,2^{-j}} - \varphi_{\sigma,2^{-j+1}}$ for $j > \ell$. Also, if $M > \ell$, further cancellation occurs so that $\sum_{j=\ell}^M \psi_{\sigma,j} = \varphi_{\sigma,2^{-M}}$. Let $x \in B = B(x_0, r(x_0)) \subset\subset \Omega$. Then, using 4.17.1) and 4.13.2), we get $\operatorname{supp}\varphi_{\sigma,2^{-M}}(\cdot - x) \subset B(x, 2^{-M+1}) \cap \Omega$ and $\int_\Omega \varphi_{\sigma,2^{-M}}(y-x)dy = 1$. In particular, if $f \in L^1_{\mathrm{loc}}(\Omega)$ is continuous at $x \in B$, then

$$f(x) = \lim_{M\to\infty} \int_\Omega \varphi_{\sigma,2^{-M}}(y-x)f(y)dy = \sum_{j=\ell}^\infty \int_\Omega \psi_{\sigma,j}(y-x)f(y)dy. \tag{4.20}$$

4.3. **Regularity of kernels in uniform domains.** We prove the following regularity result for standard kernels

$$\mathrm{K}_\Omega^{-m}(\delta) \subset \mathcal{K}_\Omega^{-m}(\delta'), \quad \text{if } 0 < \delta' < \delta < 1, \tag{4.21}$$

where $\Omega \subset \mathbb{R}^n$ is a uniform domain and $0 < m < n$. This result is a step towards the atomic decomposition of standard kernels which, in turn, leads to the extension of these kernels. Certain almost diagonality estimates turn out to be useful here. Such estimates were crucial also for the boundedness properties of WSIO's as was seen in connection with the $T\chi_\Omega$ theorem.

Parts of the estimates here originate in [HL03] for a proof that an almost diagonal operator has a kernel representation as a CZO.

A variant of a result of Gehring and Martio. Here is a useful local-to-global type Hölder estimate which is based on computations due to Gehring and Martio [GM85, pp. 206–207]. This result is useful in many occasions. For instance, while proving Hölder estimates for standard kernels the special formulation below is convenient.

Theorem 4.22. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and $0 < \delta < 1$. Denote by $a = a_\Omega$ the uniformity constant as in Definition 1.13. Let $x, y \in \Omega$ be distinct and $\gamma : [0, \ell(\gamma)] \to \Omega$ be a path joining these two points as in the Definition 1.13. Assume that $f : \Omega \to \mathbb{C}$ is such that $f \circ \gamma : [0, \ell(\gamma)] \to \mathbb{C}$ is continuous and*

$$|f(\gamma(t)+k) - f(\gamma(t))| \leq |k|^\delta \tag{4.23}$$

for every $t \in [0, \ell(\gamma)]$ and $\gamma(t) + k \in B(\gamma(t), \min(t, \ell(\gamma)-t)/c) \subset \Omega$ for a fixed $c > a_\Omega$. Then we have the endpoint-estimate

$$|f(x) - f(y)| \leq C|x-y|^\delta,$$

where C depends at most on the parameters a, c, δ.

Proof. Denote by $z = \gamma(\ell(\gamma)/2)$ the midpoint. Because

$$|f(x) - f(y)| \leq |f(z) - f(x)| + |f(y) - f(z)|$$

it suffices to prove that $|f(z) - f(x)| \le C|x-y|^\delta$. Indeed, the corresponding estimate for $|f(y) - f(z)|$ is symmetric to this. Let $0 < t_0 < \ell(\gamma)/2$. This should be thought to be close to zero so that $f(\gamma(t_0))$ is close to $f(\gamma(0)) = f(x)$ due to continuity. Assume that $t_0, \dots, t_{j-1}$ are chosen. If $t_{j-1} < \ell(\gamma)/2$ then we choose $t_j = \min(\ell(\gamma)/2, (1+1/2c)t_{j-1})$. This procedure ends after a finite number of steps when we reach $t_m = \ell(\gamma)/2$ with $m \in \mathbb{N}$. Using the triangle inequality, we have the estimate

$$|f(z) - f(\gamma(t_0))| = |f(\gamma(t_m)) - f(\gamma(t_0))| \le \sum_{j=1}^m |f(\gamma(t_j)) - f(\gamma(t_{j-1}))|. \tag{4.24}$$

Write

$$|f(\gamma(t_j)) - f(\gamma(t_{j-1}))| = |f(\gamma(t_{j-1}) + k_j) - f(\gamma(t_{j-1}))|, \quad j = 1, \dots, m,$$

where $|k_j| = |\gamma(t_j) - \gamma(t_{j-1})| \le |t_j - t_{j-1}| \le t_{j-1}/2c < \min(t_{j-1}, \ell(\gamma) - t_{j-1})/c$ by the arc-length parametrization of γ. Hence, using (4.24) and (4.23), we obtain the estimate $|f(z) - f(\gamma(t_0))| \le \sum_{j=1}^m |k_j|^\delta$. In the sequel we estimate the sum on the right-hand side. First of all, we have $|k_m|^\delta < (\ell(\gamma)/c)^\delta \le |x-y|^\delta$. If $m > 1$, we also need the estimate

$$\begin{aligned}\sum_{j=1}^{m-1} |k_j|^\delta &\le \sum_{j=1}^{m-1} |t_{j-1} - t_j||t_{j-1} - t_j|^{\delta-1} = (1+2c)^{1-\delta} \sum_{j=1}^{m-1} |t_{j-1} - t_j| t_j^{\delta-1} \\ &\le (1+2c)^{1-\delta} \int_{t_0}^{t_{m-1}} s^{\delta-1} ds \le (1+2c)^{1-\delta} \delta^{-1} a^\delta |x-y|^\delta.\end{aligned}$$

Combining the estimates beginning from (4.24) we get

$$|f(z) - f(\gamma(t_0))| \le (1 + (1+2c)^{1-\delta}\delta^{-1}a^\delta)|x-y|^\delta, \quad 0 < t_0 < \ell(\gamma)/2. \tag{4.25}$$

Using the continuity of $f \circ \gamma$ we let $t_0 \to 0$ and obtain the desired estimate as $\gamma(0) = x$ and the right-hand side of (4.25) is independent of t_0. □

A dyadic resolution of kernel. We continue with a dyadic resolution of a standard kernel $K \in \mathrm{K}_\Omega^{-m}(\delta)$, where $\Omega \subset \mathbb{R}^n$ is a uniform domain. Let $x_0, y_0 \in \Omega$ be distinct points and let $\{\psi_{\sigma,j}\}_{j \ge \ell}, \{\psi_{\rho,k}\}_{k \ge \ell}$ be m-regular dyadic resolutions of unity along quasihyperbolic geodesics $\sigma : x_0 \curvearrowright y_0$ and $\rho = \sigma^{-1} : y_0 \curvearrowright x_0$, respectively, see Section 4.2. Denote

$$\Omega(x_0, y_0) = B(x_0, r(x_0) \wedge (|x_0 - y_0|/4b)), \quad r(x_0) = \mathrm{dist}(x_0, \partial\Omega)/4b,$$

where the constant $b \ge 1$ is defined in Lemma 4.16. Then, if $j, k \ge \ell = \ell(x_0, y_0)$ and $(x, y) \in \Omega(x_0, y_0) \times \Omega(y_0, x_0) \subset \Omega \times \Omega$, we denote

$$K_{j,k}^{\sigma,\rho}(x, y) = \int_\Omega \psi_{\sigma,j}(\alpha - x) \int_\Omega K(\alpha, \omega)\psi_{\rho,k}(\omega - y) d\omega d\alpha. \tag{4.26}$$

Now have the following decomposition of the kernel

$$K(x,y)=\sum_{j=\ell}^{\infty}\sum_{k=\ell}^{\infty}K_{j,k}^{\sigma,\rho}(x,y),\quad (x,y)\in\Omega(x_0,y_0)\times\Omega(y_0,x_0). \tag{4.27}$$

To prove (4.27), let us fix $x\in\Omega(x_0,y_0)$ and $y\in\Omega(y_0,x_0)$. Then, using the identity (4.20), we have

$$K(x,y)=\lim_{M\to\infty}\int_{\Omega}\varphi_{\sigma,2^{-M}}(\alpha-x)K(\alpha,y)d\alpha=\sum_{j=\ell}^{\infty}\int_{\Omega}\psi_{\sigma,j}(\alpha-x)K(\alpha,y)d\alpha \tag{4.28}$$

Expanding the quantities $K(\alpha,y)$ on the right-hand side of (4.28) we are lead to

$$K(x,y)=\sum_{j=\ell}^{\infty}\int_{\Omega}\psi_{\sigma,j}(\alpha-x)\sum_{k=\ell}^{\infty}\int_{\Omega}K(\alpha,\omega)\psi_{\rho,k}(\omega-y)d\omega d\alpha.$$

The dominated convergence theorem applies to the inner summation and this establishes (4.27). Indeed, if $M\ge\ell$ and $\alpha\in\Omega\setminus\{y\}$, we have

$$\Big|\sum_{k=\ell}^{M}\int_{\Omega}K(\alpha,\omega)\psi_{\rho,k}(\omega-y)d\omega\Big|=\Big|\int_{\Omega}K(\alpha,\omega)\varphi_{\rho,2^{-M}}(\omega-y)d\omega\Big|\le C|\alpha-y|^{-n+m}$$

with C independent of M,α,y. This follows from a case study with $|\alpha-y|\le 2^{-M+2}$ and $|\alpha-y|\ge 2^{-M+2}$. Within these cases we apply Lemma 4.17 and the kernel size estimate $|K(\alpha,\omega)|\le C_K|\alpha-\omega|^{-n+m}$.

Proving regularity of kernels using dyadic resolution. Using the dyadic resolution of a standard kernel $K\in \mathrm{K}_{\Omega}^{-m}(\delta)$ we prove that this kernel has continuous derivatives up to oder m and Hölder continuous derivatives of order m, that is, we have $K\in\mathcal{K}_{\Omega}^{-m}(\delta')$. We rely on almost diagonality estimates that appear also in connection with the $T\chi_\Omega$ theorem.

First we establish molecule-like estimates for images of certain atoms under weakly singular integral operators. Analogous estimates were already obtained in Lemma 3.8 but here the atoms are understood as differences of bump functions composing the m-regular dyadic resolution of unity.

Lemma 4.29. *Let $\Omega\subset\mathbb{R}^n$ be a uniform domain and $T\in \mathrm{SK}_{\Omega}^{-m}(\delta)$ be associated with a kernel $K\in \mathrm{K}_{\Omega}^{-m}(\delta)$ that is decomposed as in (4.27). Let $k>\ell=\ell(y_0,x_0)$, $y\in\Omega(y_0,x_0)$, $z\in\Omega$, and $\beta\in\mathbb{N}_0^n$. Then*

$$|T(\partial_y^{\beta}(\psi_{\rho,k}(\cdot-y)))(z)|\le C2^{k(n+|\beta|-m)}(1+2^k|z-y|)^{-n-\delta},$$

where the constant C depends at most on the parameters n,m,β,K,Ω.

Proof. Denote $\psi(\omega)=\partial_y^{\beta}(\psi_{\rho,k}(\omega-y))$ if $\omega\in\mathbb{R}^n$. It suffices to prove the following

$$|T\psi(z)|\le C2^{k(n+|\beta|-m)}, \tag{4.30}$$

$$|T\psi(z)|\le C2^{k(n+|\beta|-m)}(2^k|z-y|)^{-n-\delta},\quad \text{if } |z-y|\ge(2+C_K)2^{-k+1}, \tag{4.31}$$

where C_K denotes the constant in (1.17). First we prove the inequality (4.30). A trivial modification of the inequality (3.6) and Lemma 4.17 combined show that

$$|T\psi(z)| \le CM(\psi)(z)^{1-m/n}||\psi||_1^{m/n} \le C2^{k(n+|\beta|-m)}.$$

Then we prove (4.31). For this purpose we denote

$$Q_q = Q(\rho(t(k,q)) + y - y_0, 2^{-k}/2b\sqrt{n}) \subset\subset \Omega, \quad \text{if } q \in \{1,2,\dots,p-1\}.$$

Then, applying Lemma 4.17, we get

$$\begin{aligned} |T\psi(z)| &= \left| \int_\Omega K(z,\omega)\psi(\omega)d\omega \right| \\ &\le \sum_{q=1}^{p-1} \left| \int_\Omega K(z,\omega)\partial_y^\beta(\varphi_{\rho,t(k,q)}(\omega-y) - \varphi_{\rho,t(k,q+1)}(\omega-y))d\omega \right|. \end{aligned}$$

Lemma 4.18 and Lemma 4.17 combined allows us to continue as follows

$$\begin{aligned} |T\psi(z)| &= \sum_{q=1}^{p-1} \inf_{P_q\in\mathcal{P}_m(\mathbb{R}^n)} \left| \int_\Omega (K(z,\omega) - P_q(\omega)) \right. \\ &\qquad\qquad \left. \times \partial_y^\beta(\varphi_{\rho,t(k,q)}(\omega-y) - \varphi_{\rho,t(k,q+1)}(\omega-y))d\omega \right| \\ &\le C2^{k(n+|\beta|)} \sum_{q=1}^{p-1} \inf_{P_q\in\mathcal{P}_m(\mathbb{R}^n)} \int_{Q_q} |K(z,\omega) - P_q(\omega)|d\omega. \end{aligned}$$

Let $q \in \{1,2,\dots,p-1\}$. Using the inequality $t(k,q) \le 2^{-k+1}$ it is simple to verify that

$$\max\{|z-y|/2, C_K \operatorname{diam}(Q_q)\} \le |z-y^q|, \quad y^q = \rho(t(k,q)) + y - y_0. \tag{4.32}$$

Applying Theorem 3.5 and utilizing the estimate (1.17) with the aid of (4.32), we get

$$\begin{aligned} |T\psi(z)| &\le C2^{k(n+|\beta|)} \sum_{q=1}^{p-1} \sup_{|h|\le \operatorname{diam}(Q_q)} \int_{Q_q} |\Delta_h^{m+1}(K(z,\cdot),Q_q,\omega)|d\omega \\ &\le C2^{k(|\beta|-m-\delta)}|z-y|^{-n-\delta} = C2^{k(n+|\beta|-m)}(2^k|z-y|)^{-n-\delta}. \end{aligned}$$

This is the required estimate (4.31). □

We continue in the spirit of almost diagonality, Lemma 3.26 to be more precise. One important difference is that the cancellation conditions $T1 = 0 = T^*1$ are not needed. This is because of the restriction $j,k \ge \ell$ below.

Lemma 4.33. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and $T \in \mathrm{SK}_\Omega^{-m}(\delta)$ be associated with a kernel $K \in \mathrm{K}_\Omega^{-m}(\delta)$ that is decomposed as in (4.27). Let $j,k \ge \ell = \ell(x_0,y_0)$. Then the summands in this decomposition enjoy the regularity*

$$K_{j,k}^{\sigma,\rho} \in C^\infty(\Omega(x_0,y_0)\times\Omega(y_0,x_0))$$

and, if $\alpha, \beta \in \mathbb{N}_0^n$ *and* $(x, y) \in \Omega(x_0, y_0) \times \Omega(y_0, x_0)$*, they satisfy the estimate*

$$|\partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(x, y)| \le C 2^{-\delta|j-k|+n(j\wedge k)-(m-|\alpha|-|\beta|)(j\vee k)}\big(1 + (2^j \wedge 2^k)|x - y|\big)^{-n-\delta}, \tag{4.34}$$

where the constant C *depends at most on* $n, m, \alpha, \beta, K, \Omega$.

Proof. Differentiating (4.26) under the integral signs we obtain the identity

$$\partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(x, y) = \langle \partial_x^\alpha(\psi_{\sigma,j}(\cdot - x)), T(\partial_y^\beta(\psi_{\rho,k}(\cdot - y)))\rangle.$$

Assume first that $k = j = \ell$. Then, applying Lemma 4.17 and computing as in the proof of the inequality (4.30), we get

$$|\partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(x, y)| \le C 2^{|\alpha|j+|\beta|k} 2^{k(n-m)}.$$

This is as desired because $2^\ell|x - y| \le 2^{\ell+1}|x_0 - y_0| \le 64$. Then we assume that $k \ge j \ge \ell$, $k \ne \ell$, and note that remaining case, where $j \ge k \ge \ell$ and $j \ne \ell$, is completely symmetric because the transpose kernel $K^t = (x, y) \mapsto K(y, x)$ belongs also to the class $\mathrm{K}_\Omega^{-m}(\delta)$. Denote

$$T_{\rho,k} : \mathbb{R}^n \to \mathbb{C} : z \mapsto T(\partial_y^\beta(\psi_{\rho,k}(\cdot - y)))(z)\chi_\Omega(z).$$

Then we do a change of variables

$$\begin{aligned} \partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(x, y) &= (-1)^{|\alpha|} \int_\Omega (\partial^\alpha \psi_{\sigma,j})(z - x) T_{\rho,k}(z) dz \\ &= (-1)^{|\alpha|} 2^{-nk} \int_{\mathbb{R}^n} (\partial^\alpha \psi_{\sigma,j})(y - x - 2^{-k}z) T_{\rho,k}(y - 2^{-k}z) dz \\ &= (-1)^{|\alpha|} 2^{nj-mk} \int_{\mathbb{R}^n} f\big(R^{-1}(z_0 - z)\big) g(z) dz, \end{aligned} \tag{4.35}$$

where $R = 2^{k-j}$, $z_0 = 2^k(y - x)$,

$$f(z) = 2^{-nj}(\partial^\alpha \psi_{\sigma,j})(2^{-j}z), \quad g(z) = 2^{-k(n-m)} T_{\rho,k}(y - 2^{-k}z).$$

Lemma 4.29 implies the estimate $|g(z)| \le C 2^{k|\beta|}(1 + |z|)^{-n-\delta}$ if $z \in \mathbb{R}^n$. Applying Lemma 4.17, we have

$$\operatorname{supp} f \subset 2^j \operatorname{supp}(\partial^\alpha \psi_{\sigma,j}) \subset 2^j B(0, 2^{-j+2}) = B(0, 4)$$

and $||f||_{L^\infty(\mathbb{R}^n)} \le C 2^{j|\alpha|}$. Taking also the inequality $4R \le |z_0|/2$ into account we have $\operatorname{supp} f(R^{-1}(z_0 - \cdot)) \subset B(z_0, 4R) \subset B(z_0, |z_0|/2)$. Using these estimates, we have

$$\left| \int_{\mathbb{R}^n} f\big(R^{-1}(z - z_0)\big) g(z) dz \right| \le C 2^{j|\alpha|+k|\beta|} |z_0|^{-n-\delta} R^n = C 2^{\delta(j-k)+j|\alpha|+k|\beta|} (2^j|x - y|)^{-n-\delta}.$$

This combined with (4.35) and the inequalities

$$j|\alpha| \le k|\alpha|, \quad 2^j|x - y| \ge 2^{j-1}|x_0 - y_0| \ge 2^{\ell-1}|x_0 - y_0| \ge 8$$

implies the desired estimate (4.34). □

In addition to the almost diagonality estimates we need the following simple lemma which is used later with the Weierstrass M–test.

Lemma 4.36. *Let $\beta > \alpha > 0$ and $\lambda, \varepsilon > 0$. Then*

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-\varepsilon|j-k|+\alpha(j\wedge k)}\big(1+(2^k\wedge 2^j)\lambda\big)^{-\beta} \le C\lambda^{-\alpha},$$

where the constant C depends at most on the parameters $\alpha, \beta, \varepsilon$.

Proof. Denote $s_{kj} = 2^{-\varepsilon|j-k|+\alpha(j\wedge k)}\big(1+(2^k\wedge 2^j)\lambda\big)^{-\beta}$. Notice that $s_{jk} = s_{kj}$ if $j, k \in \mathbb{Z}$. Therefore

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} s_{jk} = \sum_{j\in\mathbb{Z}}\Big(\sum_{k\le j}+\sum_{k>j}\Big)s_{jk} = \sum_{j\in\mathbb{Z}}\sum_{k\le j} s_{jk} + \sum_{j\in\mathbb{Z}}\sum_{k<j} s_{jk} \le 2\sum_{j\in\mathbb{Z}}\sum_{k\le j} s_{jk}.$$

It remains to estimate the sum on the right hand side; we have

$$\sum_{j\in\mathbb{Z}}\sum_{k\le j} s_{jk} = \sum_{j\in\mathbb{Z}}\sum_{k\le j} 2^{-\varepsilon|j-k|+k\alpha}(1+2^k\lambda)^{-\beta} \le C_\varepsilon \sum_{k\in\mathbb{Z}} 2^{k\alpha}(1+2^k\lambda)^{-\beta}.$$

Now choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \le \lambda^{-1} < 2^{k_0+1}$. Next we write

$$\sum_{k\in\mathbb{Z}} 2^{k\alpha}(1+2^k\lambda)^{-\beta} = \Big(\sum_{k\le k_0}+\sum_{k>k_0}\Big)2^{k\alpha}(1+2^k\lambda)^{-\beta} = \Sigma_1+\Sigma_2.$$

For the first term we have the estimate

$$\Sigma_1 \le \sum_{k\le k_0} 2^{k\alpha} = 2^{k_0\alpha}\sum_{k\le k_0} 2^{(k-k_0)\alpha} = C_\alpha 2^{k_0\alpha} \le C_\alpha\lambda^{-\alpha}.$$

Note that we used the inequality $\alpha > 0$. For the second term we have

$$\begin{aligned}\Sigma_2 &\le \sum_{k>k_0} 2^{k(\alpha-\beta)}(2^{-k}+\lambda)^{-\beta} \le \lambda^{-\beta}\sum_{k>k_0} 2^{k(\alpha-\beta)}\\ &\le \lambda^{-\beta}2^{k_0(\alpha-\beta)}\sum_{k>k_0} 2^{(k-k_0)(\alpha-\beta)} \le C_{\alpha,\beta}\lambda^{-\beta+\beta-\alpha} = C_{\alpha,\beta}\lambda^{-\alpha}.\end{aligned}$$

Note that we used the inequality $\beta > \alpha$. Combining the estimates above for Σ_1 and Σ_2 we find that the desired conclusion holds true. □

We are ready for the proof of kernel regularity.

Theorem 4.37. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and $K \in \mathrm{K}_\Omega^{-m}(\delta)$ be a standard kernel so that $0 < m < n$ and $0 < \delta < 1$. Then K is smooth, that is,*

$$K \in \mathcal{K}_\Omega^{-m}(\delta'), \qquad \textit{if } 0 < \delta' < \delta.$$

As a consequence, if $\Omega = \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$ satisfy $|\alpha|+|\beta| = m$, then

$$\partial_x^\alpha \partial_y^\beta K : \mathbb{R}^n\times\mathbb{R}^n \setminus \{(x,x)\} \to \mathbb{C}$$

is a Calderón–Zygmund standard kernel. These are defined in connection with (1.9).

Proof. Throughout the proof C denotes a constant depending on $m, n, \delta, \delta', K, \Omega$. Let $x_0, y_0 \in \Omega$ be distinct and $\ell = \ell(x_0, y_0)$. We use the decomposition (4.27) extensively. Let $(x, y) \in \Omega(x_0, y_0) \times \Omega(y_0, x_0)$ and $|\alpha| + |\beta| \leq m$. Then $|x - y| \geq |x_0 - y_0|/2$ and, combining this estimate with Lemma 4.33 and Lemma 4.36, we obtain

$$\begin{aligned}
&\sum_{j=\ell}^{\infty}\sum_{k=\ell}^{\infty} |\partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(x,y)| \\
&\leq C \sum_{j=\ell}^{\infty}\sum_{k=\ell}^{\infty} 2^{-\delta|j-k|+n(j\wedge k)-(m-|\alpha|-|\beta|)(j\vee k)} \big(1 + (2^j \wedge 2^k)|x-y|\big)^{-n-\delta} \\
&\leq C \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-\delta|j-k|+(n+|\alpha|+|\beta|-m)(j\wedge k)} \big(1 + (2^j \wedge 2^k)|x_0-y_0|\big)^{-n-\delta} \\
&\leq C|x_0 - y_0|^{m-n-|\alpha|-|\beta|}.
\end{aligned} \tag{4.38}$$

The Weierstrass M–test, combined with the identity (4.27), shows that

$$K|(\Omega(x_0,y_0)\times\Omega(y_0,x_0)) = \sum_{j=\ell}^{\infty}\sum_{k=\ell}^{\infty} K_{j,k}^{\sigma,\rho} \in C^m(\Omega(x_0,y_0)\times\Omega(y_0,x_0)) \tag{4.39}$$

and the series can be differentiated termwise up to the order m. As a consequence of this identity we have the regularity $K \in C^m(\Omega\times\Omega\setminus\{(x,x)\}$ and, by using (4.38), we also have the estimate

$$|\partial_x^\alpha \partial_y^\beta K(x_0,y_0)| \leq C|x_0-y_0|^{m-n-|\alpha|-|\beta|}, \quad |\alpha|+|\beta| \leq m, \tag{4.40}$$

which is the required size-estimate for smooth kernels.

We turn to Hölder-regularity estimates which are required for smooth kernels. Due to symmetry it suffices to consider differences in the first $\mathbb{R}^n$-variable only. To begin with consider the situation, where $x_0, y_0 \in \Omega$ are distinct points, $|\alpha|+|\beta| = m$, and $h \in \mathbb{R}^n$ is close to x_0 so that $x_0 + h \in \Omega(x_0, y_0)$. Fix $j, k \geq \ell = \ell(x_0, y_0)$ and denote

$$\Delta_h^1(\partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(\cdot,y_0),x_0) = \partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(x_0+h,y_0) - \partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(x_0,y_0).$$

Applying the mean value theorem and Lemma 4.33 we see that there is a point $\xi \in \mathbb{R}^n$, belonging to the line segment $[x_0, x_0 + h] \subset \Omega(x_0, y_0)$, so that $|\xi - y_0| \geq |x_0 - y_0|/2$ and

$$\begin{aligned}
|\Delta_h^1(\partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(\cdot,y_0),x_0)| &\leq |h||\nabla_x(\partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(\xi,y_0))| \\
&\leq C|h|2^{-\delta|j-k|+n(j\wedge k)+(j\vee k)} \big(1+(2^j\wedge 2^k)|x_0-y_0|\big)^{-n-\delta}.
\end{aligned} \tag{4.41}$$

Using the triangle inequality and Lemma 4.33, we also have the estimate

$$|\Delta_h^1(\partial_x^\alpha \partial_y^\beta K_{j,k}^{\sigma,\rho}(\cdot,y_0),x_0)| \leq C2^{-\delta|j-k|+n(j\wedge k)} \big(1+(2^j\wedge 2^k)|x_0-y_0|\big)^{-n-\delta}. \tag{4.42}$$

Multiplying suitable powers of these two estimates (4.41) and (4.42), we get

$$\begin{aligned}(4.43)\qquad &|\Delta_h^1(\partial_x^\alpha\partial_y^\beta K_{j,k}^{\sigma,\rho}(\cdot,y_0),x_0)| \\ &\le C|h|^{\delta'}2^{-\delta|j-k|+n(j\wedge k)+\delta'(j\vee k)}\big(1+(2^j\wedge 2^k)|x_0-y_0|\big)^{-n-\delta},\end{aligned}$$

where

$$-\delta|j-k|+n(j\wedge k)+\delta'(j\vee k)=-(\delta-\delta')|j-k|+(n+\delta')(j\wedge k).$$

Summing the estimates (4.43) and applying Lemma 4.36 we have

$$\begin{aligned}&\sum_{j=\ell}^{\infty}\sum_{k=\ell}^{\infty}|\partial_x^\alpha\partial_y^\beta K_{j,k}^{\sigma,\rho}(x_0+h,y_0)-\partial_x^\alpha\partial_y^\beta K_{j,k}^{\sigma,\rho}(x_0,y_0)| \\ (4.44)\qquad &\le C|h|^{\delta'}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}2^{-(\delta-\delta')|j-k|+(n+\delta')(j\wedge k)}\big(1+(2^j\wedge 2^k)|x_0-y_0|\big)^{-n-\delta} \\ &\le C|h|^{\delta'}|x_0-y_0|^{-n-\delta'}.\end{aligned}$$

Combining (4.39) and (4.44) it follows that

(4.45)

$$|\partial_x^\alpha\partial_y^\beta K(x_0+h,y_0)-\partial_x^\alpha\partial_y^\beta K(x_0,y_0)|\le C|h|^{\delta'}|x_0-y_0|^{-n-\delta'},\quad x_0+h\in\Omega(x_0,y_0).$$

Due to limitations on h this does not suffice for the Hölder estimate which is required for smooth kernels. We prove the full Hölder-estimate with the aid of (4.45) and geometric properties of uniform domains, captured in Theorem 4.22. Fix $h\in\mathbb{R}^n$ such that $|h|\le|x_0-y_0|/4b$ and $x_0+h\in\Omega$. Join x_0 and x_0+h with a path γ as in Definition 1.13. It is straightforward to verify that, if $t\in[0,\ell(\gamma)]$ and $k\in B(0,\min(t,\ell(\gamma)-t)/4b^2)$, we have

- $\gamma(t)+k\in\Omega(\gamma(t),y_0)$,
- $|\gamma(t)-y_0|^{-n-\delta'}\le C_n|x_0-y_0|^{-n-\delta'}$.

Thus, applying (4.45), we get the estimate

$$|\partial_x^\alpha\partial_y^\beta K(\gamma(t)+k,y_0)-\partial_x^\alpha\partial_y^\beta K(\gamma(t),y_0)|\le C|k|^{\delta'}|x_0-y_0|^{-n-\delta'}$$

for $t\in[0,\ell(\gamma)]$ and $k\in B(0,\min(t,\ell(\gamma)-t)/4b^2)$. Also, the function

$$s\mapsto\partial_x^\alpha\partial_y^\beta K(\gamma(s),y_0):[0,\ell(\gamma)]\to\mathbb{C}$$

is continuous. Therefore we can invoke Theorem 4.22 to conclude that (4.45) holds true if $x_0,y_0,x_0+h\in\Omega$ and $|h|\le|x_0-y_0|/4b$. In the remaining case $|x_0-y_0|/4b\le|h|\le|x_0-y_0|/2$ we use (4.40). □

Combining Proposition 4.6 and Theorem 4.37, we get the following corollary.

Corollary 4.46. *Let $\Omega\subset\mathbb{R}^n$ be a uniform domain and $0<m<n$. Then*

$$\bigcup_{0<\delta<1}\mathcal{K}_\Omega^{-m}(\delta)=\bigcup_{0<\delta<1}k_{\mathrm{loc}}^{m+\delta}(\Omega)=\bigcup_{0<\delta<1}\mathrm{K}_\Omega^{-m}(\delta).$$

5. Extension of smooth kernels

We arrive at the second part of our solution to the kernel extension problem. This part consists of an extension result for the smooth kernels $\mathcal{K}_\Omega^{-m}(\delta)$. Decomposing these kernels using a partition of unity, subordinate to the Whitney decomposition of the open set

$$\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\} \subset \mathbb{R}^{2n},$$

yields a characterization of smooth kernels in terms of so called kernel atoms. The purpose of this characterization is that it can be used to reduce the kernel extension problem to the Hölder extension of individual kernel atoms.

We begin with the definition of Hölder spaces and formulate the required Hölder extension results when the underlying domain is uniform.

5.1. **Hölder spaces.**

Here we define Hölder spaces on general domains and then establish extension results for these spaces on uniform domains via certain pointwise error estimates in polynomial approximation. We also consider a measure theoretical polynomial approximation, leading to the so called local smoothness spaces. These spaces are useful in showing inclusions of Hölder spaces to BMO-type spaces.

Definition 5.1. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a domain and $0 < \delta < 1$. Define the *δ-Hölder seminorm* of $f : \Omega \to \mathbb{C}$ by

$$|f|_{C^\delta(\Omega)} = \sup\left\{ \frac{|f(x) - f(y)|}{|x-y|^\delta} \,:\, x, y \in \Omega,\, x \neq y \right\}.$$

Define the *local δ-Hölder seminorm* of $f : \Omega \to \mathbb{C}$ by

$$|f|_{C^\delta_{\mathrm{loc}}(\Omega)} = \sup_{Q \subset\subset \Omega} |f|_{C^\delta(Q)},$$

where the supremum is over all of the open cubes Q compactly contained in Ω.

Definition 5.2. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a domain. Let $m \in \mathbb{N}_0$ and $0 < \delta < 1$. The *local Hölder space* $C^{m,\delta}_{\mathrm{loc}}(\Omega)$ is the Banach space of complex-valued functions $f \in C^m(\Omega)$ satisfying

$$||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)} = \sum_{|\alpha| \le m} ||\partial^\alpha f||_{L^\infty(\Omega)} + \sum_{|\alpha| = m} |\partial^\alpha f|_{C^\delta_{\mathrm{loc}}(\Omega)} < \infty.$$

Definition 5.3. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a domain and $m \in \mathbb{N}_0$. Denote by $C^m(\overline{\Omega})$ the space of continuous complex-valued functions $f : \overline{\Omega} \to \mathbb{C}$ so that $f|\Omega \in C^m(\Omega)$ and the derivatives $\partial^\alpha(f|\Omega)$, $|\alpha| \le m$, have extensions to continuous functions $\overline{\Omega} \to \mathbb{C}$ that are also denoted by $\partial^\alpha f$. Let $0 < \delta < 1$. The *Hölder space* $C^{m,\delta}(\overline{\Omega})$ is the Banach space of functions $f \in C^m(\overline{\Omega})$ satisfying

$$||f||_{C^{m,\delta}(\overline{\Omega})} = \sum_{|\alpha| \le m} ||\partial^\alpha f||_{L^\infty(\Omega)} + \sum_{|\alpha| = m} |\partial^\alpha f|_{C^\delta(\Omega)} < \infty.$$

Remark 5.4. Let $f \in C^{m,\delta}(\overline{\Omega})$. Then $||\partial^\alpha f||_{L^\infty(\Omega)} = ||\partial^\alpha f||_{L^\infty(\overline{\Omega})}$ if $|\alpha| \le m$ by using the continuity of $\partial^\alpha f : \overline{\Omega} \to \mathbb{C}$. Using the continuity of order m derivatives, we also have their δ-Hölder continuity up to the boundary, that is, if $|\alpha| = m$ then

$$|\partial^\alpha f(x) - \partial^\alpha f(y)| \le ||f||_{C^{m,\delta}(\overline{\Omega})}|x-y|^\delta, \quad \text{if } x, y \in \overline{\Omega}.$$

Extension of Hölder functions. Let $\Omega \subset \mathbb{R}^n$ be a domain and $f \in C^{m,\delta}_{\mathrm{loc}}(\Omega)$. Let also $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le m$, and $y \in Q \subset\subset \Omega$, where Q is an open cube. Then $\partial^\alpha f$ admits a polynomial approximation by the Taylor polynomials $P_\alpha f(\cdot, y) \in \mathcal{P}_{m-|\alpha|}(\mathbb{R}^n)$ so that, if $x \in Q$, we have

$$\partial^\alpha f(x) = P_\alpha f(x,y) + R_\alpha f(x,y), \quad |R_\alpha f(x,y)| \le C||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}|x-y|^{m+\delta-|\alpha|}. \tag{5.5}$$

These estimates follow by using the Taylor formula along the line-segment joining x and y in the cube Q.

It turns out that, in the case of uniform domains, these local estimates bootstrap the corresponding non-local estimates so that (5.5) holds true for every $x \in \Omega$. These non-local estimates serve as a starting point for the following results: assuming that $\Omega \subset \mathbb{R}^n$ is a uniform domain, there are bounded extension operators

$$C^{m,\delta}_{\mathrm{loc}}(\Omega) \to C^{m,\delta}(\overline{\Omega}), \quad C^{m,\delta}(\overline{\Omega}) \to C^{m,\delta}(\mathbb{R}^n).$$

First we define the Taylor polynomials and the corresponding error terms.

Definition 5.6. Let $\emptyset \ne \Omega \subset \mathbb{R}^n$ be a domain and $f \in C^{m,\delta}_{\mathrm{loc}}(\Omega)$. Let $\alpha \in \mathbb{N}_0^n$ satisfy $|\alpha| \le m$. Then the Taylor polynomial $P_\alpha f$ and the error term $R_\alpha f$ are defined in terms of the identity

$$\partial^\alpha f(x) = \sum_{|\alpha+\beta| \le m} \frac{\partial^{\alpha+\beta} f(y)}{\beta!}(x-y)^\beta + R_\alpha f(x,y) = P_\alpha f(x,y) + R_\alpha f(x,y), \tag{5.7}$$

where $x, y \in \Omega$. Assuming that $f \in C^{m,\delta}(\overline{\Omega})$ and $|\alpha| \le m$, the Taylor polynomial $P_\alpha f$ and error term $R_\alpha f$ are defined by (5.7) if $x, y \in \overline{\Omega}$.

Let $f \in C^{m,\delta}_{\mathrm{loc}}(\Omega)$. Then, if $|\alpha| \le m$ and $x, y \in \Omega$, we have the identity $P_\alpha f(x,y) = \partial_x^\alpha P_{(0,\dots,0)} f(x,y)$. Differences of this Taylor polynomial (in the second $\mathbb{R}^n$-variable) are related to the error terms as follows

$$P_\alpha f(x,a) - P_\alpha f(x,b) = \sum_{|\alpha+\beta| \le m} \frac{R_{\alpha+\beta} f(a,b)}{\beta!}(x-a)^\beta, \quad x, a, b \in \Omega. \tag{5.8}$$

The identity (5.8) is the Taylor expansion of the polynomial $P_\alpha f(\cdot, a) - P_\alpha f(\cdot, b)$ about the point a. Indeed, if $|\alpha + \beta| \le m$, we have

$$\begin{aligned} R_{\alpha+\beta} f(a,b) &= \partial^{\alpha+\beta} f(a) - P_{\alpha+\beta} f(a,b) \\ &= P_{\alpha+\beta} f(a,a) - P_{\alpha+\beta} f(a,b) = \partial_x^\beta \big(P_\alpha f(x,a) - P_\alpha f(x,b)\big)\big|_{x=a}. \end{aligned}$$

For $f \in C^{m,\delta}(\overline{\Omega})$ the identity (5.8) holds true if $x, a, b \in \overline{\Omega}$. The following lemma provides semilocal control for the error terms.

Lemma 5.9. *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a domain and $f \in C^{m,\delta}_{\mathrm{loc}}(\Omega)$. Assume that $x, y \in Q$ are two points in an open cube $Q \subset\subset \Omega$. Then, assuming that $\alpha \in \mathbb{N}_0^n$ satisfies $|\alpha| \leq m$, the error term $R_\alpha f(x,y)$ defined in* (5.7) *satisfies*

$$|R_\alpha f(x,y)| \leq 2||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}|x-y|^{m+\delta-|\alpha|}.$$

Proof. Looking at real and imaginary parts separately we can assume that f is real-valued. The line-segment L, joining the points x, y, is cointained in the cube $Q \subset\subset \Omega$. Hence the multivariate Taylor's formula, applied to the real-valued function $\partial^\alpha f \in C^{m-|\alpha|}(\Omega)$, implies that there exists a point $\xi \in L \subset Q$ such that

$$\begin{aligned}\partial^\alpha f(x) &= \sum_{|\alpha+\beta|\leq m-1} \frac{\partial^{\alpha+\beta} f(y)}{\beta!}(x-y)^\beta + \sum_{|\alpha+\beta|=m} \frac{\partial^{\alpha+\beta} f(\xi)}{\beta!}(x-y)^\beta \\ &= P_\alpha f(x,y) + \sum_{|\alpha+\beta|=m} \left(\frac{\partial^{\alpha+\beta} f(\xi) - \partial^{\alpha+\beta} f(y)}{\beta!}\right)(x-y)^\beta.\end{aligned}$$

As a consequence, we have

$$\begin{aligned}|R_\alpha f(x,y)| &= |\partial^\alpha f(x) - P_\alpha f(x,y)| \\ &= \Big| \sum_{|\alpha+\beta|=m} \left(\frac{\partial^{\alpha+\beta} f(\xi) - \partial^{\alpha+\beta} f(y)}{\beta!}\right)(x-y)^\beta \Big| \\ &\leq \sum_{|\alpha+\beta|=m} \frac{|\partial^{\alpha+\beta} f(\xi) - \partial^{\alpha+\beta} f(y)|}{\beta!}|x-y|^{|\beta|} \leq ||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}|x-y|^{m+\delta-|\alpha|}.\end{aligned}$$

This is as required. □

In case of uniform domains these semilocal error estimates imply the corresponding error estimates uniformly in the whole the domain. This result, which is formulated and proven below, can be interpreted as a higher order analogue of that the identity operator maps

$$\mathrm{loc}\,\mathrm{Lip}_\delta(\Omega) \to \mathrm{Lip}_\delta(\overline{\Omega})$$

boundedly if $0 < \delta \leq 1$. This latter result is due to Gehring and Martio [GM85]. We omit the formal definition of the local spaces $\mathrm{loc}\,\mathrm{Lip}_\delta(\Omega)$ but later we define the non-local spaces $\mathrm{Lip}_\delta(F)$ on general closed sets $F \subset \mathbb{R}^n$.

To prepare for the following proof we denote by $Q(x,r) \subset \mathbb{R}^n$ the unique open cube, sides parallel to the coordinate axes, and satisfying

$$B(x,r) \subset Q(x,r) \subset B(x,\sqrt{n}r). \tag{5.10}$$

Hence $Q(x,r)$ is centered at the point x and has side-length $2r$.

Theorem 5.11. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and denote by $a_\Omega \geq 1$ the uniformity constant as in Definition 1.13. Let $f \in C^{m,\delta}_{\mathrm{loc}}(\Omega)$, $|\alpha| \leq m \in \mathbb{N}_0$, and $x, y \in \Omega$. Then the error term $R_\alpha f(x,y)$ satisfies*

$$|R_\alpha f(x,y)| \leq C||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}|x-y|^{m+\delta-|\alpha|}, \tag{5.12}$$

where the constant C depends at most on the parameters n, m, δ, a_Ω. As a consequence, if $f \in C^{m,\delta}(\overline{\Omega})$, $|\alpha| \le m$, and $x, y \in \overline{\Omega}$, then the error estimate

$$|R_\alpha f(x,y)| \le C||f||_{C^{m,\delta}(\overline{\Omega})}|x-y|^{m+\delta-|\alpha|} \tag{5.13}$$

holds true with the same constant C as above.

Proof. We verify (5.12) assuming that $f \in C^{m,\delta}_{\mathrm{loc}}(\Omega)$ and $x, y \in \Omega$. The estimate (5.13) for $C^{m,\delta}(\overline{\Omega})$ and $x, y \in \overline{\Omega}$ then follows from (5.12) by continuity.

Let $\gamma : [0, \ell(\gamma)] \to \Omega$ be a path as in Definition 1.13 with $\gamma(0) = x$ and $\gamma(\ell(\gamma)) = y$. Define

$$g : \Omega \to \mathbb{C} : g(z) = R_\alpha f(x,z) = \partial^\alpha f(x) - P_\alpha f(x,z).$$

Clearly the composition $g \circ \gamma : [0, \ell(\gamma)] \to \mathbb{C}$ is continuous. Let $t \in [0, \ell(\gamma)]$ and consider any $k \in \mathbb{R}^n$ with the property $[\gamma(t)+k \in B(\gamma(t), \min(t, \ell(\gamma)-t)/(2a_\Omega\sqrt{n}))$. According to (5.10) and the Definition 1.13 of uniformity, we have

$$\gamma(t) + k \in Q(\gamma(t), \min(t, \ell(\gamma)-t)/(2a_\Omega\sqrt{n})) \subset \bar{B}(\gamma(t), \min(t, \ell(\gamma)-t)/2a_\Omega) \subset \Omega.$$

Furthermore, applying the identity (5.8) and Lemma 5.9, we have the estimate

$$\begin{aligned}|g(\gamma(t)+k) - g(\gamma(t))| &= \Big| \sum_{|\alpha+\beta|\le m} \frac{R_{\alpha+\beta} f(\gamma(t), \gamma(t)+k)}{\beta!}(x-\gamma(t))^\beta \Big| \\ &\le 2||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)} \sum_{|\alpha+\beta|\le m} |k|^{m+\delta-|\alpha|-|\beta|}\ell(\gamma)^{|\beta|} \le C_m a_\Omega^{m-|\alpha|}||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}|x-y|^{m-|\alpha|}|k|^\delta.\end{aligned}$$

Applying Theorem 4.22 with $c = 2a_\Omega\sqrt{n} > a_\Omega$, we get the required estimate

$$|R_\alpha f(x,y)| = |R_\alpha f(x,x) - R_\alpha f(x,y)| = |g(x) - g(y)| \le C||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}|x-y|^{m+\delta-|\alpha|},$$

where the constant C depends at most on the parameters n, m, δ, a_Ω. □

The uniform error estimates above imply the first extension result.

Theorem 5.14. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with uniformity constant $a_\Omega \ge 1$. Let $f \in C^{m,\delta}_{\mathrm{loc}}(\Omega)$, where $m \in \mathbb{N}_0$ and $0 < \delta < 1$. There is $g \in C^{m,\delta}(\overline{\Omega})$ such that $g|\Omega = f$ and $||g||_{C^{m,\delta}(\overline{\Omega})} \le C||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}$, where the constant C depends at most on the parameters n, m, δ, a_Ω.*

Proof. Let $\alpha \in \mathbb{N}_0^n$ be such that $|\alpha| \le m$. Assume that $x, y \in \Omega$ satisfy $|x-y| \le 1$. Then using Theorem 5.11, we get the following estimate

$$\begin{aligned}|\partial^\alpha f(x) - \partial^\alpha f(y)| &\le \sum_{|\alpha|<|\alpha+\beta|\le m} \frac{|\partial^{\alpha+\beta} f(y)|}{\beta!}|x-y|^{|\beta|} + |R_\alpha f(x,y)| \\ &\le C||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}(|x-y| + |x-y|^{m+\delta-|\alpha|}),\end{aligned}$$

Hence $\partial^\alpha f : \Omega \to \mathbb{C}$ is uniformly continuous and, as such, it has a unique extension to a continuous function $\overline{\Omega} \to \mathbb{C}$. Denote by g the extension of f to a continuous function $\overline{\Omega} \to \mathbb{C}$. Using Theorem 5.11, we see that order m derivatives of g satisfy the non-local Hölder estimate. Indeed, if $x, y \in \Omega$ and $|\alpha| = m$, we have

$$|\partial^\alpha g(x) - \partial^\alpha g(y)| = |\partial^\alpha f(x) - \partial^\alpha f(y)| = |R_\alpha f(x,y)| \le C||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}|x-y|^\delta.$$

The required properties of g follow easily from this estimate and the definitions. □

Theorem 5.11 combined with the classical Whitney extension implies the existence of a bounded and linear extension operator $C^{m,\delta}(\overline{\Omega}) \to C^{m,\delta}(\mathbb{R}^n)$ if $\Omega \subset \mathbb{R}^n$ is a uniform domain. We will prove this next. The following Whitney extension results for Lipschitz functions are from Stein [Ste70, pp. 166–180] but the original work is that of Whitney [Whi34].

The Lipschitz space $\mathrm{Lip}_{m+\delta}(F)$, where $m \in \mathbb{N}_0$, $0 < \delta < 1$, and $F \subset \mathbb{R}^n$ is closed, consists of $f : F \to \mathbb{C}$ for which there are functions $\{f^\alpha : F \to \mathbb{C}\}_{|\alpha|\le m}$ with $f^0 = f$ and

$$f^\alpha(x) = \sum_{|\alpha+\beta|\le m} \frac{f^{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + r_\alpha(x,y), \tag{5.15}$$

where

$$|f^\alpha(x)| \le M \text{ and } |r_\alpha(x,y)| \le M|x-y|^{m+\delta-|\alpha|} \tag{5.16}$$

if $x, y \in F$ and $|\alpha| \le m$. The norm $||f||_{\mathrm{Lip}_{m+\delta}(F)}$ is taken to be the infimum over M for which (5.16) holds for some functions $\{f^\alpha : F \to \mathbb{C}\}_{|\alpha|\le m}$ as above. The extension result of Whitney is that there is a bounded and linear extension operator

$$\mathscr{E}_m f : \mathrm{Lip}_{m+\delta}(F) \to \mathrm{Lip}_{m+\delta}(\mathbb{R}^n) \tag{5.17}$$

whose operator norm is independent of the closed set F. See [Ste70, Theorem 4].

Assume that $f \in \mathrm{Lip}_{m+\delta}(\mathbb{R}^n)$. In this case $f \in C^m(\mathbb{R}^n)$ so that the associated functions $\{f^\alpha : \mathbb{R}^n \to \mathbb{C}\}_{|\alpha|\le m}$ are unique and they are given by $f^\alpha = \partial^\alpha f$. Furthermore, the order m derivatives $f^\alpha = \partial^\alpha f$ are δ-Hölder continuous. To summarize $\mathrm{Lip}_{m+\delta}(\mathbb{R}^n) \subset C^{m,\delta}(\mathbb{R}^n)$ and we have the norm estimate

$$||f||_{C^{m,\delta}(\mathbb{R}^n)} \le C_{n,m}||f||_{\mathrm{Lip}_{(m+\delta}(\mathbb{R}^n)}, \quad f \in \mathrm{Lip}_{m+\delta}(\mathbb{R}^n). \tag{5.18}$$

For the converse, assume that $\Omega \subset \mathbb{R}^n$ is a uniform domain and $F = \overline{\Omega}$. Fix $f \in C^{m,\delta}(\overline{\Omega})$ and denote $f^\alpha = \partial^\alpha f : \overline{\Omega} \to \mathbb{C}$ if $|\alpha| \le m$. Then, using Definition 5.3 and the estimate (5.13) in Theorem 5.11, we see that $f \in \mathrm{Lip}_{m+\delta}(\overline{\Omega})$ and

$$||f||_{\mathrm{Lip}_{m+\delta}(\overline{\Omega})} \le C_{n,m,\delta,a_\Omega}||f||_{C^{m,\delta}(\overline{\Omega})}. \tag{5.19}$$

Finally, using the extension operator (5.17) and the norm-estimates (5.18) and (5.19), we finish the proof of the following extension result.

Theorem 5.20. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with uniformity constant $a_\Omega \ge 1$. Let $m \in \mathbb{N}_0$, $0 < \delta < 1$, and $f \in C^{m,\delta}(\overline{\Omega})$. There is a function $\mathscr{E}_m f : \mathbb{R}^n \to \mathbb{C}$ satisfying $\mathscr{E}_m f|\overline{\Omega} = f$, $\mathscr{E}_m f \in C^{m,\delta}(\mathbb{R}^n)$, and*

$$||\mathscr{E}_m f||_{C^{m,\delta}(\mathbb{R}^n)} \le C||f||_{C^{m,\delta}(\overline{\Omega})}, \tag{5.21}$$

where C depends at most on the parameters n, m, δ, a_Ω. The induced relation $f \mapsto \mathscr{E}_m f$ is a bounded and linear extension operator $\mathcal{E}_m : C^{m,\delta}(\overline{\Omega}) \to C^{m,\delta}(\mathbb{R}^n)$.

Remark 5.22. We need another local-to-global type estimate for Hölder seminorms. Let $\Omega \subset \mathbb{R}^n$ be a uniform domain and $0 < \delta < 1$. Assume that $f : \Omega \to \mathbb{C}$. Proceeding as in the proof of Theorem 5.11, we have $|f|_{C^\delta(\Omega)} \le C|f|_{C^\delta_{\mathrm{loc}}(\Omega)}$, where the constant C depends at most on the parameters δ, Ω.

Local smoothness spaces. Measure theoretical approach to Hölder spaces is furnished by the so called local smoothness spaces $\mathscr{C}^{m+\delta}_\infty(\Omega)$ of DeVore and Sharpley [DS84]. These spaces are based on a generalization of the sharp maximal function which measures the error in local polynomial approximation. Local smoothness spaces emerge naturally while proving inclusions of the form

$$C^{m,\delta}_{\mathrm{loc}}(\Omega) \subset \dot{F}^{m,2}_\infty(\Omega). \tag{5.23}$$

Such inclusions are convenient since the spaces $\dot{F}^{m,2}_\infty(\Omega)$, or their sequence counterparts, appear in the assumptions of our main result which is a $T1$ theorem on admissible domains. In some cases these assumptions can be verified by using (5.23), see later Example 6.21.

If $Q \subset \mathbb{R}^n$ is a cube and $f \in L^1(Q)$, we denote

$$E_m(f, Q) = \inf_{P \in \mathcal{P}_m(\mathbb{R}^n)} \int_Q |f(x) - P(x)|dx, \quad m \in \mathbb{N}_0.$$

The local smoothness spaces are defined in terms of these error measurements.

Definition 5.24. Let $\emptyset \ne \Omega \subset \mathbb{R}^n$ be a domain, $m \in \mathbb{N}_0$, and $0 < \delta < 1$. Let $f \in L^1_{\mathrm{loc}}(\Omega)$ and define a seminorm

$$|f|_{\mathscr{C}^{m+\delta}_\infty(\Omega)} = \sup_{Q \subset\subset \Omega} |Q|^{-(1+(m+\delta)/n)} E_m(f, Q),$$

where the supremum is taken over all cubes Q, compactly contained in Ω. Define a norm $||f||_{\mathscr{C}^{m+\delta}_\infty(\Omega)} = ||f||_{L^\infty(\Omega)} + |f|_{\mathscr{C}^{m+\delta}_\infty(\Omega)}$. The *local smoothness space* $\mathscr{C}^{m+\delta}_\infty(\Omega)$ consists of $f \in L^1_{\mathrm{loc}}(\Omega)$ for which $||f||_{\mathscr{C}^{m+\delta}_\infty(\Omega)} < \infty$.

We use the following relation between the local smoothness and Hölder spaces.

Theorem 5.25. *Let $\emptyset \ne \Omega \subset \mathbb{R}^n$ be a domain, $m \in \mathbb{N}_0$, and $0 < \delta < 1$. Then we have a bounded inclusion*

$$C^{m,\delta}_{\mathrm{loc}}(\Omega) \subset \mathscr{C}^{m+\delta}_\infty(\Omega),$$

where the implicit constant depends at most on the parameters n, m.

Proof. We invoke the Whitney approximation theorem 3.5. It implies that there exists a constant C depending at most on m and n such that, for each open cube $Q \subset \mathbb{R}^n$ and each $f \in L^1(Q)$,

$$E_m(f, Q) \le C_{n,m}|Q| \sup_{|h| \le \mathrm{diam}(Q)} ||\Delta^{m+1}_h(f, Q, \cdot)||_{L^\infty(Q)}. \tag{5.26}$$

Let $f \in C^{m,\delta}_{\mathrm{loc}}(\Omega)$. Fix an open cube $Q \subset\subset \Omega$ and $x, h \in \mathbb{R}^n$ so that $\Delta_h^{m+1}(f, Q, x) \neq 0$. Then it must be that $\{x, x+h, \cdots, x+(m+1)h\} \subset Q$, and the convexity of Q implies that the line-segment connecting x to $x+(m+1)h$ is contained in Q. This allows us to iterate the identity $f(x+h) - f(x) = \int_0^1 h \cdot \nabla f(x+\theta h) d\theta$ for

(5.27)
$$\Delta_h^{m+1}(f, Q, x) = \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} h_{j_1} \cdots h_{j_m} A_{j_1,\ldots,j_m},$$
$$A_{j_1,\ldots,j_m} = \int_0^1 \cdots \int_0^1 \Delta_h^1(\partial_{j_1} \cdots \partial_{j_m} f, x + (\theta_1 + \cdots + \theta_m)h) d\theta_1 \cdots d\theta_m.$$

Taking the absolute values, we obtain the norm-estimate

$$\sup_{h \neq 0} ||\Delta_h^{m+1}(f, Q, \cdot)||_{L^\infty(Q)} \leq |h|^{m+\delta} \sum_{|\alpha|=m} c_{\alpha,m} |\partial^\alpha f|_{C^\delta_{\mathrm{loc}}(\Omega)} \leq C_{n,m} |Q|^{(m+\delta)/n} ||f||_{C^{m,\delta}_{\mathrm{loc}}(\Omega)}.$$

Using also (5.26), we see that the inclusion $C^{m,\delta}_{\mathrm{loc}}(\Omega) \subset \mathscr{C}^{m+\delta}_\infty(\Omega)$ is bounded. □

Remark 5.28. If Ω is a uniform domain, then the dyadic resolution of unity can be utilized to show that the spaces $\mathscr{C}^{m+\delta}_\infty(\Omega)$ and $C^{m,\delta}_{\mathrm{loc}}(\Omega)$ are isomorphic. This result corresponds to the kernel regularity result given in Section 4.

The extension problem for local smoothness spaces is treated in [DS84, DS93, Miy93] some of which are based on Sobolev extension techniques developed by Jones [Jon81]. By using those results, the theory of local smoothness spaces can be utilized to establish extension results for Hölder functions on uniform domains. But this approach to the Hölder extension leads to a more technical treatment than the previously described approach based on Lipschitz extension.

Example 5.29. The space $\mathrm{BMO}(\Omega)$, defined in 3.48, is also related to the local smoothness spaces. First of all, the inclusions $\mathscr{C}^{m+\delta}_\infty(\Omega) \subset L^\infty(\Omega) \subset \mathrm{BMO}(\Omega)$ are trivially bounded if $m \in \mathbb{N}_0$ and $0 < \delta < 1$. But there is more to this relation. Assume that $f \in L^1_{\mathrm{loc}}(\Omega)$, $Q \subset\subset \Omega$ is a cube, and $f_Q = |Q|^{-1} \int_Q f$. Then

$$|Q|^{-1} E_0(f, Q) \leq |Q|^{-1} \int_Q |f(x) - f_Q| dx \leq 2|Q|^{-1} E_0(f, Q).$$

Hence, if $\Omega \subset \mathbb{R}^n$ is bounded, we have $||f||_{\mathrm{BMO}(\Omega)} \leq 2 \operatorname{diam}(\Omega)^\delta |f|_{\mathscr{C}^\delta_\infty(\Omega)}$ if $0 < \delta < 1$ and $f \in L^1_{\mathrm{loc}}(\Omega)$.

Here is a modification of Example 5.29.

Theorem 5.30. *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain. Let $m \in \mathbb{N}$ and $0 < \delta < 1$. Then, assuming that $f \in L^1_{\mathrm{loc}}(\Omega)$, we have*

(5.31)
$$||f||_{\dot{F}^{m,2}_\infty(\Omega)} \leq C_{m,n,\delta} ||f||_{\mathscr{C}^{m+\delta}_\infty(\Omega)}.$$

Hence the inclusion $\mathscr{C}_\infty^{m+\delta}(\Omega) \subset \dot F_\infty^{m,2}(\Omega)$ *is bounded. Assuming also that* Ω *is bounded, we then have the estimate*

$$||f||_{\dot F_\infty^{m,2}(\Omega)} \le C_{m,n,\delta}\,\mathrm{diam}(\Omega)^\delta |f|_{\mathscr{C}_\infty^{m+\delta}(\Omega)}. \tag{5.32}$$

Proof. We only verify (5.31); the verification of (5.32) is similar to this. Fix $f \in \mathscr{C}_\infty^{m+\delta}(\Omega)$. Let $(Q,\varepsilon) \in \mathcal{D}_I^m(\Omega) \times \mathcal{E}$ and denote $f(Q,\varepsilon) = \langle f \mid \psi_Q^\varepsilon\rangle$. Using B3) in Appendix B and Remark 3.46, stating that $\mathrm{supp}\,\psi_Q^\varepsilon \subset C_{m+1}Q \subset\subset \Omega$, we get

$$|f(Q,\varepsilon)| = \inf_{P\in\mathcal{P}_m(\mathbb{R}^n)} \left| \int_\Omega (f(x)-P(x))\overline{\psi_Q^\varepsilon(x)}dx\right|.$$

Taking also the estimate B5) into account, that is $||\psi_Q^\varepsilon||_{L^\infty(\mathbb{R}^n)} \le C_{m+1}|Q|^{-1/2}$, we reach the following estimate after elementary manipulations

$$|f(Q,\varepsilon)| \le C_{m,n,\delta} \times \min\{|Q|^{1/2+(m+\delta)/n}|f|_{\mathscr{C}_\infty^{m+\delta}(\Omega)}, |Q|^{1/2}||f||_{L^\infty(\Omega)}\}. \tag{5.33}$$

Fix $P \in \mathcal{D}_I^m(\Omega)$ and consider the following summations

$$\begin{aligned}\Sigma_P &= \frac{1}{|P|}\sum_{Q\subset P}\sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n}|f(Q,\varepsilon)|^2 \\ &= \frac{1}{|P|}\sum_{Q\in\mathcal{A}}\sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n}|f(Q,\varepsilon)|^2 + \frac{1}{|P|}\sum_{Q\in\mathcal{B}}\sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n}|f(Q,\varepsilon)|^2,\end{aligned}$$

where we have partitioned $\{Q \in \mathcal{D} : Q \subset P\}$ as follows: $\mathcal{A} = \{Q \subset P : \ell(Q) < 1\}$ and $\mathcal{B} = \{Q \subset P : \ell(Q) \ge 1\}$. To estimate the sum over the family $\mathcal{A}$, apply the first estimate in (5.33). To estimate the sum over the family $\mathcal{B}$, apply the second estimate in (5.33). Then we obtain the estimate $\Sigma_P \le C_{m,n,\delta}||f||^2_{\mathscr{C}_\infty^{m+\delta}(\Omega)}$, where the right-hand side is independent of $P \in \mathcal{D}_I^m(\Omega)$. According to Definition 3.47 we have established (5.31). □

Combining Theorem 5.25 and Theorem 5.30, we get the following corollary.

Corollary 5.34. *Let* $\emptyset \ne \Omega \subset \mathbb{R}^n$, $n \ge 2$, *be a domain. Let* $m \in \mathbb{N}$ *and* $0 < \delta < 1$. *Then we have the following bounded restriction and inclusion, respectively,*

$$C^{m,\delta}(\overline{\Omega}) \hookrightarrow C^{m,\delta}_{\mathrm{loc}}(\Omega) \subset \dot F_\infty^{m,2}(\Omega).$$

The implicit constant in the inclusion depends at most on the parameters n, m, δ.

5.2. **Uniformity and products.** In order to extend standard kernels we need to extend so called kernel atoms which are Hölder regular functions defined in the product domain $\Omega \times \Omega$. The extension of such functions is possible in the case of uniform product domains and for this reason we prove that a domain $\Omega \subset \mathbb{R}^n$ is uniform if, and only if, $\Omega \times \Omega \subset \mathbb{R}^{2n}$ is uniform.

The proof relies on the following characterization from [Väi88] involving certain continua that are referred to as *distance cigars*.

Theorem 5.35. *Let $n \geq 2$ and $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a domain, that is, an open and connected set in $\mathbb{R}^n$. Then Ω is uniform if, and only if, it satisfies the following condition: There is a constant $c > 1$ so that for every pair $x, y \in \Omega$ there exists a continuum $E \subset \Omega$, a connected and compact set, containing these two points such that* $\operatorname{diam}(E) \leq c|x-y|$ *and that every point $z \in E$ satisfies*

$$\min\{|z-x|, |z-y|\} \leq c \operatorname{dist}(z, \partial\Omega).$$

The following proof is ours but there are similar results in the literature when Ω is bounded and uniformity is replaced by inner uniformity [BS01].

Theorem 5.36. *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain. Then Ω is uniform if, and only if, the product domain $\Omega \times \Omega \subset \mathbb{R}^{2n}$ is uniform.*

Proof. First assume that $\Omega \times \Omega$ is uniform. Let $x_1, y_1 \in \Omega$. Denote $x = (x_1, x_1) \in \Omega \times \Omega$ and $y = (y_1, x_1) \in \Omega \times \Omega$. Applying Theorem 5.35 to the points $x, y \in \Omega \times \Omega$ we obtain a continuum $E \subset \Omega \times \Omega$ containing these two points with the associated constant c independent of them. Denote $E_1 = \pi_1(E) \subset \Omega$ where $\pi_1 : \mathbb{R}^{2n} \to \mathbb{R}^n$ denotes the projection to the first n coordinates. Then E_1 is a continuum containing the points x_1, y_1 and $\operatorname{diam}(E_1) \leq \operatorname{diam}(E) \leq c|x-y| = c|x_1 - y_1|$. If $z_1 \in E_1$ then $z_1 = \pi_1(z)$ for some $z \in E$, and we have

$$\begin{aligned}\min\{|z_1 - x_1|, |z_1 - y_1|\} &\leq \min\{|z-x|, |z-y|\} \\ &\leq c \operatorname{dist}(z, \partial(\Omega \times \Omega)) \leq c \operatorname{dist}(z_1, \partial\Omega).\end{aligned}$$

Since the constant c is independent of the points $x_1, y_1 \in \Omega$ we can invoke Theorem 5.35 to conclude that Ω is uniform.

Then we assume that Ω is uniform. Let $x = (x_1, x_2) \in \Omega \times \Omega$ and $y = (y_1, y_2) \in \Omega \times \Omega$. We will invoke Theorem 5.35 and for this purpose we need to construct distance cigars containing these two points x and y. Without loss of generality we can assume that $|x_1 - y_1| \geq |x_2 - y_2|$. Fix a path $\gamma_1 : [0, \ell(\gamma_1)] \to \Omega$ joining x_1 to y_1 as in the Definition 1.13 of uniform domains. There is $z \in \{x_1, y_1\} \subset \Omega$ so that $|x_1 - y_1|/2 \leq |x_2 - z|$. Let $\gamma_z : [0, \ell(\gamma_z)] \to \Omega$ be a path joining x_2 to z as in Definition 1.13. Let $t_z = |x_1 - y_1|/4 \leq \ell(\gamma_z)/2$ and $w = \gamma_z(t_z) \in \Omega$. Denote by $\gamma_w : [0, \ell(\gamma_w)] \to \Omega$ a path joining w to y_2 as in Definition 1.13. We record the following useful facts for later purposes:

a) The properties of γ_z imply that $|x_1 - y_1|/4a \leq \operatorname{dist}(w, \partial\Omega)$,
b) Let $t \in [0, |x_1 - y_1|/8a] \cap [0, \ell(\gamma_w)/2]$. Then $|\gamma_w(t) - w| \leq |x_1 - y_1|/8a$ and, combining this with a), we have $|x_1 - y_1|/8a \leq \operatorname{dist}(\gamma_w(t), \partial\Omega)$,
c) Let $t \in [|x_1 - y_1|/8a, \infty) \cap [0, \ell(\gamma_w)/2]$ (if such exists). Then the properties of γ_w imply that $|x_1 - y_1|/8a^2 \leq \operatorname{dist}(\gamma_w(t), \partial\Omega)$.

Let $\gamma_2 : [0, t_z + \ell(\gamma_w)] \to \Omega$ be the path joining x_2 to y_2 that is defined by the rule

$$\gamma_2(t) = \begin{cases} \gamma_z(t), & t \in [0, t_z], \\ \gamma_w(t - t_z), & t \in [t_z, t_z + \ell(\gamma_w)]. \end{cases}$$

Then γ_2 is parametrized by the arc length measured from x_2 and $|x_1 - y_1|/4 \le \ell(\gamma_2) = t_z + \ell(\gamma_w) \le (a/4 + a + 1/4)|x_1 - y_1|$. Using b) and c) above, properties of γ_z and γ_w, and the definition of γ_2, we get

$$\min(t, \ell(\gamma_2) - t) \le C_a \operatorname{dist}(\gamma_2(t), \partial\Omega), \quad t \in [0, \ell(\gamma_2)]. \tag{5.37}$$

Denote $\varrho = \ell(\gamma_2)\ell(\gamma_1)^{-1}$ and $\sigma(t) = \varrho t \in [0, \ell(\gamma_2)]$ if $t \in [0, \ell(\gamma_1)]$. Using this we define

$$\Gamma = \Gamma_1 \times \Gamma_2 = \gamma_1 \times (\gamma_2 \circ \sigma) : [0, \ell(\gamma_1)] \to \Omega \times \Omega.$$

Now $E = \Gamma[0, \ell(\gamma_1)] \subset \Omega \times \Omega$ is a continuum containing the points x and y.

We verify that E satisfies the conditions of Theorem 5.35 with constant depending at most on the domain Ω. The diameter of E is estimated as follows. If $t \in [0, \ell(\gamma_1)]$ then

$$\begin{aligned} |x - \Gamma(t)| &\le |x_1 - \Gamma_1(t)| + |x_2 - \Gamma_2(t)| \le \ell(\gamma_1) + t_z + \ell(\gamma_w) \\ &\le a|x_1 - y_1| + (a/4 + a + 1/4)|x_1 - y_1| \le 4a|x_1 - y_1| \le 4a|x - y|. \end{aligned}$$

Hence $\operatorname{diam}(E) \le 8a|x - y|$.

We verify the cigar condition. Let $t \in [0, \ell(\gamma_1)]$ and

$$m = \min\{|x - \Gamma(t)|, |y - \Gamma(t)|\}.$$

Then, using the arc length parametrization of γ_1 and γ_2, we have the estimate

$$m \le |x - \Gamma(t)| \le |x_1 - \Gamma_1(t)| + |x_2 - \Gamma_2(t)| \le (1 + \varrho)t,$$

implying that $m/(1 + \varrho) \le t$. In a similar fashion, we have

$$m \le |y - \Gamma(t)| \le |y_1 - \Gamma_1(t)| + |y_2 - \Gamma_2(t)| \le \ell(\gamma_1) - t + \varrho(\ell(\gamma_1) - t),$$

implying that $m/(1 + \varrho) \le \ell(\gamma_1) - t$. Combining these estimates for t with the inequality (5.37) and the properties of γ_1, we get the following

$$\begin{cases} m/(1 + \varrho) \le \min(t, \ell(\gamma_1) - t) \le a \operatorname{dist}(\gamma_1(t), \partial\Omega), \\ \varrho m/(1 + \varrho) \le \min(\varrho t, \ell(\gamma_2) - \varrho t) \le C_a \operatorname{dist}(\gamma_2(\varrho t), \partial\Omega). \end{cases}$$

Note that $\min(1/a(1+\varrho), \varrho/C_a(1+\varrho)) \ge c_a$ for some $c_a > 0$ which depends at most on the uniformity constant a. Using also that $m = \min\{|x - \Gamma(t)|, |y - \Gamma(t)|\}$, we have

$$\begin{aligned} \operatorname{dist}(\Gamma(t), \partial(\Omega \times \Omega)) &= \operatorname{dist}((\gamma_1(t), \gamma_2(\varrho t)), \partial(\Omega \times \Omega)) \\ &\ge c_a \min\{|x - \Gamma(t)|, |y - \Gamma(t)|\}. \end{aligned}$$

It follows that, if $x, y \in \Omega \times \Omega$, there is a continuum $E \subset \Omega \times \Omega$ containing these two points and satisfying the diameter and distance cigar conditions in Theorem 5.35 with constant $c = \max\{8a, 1/c_a\}$ independent of the points x, y. As a consequence, $\Omega \times \Omega$ is a uniform domain. □

5.3. **Atomic decomposition and kernel extension.** We come to a characterization of smooth kernels in terms of an atomic decomposition. This characterization, combined with Hölder extension results, is then applied to show an extension result for the smooth kernels. This section is based on our research but we strongly rely on the previous extension results for Hölder functions.

Kernel atoms and their extension. Here we define the kernel atoms and establish their extension properties on uniform domains. We begin with notation. Denote $x = (x_1, x_2) \in \mathbb{R}^{2n}$, where $x_1, x_2 \in \mathbb{R}^n$ and $n \geq 2$. The *diagonal set* is denoted by

$$\Delta = \{(x, x) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{2n}.$$

Notice that $\Delta \subset \mathbb{R}^{2n}$ is closed, $\partial(\mathbb{R}^{2n} \setminus \Delta) = \Delta$, and $\mathbb{R}^{2n} \setminus \Delta$ is a domain. Let $\mathcal{F}_\Delta$ be the Whitney decomposition of $\mathbb{R}^{2n} \setminus \Delta$ as described in Appendix C. Recall the properties of the associated partition of unity $\{\varphi_Q\}_{Q\in\mathcal{F}_\Delta}$ and the definition of cubes $Q \subset Q^* \subset Q^{**}$ therein. All the references C1)–C14) are to Appendix C.

Definition 5.38. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain. Let $m \in \mathbb{N}$, $0 < \delta < 1$, $Q \in \mathcal{F}_\Delta$, and $R \in \{Q^*, Q^{**}\}$. Then $K_Q : \Omega \times \Omega \to \mathbb{C}$ is an (R, Ω, m, δ) *kernel atom*, if it satisfies A1)–A4) below

A1) $\operatorname{supp}_{\Omega\times\Omega} K_Q \subset R$,
A2) $K_Q \in C^m(\Omega \times \Omega)$,
A3) $||\partial^\alpha K_Q||_{L^\infty(\Omega\times\Omega)} \leq \operatorname{diam}(Q)^{m-n-|\alpha|}$ if $\alpha \in \mathbb{N}_0^{2n}$, $|\alpha| \leq m$,
A4) $|\partial^\alpha K_Q|_{C^\delta_{\mathrm{loc}}(\Omega\times\Omega)} \leq \operatorname{diam}(Q)^{-n-\delta}$ if $\alpha \in \mathbb{N}_0^{2n}$, $|\alpha| = m$.

In A1) $\operatorname{supp}_{\Omega\times\Omega}$ stands for the closure of $\{x \in \Omega \times \Omega : K_Q(x) \neq 0\}$ in $\Omega \times \Omega$. Notice also that it is possible that $R \subsetneq \Omega$. Here is an extension result regarding kernel atoms in uniform domains.

Theorem 5.39. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain. Assume that $Q \in \mathcal{F}_\Delta$ and K_Q is an (Q^*, Ω, m, δ) kernel atom for $m \in \mathbb{N}$ and $0 < \delta < 1$. Then there exists an $(Q^{**}, \mathbb{R}^n, m, \delta)$ kernel atom $\hat{K}_Q$ satisfying*

$$K_Q = (\kappa \hat{K}_Q)|\Omega \times \Omega,$$

where the constant $\kappa > 0$ depends at most on the parameters n, m, δ and the uniformity constant $a_{\Omega\times\Omega}$ of the uniform domain $\Omega \times \Omega \subset \mathbb{R}^{2n}$.

Proof. Theorem 5.36 implies that the domain $\Omega \times \Omega \subset \mathbb{R}^{2n}$ is uniform. Denote by $a_{\Omega\times\Omega} \geq 1$ its uniformity constant. It easy to verify that this uniformity constant is invariant under dilatations and, in particular, the dilatated domain

$$\omega = \operatorname{diam}(Q)^{-1}(\Omega \times \Omega) = \{\operatorname{diam}(Q)^{-1}x : x \in \Omega \times \Omega\}$$

is uniform in $\mathbb{R}^{2n}$ with the constant $a_\omega = a_{\Omega\times\Omega}$. In what follows we use the convention that C denotes a constant that depends at most on the parameters n, m, δ, and $a_\omega = a_{\Omega\times\Omega}$.

First we define certain auxiliary functions. Define $f : \omega \to \mathbb{C}$ by

$$f(x) = \operatorname{diam}(Q)^{n-m} K_Q(\operatorname{diam}(Q)x).$$

Then using the estimates A2)–A4) for K_Q it is straightforward to verify that $f \in C^{m,\delta}_{\mathrm{loc}}(\omega)$ and $||f||_{C^{m,\delta}_{\mathrm{loc}}(\omega)} \le 1$. In what follows we extend f twice. First of all, using Theorem 5.14, we infer that f has a continuous extension to the closure of the domain and the order m derivatives of this extension are Hölder regular in the whole closed set $\overline{\omega}$. To put this otherwise, there is $g \in C^{m,\delta}(\overline{\omega})$ such that $g|\omega = f$ and

$$||g||_{C^{m,\delta}(\overline{\omega})} \le C||f||_{C^{m,\delta}_{\mathrm{loc}}(\omega)}. \tag{5.40}$$

Applying Theorem 5.20 and (5.40), we obtain $G = \mathscr{E}_m g \in C^{m,\delta}(\mathbb{R}^{2n})$ satisfying the identity $G|\omega = g|\omega = f$ and the norm-estimate

$$||G||_{C^{m,\delta}(\mathbb{R}^{2n})} \le C||g||_{C^{m,\delta}(\overline{\omega})} \le C||f||_{C^{m,\delta}_{\mathrm{loc}}(\omega)} = C.$$

Define $h : \mathbb{R}^{2n} \to \mathbb{C}$ by $h(x) = \psi_Q(\mathrm{diam}(Q)x)G(x)$, where $\psi_Q : \mathbb{R}^{2n} \to \mathbb{R}$ satisfies the properties C13)–C14) for $Q \in \mathcal{F}_\Delta$. Using C14), we get

$$||h||_{C^{m,\delta}(\mathbb{R}^{2n})} \le C||\psi_Q(\mathrm{diam}(Q)\cdot)||_{C^{m,\delta}(\mathbb{R}^{2n})}||G||_{C^{m,\delta}(\mathbb{R}^{2n})} \le \kappa, \tag{5.41}$$

where the constant κ depends at most on the parameters n, m, δ, and $a_\omega = a_{\Omega\times\Omega}$. Now we define the function $\hat{K}_Q : \mathbb{R}^{2n} \to \mathbb{C}$ by the rule

$$\hat{K}_Q(x) = \frac{\mathrm{diam}(Q)^{m-n}h(\mathrm{diam}(Q)^{-1}x)}{\kappa}, \quad x \in \mathbb{R}^{2n}.$$

Then using C13) and the norm-estimate (5.41), it is straightforward to verify that $\hat{K}_Q$ is an $(Q^{**}, \mathbb{R}^n, m, \delta)$ kernel atom. Furthermore, if $x \in \Omega \times \Omega$, then using A1) for K_Q and C13), we have

$$\kappa\hat{K}_Q(x) = \psi_Q(x)\,\mathrm{diam}(Q)^{m-n}G(\mathrm{diam}(Q)^{-1}x) = \psi_Q(x)K_Q(x) = K_Q(x).$$

All in all, $\hat{K}_Q$ is as required. □

Atomic decomposition of smooth kernels. Here we establish the so called atomic decomposition of the smooth kernels. The proof of this characterization of smooth kernels involves technicalities and to clarify we outline an argument first, showing how such decompositions arise.

It is natural to treat a given smooth kernel $K \in \mathcal{K}^{-m}_\Omega(\delta)$ as of being defined in the domain $\omega = \Omega \times \Omega \setminus \{(x, x)\} \subset \mathbb{R}^{2n}$. Let us indicate this shift in the viewpoint even further. Notice that

$$\partial^\gamma K(x) = \partial^\alpha_{x_1}\partial^\beta_{x_2}K(x_1, x_2),$$

if $x = (x_1, x_2) \in \omega$ and $\gamma = (\alpha, \beta) \in \mathbb{N}^{2n}_0$ satisfies $|\gamma| = |\alpha| + |\beta| \le m$. This notation is utilized in the sequel. Assume a regular situation: $K \in C^{m+1}(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\}$ satisfies the homogeneity estimates

$$|\partial^\gamma K(x)| \le C_K|x_1 - x_2|^{m-n-|\gamma|}, \quad |\gamma| \le m + 1 \le n. \tag{5.42}$$

Fix $Q \in \mathcal{F}_\Delta$. Then, according to later Lemma 5.45, $|x_1 - x_2| \geq C_n \operatorname{diam}(Q)$ if $x \in Q^*$. Also the function φ_Q, defined in Appendix C, satisfies $\operatorname{supp} \varphi_Q \subset Q^*$ and

$$||\partial^\gamma \varphi_Q|| \leq C_{n,\gamma} \operatorname{diam}(Q)^{-|\gamma|}.$$

Combining these facts and using (5.42), we get the following for $\gamma \in \mathbb{N}_0^{2n}$, $|\gamma| \leq m+1$,

$$\begin{aligned} ||\partial^\gamma(\varphi_Q K)||_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} &\leq \sum_{\sigma\leq\gamma} c_{\sigma,\gamma} ||\partial^{\gamma-\sigma}\varphi_Q||_{L^\infty(Q^*)} ||\partial^\sigma K||_{L^\infty(Q^*)} \\ &\leq C_{n,m,K} \operatorname{diam}(Q)^{-(|\gamma|-|\sigma|)} \operatorname{diam}(Q)^{m-n-|\sigma|} = C_{n,m,K} \operatorname{diam}(Q)^{m-n-|\gamma|}. \end{aligned}$$

This indicates that the summands $\varphi_Q K$ in the decomposition $K = \sum_{Q\in\mathcal{F}_\Delta} \varphi_Q K$ are constant multiples of kernel atoms. To advance in the general situation, we need a simple Fubini type argument.

Lemma 5.43. *Let $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain, $0 < m < n$, and $0 < \delta < 1$. Let $K \in \mathcal{K}_\Omega^{-m}(\delta)$. Assume that $P \subset\subset \Omega \times \Omega$ is an open cube and $x, x+h \in P$ satisfy $4|h| \leq |x_1 - x_2|$. Then*

$$|\Delta_h^1(\partial^\gamma K, P, x)| = |\partial^\gamma K(x+h) - \partial^\gamma K(x)| \leq C_{n,K} |h|^\delta |x_1 - x_2|^{-n-\delta},$$

if $\gamma \in \mathbb{N}_0^{2n}$ is such that $|\gamma| = m$.

Proof. Denote $P = P_1 \times P_2$, where $P_1, P_2 \subset \Omega$ are cubes. Then $x_1, x_1 + h_1 \in P_1$ and $x_2, x_2 + h_2 \in P_2$. In particular,

$$(x_1, x_2 + h_2) \in P_1 \times P_2 = P \subset \Omega \times \Omega,$$

and we can estimate as follows

$$|\Delta_h^1(\partial^\gamma K, P, x)| \leq |\Delta_{(h_1,0)}^1(\partial^\gamma K, P, (x_1, x_2 + h_2))| + |\Delta_{(0,h_2)}^1(\partial^\gamma K, P, x)|.$$

Use $4\max\{|h_1|, |h_2|\} \leq |x_1 - x_2|$ for the estimates $2|h_1| \leq |x_1 - x_2|/2 \leq |x_1 - x_2 - h_2|$ and $2|h_2| \leq |x_1 - x_2|$. Hence we are allowed to apply the Hölder-regularity estimate satisfied by K to reach the following

$$\begin{aligned} |\Delta_h^1(\partial^\gamma K, P, x)| &\leq C_K |h_1|^\delta |x_1 - x_2 - h_2|^{-n-\delta} + C_K |h_2|^\delta |x_1 - x_2|^{-n-\delta} \\ &\leq C_{n,K} |h|^\delta |x_1 - x_2|^{-n-\delta}. \end{aligned}$$

This is the required estimate. □

We also need a certain geometric connection between the diagonal set and cubes in the Whitney decomposition $\mathcal{F}_\Delta$. Here is the first ingredient towards this connection.

Lemma 5.44. *Let $x \in \mathbb{R}^{2n}$. Then $\frac{1}{\sqrt{2}}|x_1 - x_2| \leq \operatorname{dist}(x, \Delta) \leq |x_1 - x_2|$.*

Proof. First of all, we have $|(x_1, x_2) - (x_1, x_1)| = |x_1 - x_2|$ and $\operatorname{dist}(x, \Delta) \leq |x_1 - x_2|$. Next we assume that $\Delta \cap \bar{B}(x, r) \neq \emptyset$. Then it suffices to verify that $|x_1 - x_2| \leq \sqrt{2}r$. According to the assumption there is a point z,

$$z = x + h = (x_1 + h_1, x_2 + h_2) \in \Delta \cap \bar{B}(x, r).$$

Notice that $|h_1 - h_2| = |x_1 - x_2|$ because $0 = z_1 - z_2 = x_1 + h_1 - x_2 - h_2$. Also, $|(h_1, h_2)| = |(h_2, h_1)| = |h| \le r$ and therefore $\sqrt{2}|h_1 - h_2| = |(h_2 - h_1, h_1 - h_2)| \le 2r$. The estimates above prove that $|x_1 - x_2| = |h_1 - h_2| \le \sqrt{2}r$. □

The geometric connection between the diagonal set and the cubes in the Whitney decomposition is as follows.

Lemma 5.45. *Let $Q \in \mathcal{F}_\Delta$ and $x \in Q^{**}$. Then $C_n^{-1} \operatorname{diam}(Q^{**}) \le |x_1 - x_2| \le C_n \operatorname{diam}(Q)$.*

Proof. Recall that $\partial(\mathbb{R}^{2n} \setminus \Delta) = \Delta$. By using Lemma 5.44, we see that

$$\operatorname{dist}(Q^{**}, \Delta) \le \operatorname{dist}(x, \Delta) \le |x_1 - x_2|.$$

This together with C6) shows the estimate $C_n^{-1} \operatorname{diam}(Q^{**}) \le |x_1 - x_2|$. On the other hand, using C6), we have

$$\operatorname{dist}(x, \Delta) \le \operatorname{dist}(Q^{**}, \Delta) + \operatorname{diam}(Q^{**}) \le (1 + C_n) \operatorname{diam}(Q^{**}).$$

Lemma 5.44 implies that $|x_1 - x_2| \le \sqrt{2} \operatorname{dist}(x, \Delta) \le \sqrt{2}(1 + C_n) \operatorname{diam}(Q^{**})$. At the end it suffices to use C5) for $\operatorname{diam}(Q^{**}) \le C_n \operatorname{diam}(Q)$. □

We are ready for the atomic decomposition of smooth kernels.

Theorem 5.46. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain, $0 < m < n$, and $0 < \delta < 1$. Then we have 1)–2) below*

1) *Assume that $K : \Omega \times \Omega \setminus \Delta \to \mathbb{C}$ has the atomic decomposition*

$$K(x) = \lambda \sum_{Q \in \mathcal{F}_\Delta} K_Q(x), \quad x \in \Omega \times \Omega \setminus \Delta, \tag{5.47}$$

   where $\lambda \in \mathbb{C}$ and K_Q is an $(Q^{**}, \Omega, m, \delta)$ kernel atom if $Q \in \mathcal{F}_\Delta$. Then $K \in \mathcal{K}_\Omega^{-m}(\delta)$.

2) *Assume that $K \in \mathcal{K}_\Omega^{-m}(\delta)$. Then K has the atomic decomposition* (5.47)*, where $\lambda > 0$ and K_Q is an (Q^*, Ω, m, δ) kernel atom if $Q \in \mathcal{F}_\Delta$.*

Proof. Notice that, according to Theorem 5.36, the domain $\Omega \times \Omega \subset \mathbb{R}^{2n}$ is uniform. First we prove 1). Without loss of generality we can assume that $\lambda = 1$. Fix $Q \in \mathcal{F}_\Delta$ and $x, x + h \in \Omega \times \Omega \setminus \Delta$ so that $|h| \le |x_1 - x_2|/4$. Applying Lemma 5.45 and A1)–A3) in Definition 5.38, we have

$$|\partial^\alpha K_Q(x)| \le C_n |x_1 - x_2|^{m-n-|\alpha|}, \quad |\alpha| \le m. \tag{5.48}$$

Using Lemma 5.45 and both A1) and A4) in Definition 5.38, but also the uniformity of $\Omega \times \Omega$ with Remark 5.22, we have

$$|\partial^\alpha K_Q(x + h) - \partial^\alpha K_Q(x)| \le C_{n,\delta,\Omega} |h|^\delta |x_1 - x_2|^{-n-\delta}, \quad |\alpha| = m. \tag{5.49}$$

According to C8), there are at most C_n cubes in

$$\mathcal{N}(x) = \{Q \in \mathcal{F}_\Delta : x \in Q^{**}\}.$$

Applying A1) in Definition 5.38 and the estimate (5.48), we get

$$|\partial^\alpha K(x)| \le \sum_{Q\in\mathcal{N}(x)} |\partial^\alpha K_Q(x)| \le C_n |x_1 - x_2|^{m-n-|\alpha|}, \quad |\alpha| \le m. \tag{5.50}$$

In a similar way, summing over the cubes in $\mathcal{N}(x)\cup\mathcal{N}(x+h)$ and using (5.49), we get

$$|\partial^\alpha K(x+h) - \partial^\alpha K(x)| \le 2C_{n,\delta,\Omega}|h|^\delta |x_1 - x_2|^{-n-\delta}, \quad |\alpha| = m. \tag{5.51}$$

The remaining estimate (5.51) with $x, x+h \in \Omega\times\Omega$ and $2|h| \le |x_1 - x_2| < 4|h|$ follows from the kernel size estimate (5.50). All in all, we have shown that $K \in \mathcal{K}_\Omega^{-m}(\delta)$.

Then we prove 2). First recall that $\{\varphi_Q\}_{Q\in\mathcal{F}_\Delta}$ is a partition of unity in $\mathbb{R}^{2n}\setminus\Delta$, see Appendix C. Fix a cube $Q\in\mathcal{F}_\Delta$ and define $K_Q(x) = \varphi_Q(x)K(x)$ if $x\in\Omega\times\Omega$. Hence, using C10), we have

$$K(x) = K(x)\sum_{Q\in\mathcal{F}_\Delta}\varphi_Q(x) = \sum_{Q\in\mathcal{F}_\Delta} K_Q(x), \quad x\in\Omega\times\Omega\setminus\Delta.$$

It suffices to verify that there exists $\lambda>0$ such that $\lambda^{-1}K_Q$ is an (Q^*,Ω,m,δ) kernel atom if $Q\in\mathcal{F}_\Delta$. Fix $Q\in\mathcal{F}_\Delta$. The condition A1) holds since, according to C11), we have

$$\operatorname{supp}_{\Omega\times\Omega} K_Q \subset \operatorname{supp}_{\Omega\times\Omega}\varphi_Q \subset \operatorname{supp}\varphi_Q \subset Q^*.$$

It remains to verify A2)–A4). First of all, the condition A2) holds since $\varphi_Q \in C_0^\infty(\operatorname{int}(Q^*))$, $\operatorname{dist}(Q^*,\Delta) > 0$, and $K\in C^m(\Omega\times\Omega\setminus\Delta)$. Next we verify the condition A3). Fix $\alpha\in\mathbb{N}_0^{2n}$, $|\alpha|\le m$. If $x\in\Omega\times\Omega\setminus Q^*$, then $\partial^\alpha K_Q(x) = 0$. Assuming $x\in Q^*\cap\Omega\times\Omega$, we have

$$\partial^\alpha K_Q(x) = \sum_{\beta\le\alpha} c_{\alpha,\beta}\partial^\beta\varphi_Q(x)\partial^{\alpha-\beta}K(x). \tag{5.52}$$

Fix $\beta\in\mathbb{N}_0^{2n}$ satisfying $\beta\le\alpha$. Use C12) and estimates about K, combined with Lemma 5.45, for

$$\begin{aligned}|\partial^\beta\varphi_Q(x)\partial^{\alpha-\beta}K(x)| &\le C_{n,\beta}\operatorname{diam}(Q)^{-|\beta|}C_{n,K}\operatorname{diam}(Q)^{m-n-|\alpha|+|\beta|}\\ &\le C_{n,\beta,K}\operatorname{diam}(Q)^{m-n-|\alpha|}.\end{aligned}$$

Combining this with the identity (5.52) we get A3) for $\lambda_3^{-1}K_Q$ with λ_3 depending at most on n, K. Then we prove A4). According to the Definition 5.1 we need an estimate for

$$|\Delta_h^1(\partial^\alpha K_Q, P, x)| = |\partial^\alpha K_Q(x+h) - \partial^\alpha K_Q(x)|,$$

where $|\alpha| = m$ and $P \subset\subset \Omega\times\Omega$ is an open cube so that $x, x+h\in P$. First consider the case $x\in Q^*$, $|x_1-x_2| < 4|h|$. Using the proof of A3) from above, we get

$$\begin{aligned}|\Delta_h^1(\partial^\alpha K_Q, P, x)| &\le |\partial^\alpha K_Q(x+h)| + |\partial^\alpha K_Q(x)|\\ &\le 2\lambda_3\operatorname{diam}(Q)^{-n} \le C_{n,K}|h|^\delta\operatorname{diam}(Q)^{-n-\delta}.\end{aligned} \tag{5.53}$$

In the last inequality we used the estimate $\operatorname{diam}(Q) \le C_n|h|$, which follows from Lemma 5.45 and the estimate $|x_1 - x_2| < 4|h|$ with $x \in Q^* \subset Q^{**}$. Next we consider the case $x \in Q^*$, $4|h| \le |x_1 - x_2|$. Fix $\beta \in \mathbb{N}_0^{2n}$ satisfying $\beta \le \alpha$. Then, using Lemma 5.45, the mean value theorem, and Lemma 5.43 if $|\beta| = 0$, we get the following estimate

$$\begin{aligned}
&|\partial^\beta \varphi_Q(x+h)\partial^{\alpha-\beta}K(x+h) - \partial^\beta\varphi_Q(x)\partial^{\alpha-\beta}K(x)| \\
&\qquad \le |\partial^\beta \varphi_Q(x+h)\partial^{\alpha-\beta}K(x+h) - \partial^\beta\varphi_Q(x+h)\partial^{\alpha-\beta}K(x)| \\
&\qquad\qquad + |\partial^\beta \varphi_Q(x+h)\partial^{\alpha-\beta}K(x) - \partial^\beta\varphi_Q(x)\partial^{\alpha-\beta}K(x)| \\
&\qquad \le C_{n,\beta}\operatorname{diam}(Q)^{-|\beta|}C_{n,\delta,K}|h|^\delta \operatorname{diam}(Q)^{m-n-|\alpha|+|\beta|-\delta} \\
&\qquad\qquad + C_{n,\beta,\delta}|h|^\delta \operatorname{diam}(Q)^{-|\beta|-\delta}C_{n,K}\operatorname{diam}(Q)^{m-n-|\alpha|+|\beta|} \\
&\qquad \le C_{n,\beta,\delta,K}|h|^\delta \operatorname{diam}(Q)^{-n-\delta}.
\end{aligned}$$

Combining this with (5.52) and (5.53) shows that the estimate

$$|\Delta_h^1(\partial^\alpha K_Q, P, x)| \le C_{n,\delta,K}|h|^\delta \operatorname{diam}(Q)^{-n-\delta}, \tag{5.54}$$

holds true if $x, x+h \in P$ and $x \in Q^*$. It remains to consider the case $x + h \in Q^*$. But this reduces to the estimate (5.54) since, denoting $k = -h$ and $y = x + h$, we have $y, y+k \in P$, $y \in Q^*$, and

$$|\Delta_h^1(K_Q, P, x)| = |\Delta_k^1(K_Q, P, y)| \le C_{n,\delta,K}|k|^\delta \operatorname{diam}(Q)^{-n-\delta}, \quad |k| = |h|. \tag{5.55}$$

The estimates (5.54) and (5.55) imply that there exists $\lambda_4 > 0$, depending at most on n, δ, K, such that $\lambda_4^{-1}K_Q$ satisfies A4). Denoting $\lambda = \lambda_3 + \lambda_4$, $\lambda^{-1}K_Q$ satisfies A1)–A4). □

Kernel extension via atomic decomposition. The atomic decomposition of smooth kernels is a powerful tool. Indeed, combined with the atomic extension, it provides us the desired kernel extension result.

Theorem 5.56. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain, $0 < m < n$, $0 < \delta < 1$, and $K \in \mathcal{K}_\Omega^{-m}(\delta)$ be a smooth kernel. Then there exists $\hat{K} \in \mathcal{K}_{\mathbb{R}^n}^{-m}(\delta)$ such that*

$$\hat{K}|\Omega \times \Omega \setminus \Delta = K.$$

That is, K has an extension to a smooth kernel $\hat{K} : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$.

Proof. Applying 2) in Theorem 5.47 implies that K has the atomic decomposition

$$K(x) = \lambda \sum_{Q \in \mathcal{F}_\Delta} K_Q(x), \quad x \in \Omega \times \Omega \setminus \Delta,$$

where $\lambda > 0$ and K_Q is an (Q^*, Ω, m, δ) kernel atom if $Q \in \mathcal{F}_\Delta$. Fix $Q \in \mathcal{F}_\Delta$. Then applying Theorem 5.39 to K_Q gives an $(Q^{**}, \mathbb{R}^n, m, \delta)$ kernel atom $\hat{K}_Q$ satisfying

$K_Q = (\kappa \hat{K}_Q)|\Omega \times \Omega$, where κ independent of Q. As a consequence, we have the representation

$$K(x) = \lambda\kappa \sum_{Q\in\mathcal{F}_\Delta} \hat{K}_Q(x), \quad x \in \Omega\times\Omega\setminus\Delta. \tag{5.57}$$

On the other hand, applying 1) in Theorem 5.47, we see that the right-hand side of (5.57) defines a kernel $\hat{K} \in \mathcal{K}^{-m}_{\mathbb{R}^n}(\delta)$. □

Remark 5.58. According to Corollary 4.46 this extension result applies to

$$K \in \mathrm{K}^{-m}_{\Omega}(\delta) \cup k^{m+\delta}_{\mathrm{loc}}(\Omega) \cup \mathcal{K}^{-m}_{\Omega}(\delta),$$

given that $\Omega \subset \mathbb{R}^n$ is uniform. Such kernels admit also an atomic decomposition as in Theorem 5.46. When extending kernels in the two other classes, besides smooth kernels, the Hölder-regularity decreases but this is most likely an artifact caused by our proof of the kernel regularity result.

We formulate the following most useful extension result as a corollary. It follows by combining Theorem 4.37, Theorem 5.56, and Proposition 4.6.

Corollary 5.59. *Let $\Omega \subset \mathbb{R}^n$ be a uniform domain, $0 < m < n$, $0 < \delta' < \delta < 1$, and $K \in \mathrm{K}^{-m}_{\Omega}(\delta)$ be a standard kernel. Then there exists $\hat{K} \in \mathrm{K}^{-m}_{\mathbb{R}^n}(\delta')$ such that*

$$\hat{K}|\Omega\times\Omega\setminus\{(x,x)\} = K.$$

In words, K has an extension to a standard kernel $\hat{K} : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\} \to \mathbb{C}$.

Remark 5.60.
- In a similar fashion one can also extend Calderón–Zygmund standard kernels (strictly speaking, we haven't defined these kernels on domains). However, in this case there will be no canonical way to associate an operator to this extension of the singular kernel unless the (original) kernel is, say, antisymmetric. Therefore the current approach to the boundedness via extension of the kernel does not apply to Calderón–Zygmund type operators on domains in its full generality.
- The pointwise properties of kernels are easier to establish than the boundedness properties of the corresponding WSIO's. The difference between these is that the norm estimates involved in the pointwise properties are more simple. For instance, this allows us to circumvent Calderón reproducing formulae in the context of kernel regularity and apply the dyadic resolution of unity instead. As a matter of fact, we do not know how to construct suitable Calderón reproducing formulae on domains. The standard formulae, as in [HL03, HS94], do not apply because we need further moments aside from the zeroth.

6. WSIO's on domains

The topic here is a formulation and proof of our main result in this monograph, which is a boundedness result for WSIO's on admissible domains. We begin with discussing the properties of admissible domains like invariance under quasiconformal mappings of $\mathbb{R}^n$ onto itself. Then we strengthen the $T\chi_\Omega$ theorem by utilizing kernel regularity results and the $T1$ theorem of David and Journé. Next we verify that the $T\chi_\Omega$ theorem, combined with the kernel extension, leads to the proof of our main result: $T1$ theorem on admissible domains. Given $T \in \mathrm{SK}_\Omega^{-m}(\delta)$, this result describes the boundedness of

$$\partial^\alpha T, \quad |\alpha| = m,$$

on the spaces $L^p(\Omega)$ for $1 < p < \infty$. There are also endpoint boundedness results.

6.1. **Admissible domains.** We discuss the properties of so called admissible domains where our main result applies. These domains are, according to Definition 1.15, both uniform and Whitney coplump. The main observation here is the invariance of admissible domains under quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n$. These mappings generalize conformal mappings of one complex variable to higher dimensional real spaces.

The class of uniform domains was introduced by Martio and Sarvas in late 70's in their work [MS79], where the invariance of such domains under quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ is also established: if $\Omega \subset \mathbb{R}^n$ is a uniform, then the image $f\Omega \subset \mathbb{R}^n$ is uniform. The definition for uniform domains given in [MS79] is different from the Definition 1.13, but the equivalence of these (and other) definitions was established by Martio [Mar80]. For further characterizations of uniformity see the references [Väi88, Geh87].

A domain $\Omega \subset \mathbb{R}^n$ is Whitney coplump if either $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n \setminus \Omega$ is unbounded and c-plump for some $c \geq 1$ in the sense of Definition 1.14. These domains were studied by Martio and Väisälä [MV93] in connection with the $\mathcal{A}$-harmonic measure and passability. Whitney coplump domains are invariant under quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n$: if $\Omega \subset \mathbb{R}^n$ is a Whitney coplump domain, then the image $f\Omega$ is also Whitney coplump. We prove this in Theorem 6.6. Convex domains are Whitney coplump – this follows from the definitions and existence certain hyperplanes [Roc70, p. 100]:

Theorem 6.1. *Let $\emptyset \neq \Omega \subsetneq \mathbb{R}^n$ be a convex domain and $b \in \mathbb{R}^n \setminus \Omega$. Then there exists an affine hyperplane $P \subset \mathbb{R}^n$ so that $b \in P$ and P does not separate the points in Ω.*

Corollary 6.2. *A convex domain $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n \geq 2$, is Whitney coplump.*

Uniformity of the domain does not imply the Whitney coplumpness of the domain: removing an inward cusp from the unit ball $B(0,1) \subset \mathbb{R}^3$ does not affect the uniformity but the Whitney coplumpness of the resulting domain fails. On the other hand, Whitney coplumpness does not suffice for the uniformity which is seen by looking at exterior cusps. Thus, in general, both uniformity and Whitney coplumpness need to be verified to ensure admissibility of the domain.

Next we turn to the quasiconformal invariance of admissible domains. Useful references for quasiconformal mappings are [AIM09, Väi71].

Definition 6.3. A homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is a K-*quasiconformal mapping* if, for every $x \in \mathbb{R}^n$,

$$\limsup_{r\to 0} \frac{L(x,f,r)}{l(x,f,r)} \leq K < \infty,$$

where $L(x,f,r) = \max_{|x-y|=r} |f(x) - f(y)|$ and $l(x,f,r) = \min_{|x-y|=r} |f(x) - f(y)|$.

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an admissible domain and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a K-quasiconformal mapping. Then the image $f\Omega \subset \mathbb{R}^n$ is admissible.*

Next we prove Theorem 6.4 and it suffices to verify the invariance of both uniformity and Whitney coplumpness under quasiconformal mappings $\mathbb{R}^n \to \mathbb{R}^n$. The invariance of uniformity is established in [MS79, Theorem 2.15] and we omit the proof which is based on the following quasisymmetry type estimate

$$L(x,f,r_2) \leq c_{n,K}(r_2/r_1)^{K^{1/(n-1)}} l(x,f,r_1), \tag{6.5}$$

where $x \in \mathbb{R}^n$, $0 < r_1 \leq r_2 < \infty$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping. The estimate (6.5) implies also the remaining invariance of Whitney coplump domains.

Theorem 6.6. *Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, be a Whitney coplump domain and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a K-quasiconformal mapping. Then there exists $c = c_{n,K,\Omega} \geq 1$ such that for all $x \in \mathbb{R}^n \setminus f\Omega$ and $0 < r < \infty$ there exists $z \in \bar{B}(x,r)$ such that $B(z,r/c) \subset \mathbb{R}^n \setminus f\Omega$. As a consequence,* $\operatorname{diam}(\mathbb{R}^n \setminus f\Omega) = \infty$ *and $f\Omega \subset \mathbb{R}^n$ is Whitney coplump.*

Proof. This proof relies on the estimate (6.5) applied to the K-quasiconformal mappings f and f^{-1}. Fix $x \in \mathbb{R}^n \setminus f\Omega$ and $0 < r < \infty$. Denote $l' = l(x,f^{-1},r)$, $L' = L(x,f^{-1},r)$, and $x' = f^{-1}(x) \in \mathbb{R}^n \setminus \Omega$. Because $\Omega \subsetneq \mathbb{R}^n$ is a Whitney coplump domain, there exists $z' \in \bar{B}(x',l'/2)$ and a constant $c_\Omega \geq 1$ such that

$$B(z',l'/(2c_\Omega)) \subset \mathbb{R}^n \setminus \Omega. \tag{6.7}$$

Denote $z = f(z')$. Now $z' \in \bar{B}(x',l'/2) \subset B(x',l') \subset f^{-1}B(x,r)$ and, as a consequence,

$$z = f(z') \in B(x,r) \subset \bar{B}(x,r). \tag{6.8}$$

We also have $f^{-1}B(x,r) \subset B(x',L') \subset B(z',2L')$ and therefore

$$B(x,r) \subset fB(z',2L'). \tag{6.9}$$

Denote $l = l(z',f,l'/(2c_\Omega))$ and $L = L(z',f,2L')$. Using the relation (6.9), we find two points $b_1, b_2 \in \mathbb{R}^n$ satisfying $|b_1 - b_2| \geq 2r$ and

$$b_1, b_2 \in \partial[fB(z',2L')] = f\partial[B(z',2L')] = \{f(y') \,:\, |z'-y'| = 2L'\}.$$

Applying the triangle-inequality and the definition of L we see that there is $b \in \{b_1, b_2\}$ such that $r \le |z-b| \le L$. Applying this and the estimate (6.5) twice, we get

$$r \le L \le c_{n,K}(4c_\Omega L'/l')^{K^{1/(n-1)}} l \le c_{n,K}(4c_\Omega c_{n,K})^{K^{1/(n-1)}} l. \tag{6.10}$$

Using the estimate (6.10) and the relation (6.7), we get

$$B(z, r/c_{n,K,\Omega}) \subset B(z,l) \subset fB(z', l'/(2c_\Omega)) \subset \mathbb{R}^n \setminus f\Omega, \tag{6.11}$$

where $c_{n,K,\Omega} = c_{n,K}(4c_\Omega c_{n,K})^{K^{1/(n-1)}}$ depends at most on n, K, Ω. Combining the relations (6.8) and (6.11) we find that $z \in \bar{B}(x,r)$ and $B(z, r/c_{n,K,\Omega}) \subset \mathbb{R}^n \setminus f\Omega$ as required. □

6.2. $T\chi_\Omega$ **theorem revisited.** We strengthen the $T\chi_\Omega$ theorem for restricted operators, formulated in Theorem 3.118. For the notation we refer to the notation in the sections 3.1 and 3.3.

Theorem 6.12. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a Whitney coplump domain and $T \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$, where $0 < m < n$ and $0 < \delta < 1$. Then the following conditions are equivalent*

- $\partial^\alpha T, \partial^\alpha T^* \in \mathscr{L}(L^2(\Omega))$ *if* $|\alpha| = m$,
- $\partial^\alpha T, \partial^\alpha T^* \in \mathscr{L}(L^p(\Omega))$ *if* $1 < p < \infty$ *and* $|\alpha| = m$.

Proof. The second condition implies the first. Assume the first condition. Then, due to symmetry, it suffices to consider the operator T. Theorem 3.118 shows that there exists $S \in \mathrm{SK}^{-m}_{\mathbb{R}^n}(\delta)$ such that $S \sim T$ and $\partial^\alpha S \in \mathscr{L}(L^2(\mathbb{R}^n))$ if $|\alpha| = m$. Fix $\alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| = m$. Assuming that S is associated with a kernel $\kappa \in \mathrm{K}^{-m}_{\mathbb{R}^n}(\delta)$, Theorem 4.37 implies that $\kappa \in \mathcal{K}^{-m}_{\mathbb{R}^n}(\delta')$ if $0 < \delta' < \delta$. In particular, $\kappa \in C^m(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\})$ and

$$\partial_x^\alpha \kappa : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x)\} \to \mathbb{C}$$

is a Calderón–Zygmund standard kernel. Let $f, g \in C_0^\infty(\mathbb{R}^n)$ be test functions such that their supports are disjoint. Applying Fubini's theorem and integrating by parts gives

$$\begin{aligned}\langle \partial^\alpha S f, g\rangle &= (-1)^{|\alpha|}\langle Sf \mid \partial^\alpha \bar{g}\rangle = (-1)^{|\alpha|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \kappa(x,y)\partial^\alpha g(x)dxf(y)dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_x^\alpha \kappa(x,y) g(x) dx f(y) dy.\end{aligned}$$

Applying the Fubini's theorem to the right-hand side we see that the continuous operator $\partial^\alpha S : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, induced by the bounded extension $\partial^\alpha S : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, is associated with the Calderón–Zygmund standard kernel $\partial_x^\alpha \kappa$. Hence $\partial^\alpha S \in \mathrm{SK}(\delta')$ and the $T1$ theorem of David and Journé, Theorem 1.12, shows that $\partial^\alpha S \in \mathscr{L}(L^p(\mathbb{R}^n))$ if $1 < p < \infty$. Because $S \sim T$, we also have $\partial^\alpha T \in \mathscr{L}(L^p(\Omega))$ if $1 < p < \infty$. □

6.3. $T1$ **theorem on admissible domains.** We come to the formulation and proof of our main result: a $T1$ theorem for WSIO's on admissible domains.

Let us first set up the stage. Fix $n \geq 2$ and $m \in \mathbb{N}$, $0 < m < n$. Fix $(m+1)$-regular wavelets $\{\psi_Q^\varepsilon\}$ that are defined in Appendix B. Let $\Omega \subset \mathbb{R}^n$ be an admissible domain and let $T \in \mathrm{SK}_\Omega^{-m}(\delta)$, $0 < \delta < 1$, be associated with a kernel $K \in \mathrm{K}_\Omega^{-m}(\delta)$, that is,

$$Tf(x) = \int_\Omega K(x,y)f(y)dy, \quad x \in \Omega \text{ and } f \in C_0(\Omega).$$

The adjoint operator $T^* \in \mathrm{SK}_\Omega^{-m}(\delta)$ is associated with the kernel $(x,y) \mapsto \overline{K(y,x)}$. See Introduction for further details about the classes $\mathrm{K}_\Omega^{-m}(\delta)$ and $\mathrm{SK}_\Omega^{-m}(\delta)$.

Extension of the operator. Our strategy is to apply boundedness results obtained in Section 3. For this purpose we need to extend the operator T and establish various compatibility results. To begin with, we extend the associated kernel $K \in \mathrm{K}_\Omega^{-m}(\delta)$ that is defined in $\Omega \times \Omega \setminus \{(x,x)\}$. This extension is obtained by applying Corollary 5.59 which gives a kernel $\hat{K} \in \mathrm{K}_{\mathbb{R}^n}^{-m}(\delta')$ so that

$$\hat{K}|\Omega \times \Omega \setminus \{(x,x)\} = K. \tag{6.13}$$

Define $\hat{T} \in \mathrm{SK}_{\mathbb{R}^n}^{-m}(\delta')$ to be the corresponding operator such that

$$\hat{T}f(x) = \int_{\mathbb{R}^n} \hat{K}(x,y)f(y)dy, \quad \text{if } x \in \mathbb{R}^n \text{ and } f \in C_0(\mathbb{R}^n).$$

The adjoint operator $\hat{T}^* = (\hat{T})^*$ is associated with the adjoint kernel $(x,y) \mapsto \overline{\hat{K}(y,x)}$. The estimate (3.6) shows that $\hat{T}$ and $\hat{T}^*$ induce linear operators $C_0(\mathbb{R}^n) \to (C_0(\mathbb{R}^n))^*$. Let $\mathrm{id} : C_0(\Omega) \hookrightarrow C_0(\mathbb{R}^n)$ and $\mathrm{id}^* : (C_0(\mathbb{R}^n))^* \hookrightarrow (C_0(\Omega))^*$ denote the canonical inclusions. Fix $f, g \in C_0(\Omega)$. Then, applying (6.13) and the relation $\operatorname{supp} f \cup \operatorname{supp} g \subset \Omega$, we get

$$\int_\Omega Tf(x)\overline{g(x)}dx = \int_\Omega \int_\Omega K(x,y)f(y)dy\overline{g(x)}dx = \langle \mathrm{id}^* \circ \hat{T} \circ \mathrm{id}(f) \mid g\rangle. \tag{6.14}$$

This identifies T as the operator $\mathrm{id}^* \circ \hat{T} \circ \mathrm{id} : C_0(\Omega) \to (C_0(\Omega))^*$ and T^* as the operator $\mathrm{id}^* \circ \hat{T}^* \circ \mathrm{id} : C_0(\Omega) \to (C_0(\Omega))^*$. Hence, if $|\alpha| = m$, then according to Definition (3.59) we have $\partial^\alpha T \in \mathscr{L}(L^p(\Omega))$ if, and only if, $\partial^\alpha \hat{T} \in \mathscr{L}(L^p(\Omega))$. Same holds for T^* and $\hat{T}^*$.

WSIO's and the space $\mathrm{BMO}(\Omega)$. We extend the domain of definition of T to the space $\mathrm{BMO}(\Omega)$, see Definition 3.48. Then we establish compatibility relations to the extended operator which is already defined in the space $\mathrm{BMO}(\mathbb{R}^n)$.

First recall Definition 3.45 for the cubes $\mathcal{D}_I^m(\Omega) \subset \mathcal{D}$. We define $Tb : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ for $b \in \mathrm{BMO}(\Omega)$ by the rule $Tb(Q,\varepsilon) = 0$ if $Q \in \mathcal{D} \setminus \mathcal{D}_I^m(\Omega)$ and

(6.15)

$$Tb(Q,\varepsilon) = \int_\Omega b(x)\overline{T^*\psi_Q^\varepsilon(x)}dx = \int_\Omega b(x) \int_\Omega K(y,x)\overline{\psi_Q^\varepsilon(y)}dydx, \quad Q \in \mathcal{D}_I^m(\Omega).$$

The sequence $T^*b : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ is defined similarly. The definition (6.15) induces a linear operator

$$T : \mathrm{BMO}(\Omega) \to \{\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}\}$$

given that the integrals in (6.15) are defined. To verify this, let $b \in \mathrm{BMO}(\Omega)$ and $(Q, \varepsilon) \in \mathcal{D}^m_I(\Omega) \times \mathcal{E}$. Then $\hat{T}(b\chi_\Omega) : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ is defined in connection with (3.61). Remark 3.55 implies that $\operatorname{supp} \psi^\varepsilon_Q \subset\subset \Omega$. Therefore, using also (6.13) and the definition (6.15), we get

$$Tb(Q, \varepsilon) = \hat{T}(b\chi_\Omega)(Q, \varepsilon), \quad \text{if } (Q, \varepsilon) \in \mathcal{D}^m_I(\Omega) \times \mathcal{E}. \tag{6.16}$$

As a consequence (6.15) is well defined. Another implication of (6.16) is that $Tb = \hat{T}(b\chi_\Omega)$ and $T^*b = \hat{T}^*(b\chi_\Omega)$ in $\dot{f}^{m,2}_\infty(\Omega)$ if $b \in \mathrm{BMO}(\Omega)$.

We denote $T \in \mathscr{L}(L^\infty(\Omega), \dot{f}^{m,2}_\infty(\Omega))$ if

$$||Tb||_{\dot{f}^{m,2}_\infty(\Omega)} \leq C||b||_{L^\infty(\Omega)} \tag{6.17}$$

with C independent of $b \in L^\infty(\Omega)$. Also, we denote $T \in \mathscr{L}(\mathrm{BMO}(\Omega), \dot{f}^{m,2}_\infty(\Omega))$ if

$$||Tb||_{\dot{f}^{m,2}_\infty(\Omega)} \leq C||b||_{\mathrm{BMO}(\Omega)} \tag{6.18}$$

holds with C independent of $b \in \mathrm{BMO}(\Omega)$. The identity $T\chi_\Omega = 0$ in $\dot{f}^{m,2}_\infty(\Omega)$ is necessary for (6.18) to hold because $||\chi_\Omega||_{\mathrm{BMO}(\Omega)} = 0$. Comparing definitions (3.60) and (6.17), we see that $T \in \mathscr{L}(L^\infty(\Omega), \dot{f}^{m,2}_\infty(\Omega))$ if, and only if, $\hat{T} \in \mathscr{L}(L^\infty(\Omega), \dot{f}^{m,2}_\infty(\Omega))$. Comparing definitions (6.18) and (3.61), we also have $T \in \mathscr{L}(\mathrm{BMO}(\Omega), \dot{f}^{m,2}_\infty(\Omega))$ if, and only if, $\hat{T} \in \mathscr{L}(\mathrm{BMO}(\Omega), \dot{f}^{m,2}_\infty(\Omega))$.

Main results and an application. Combining the preparations above with Theorem 3.118 and Theorem 6.12 we reach our main result in this monograph.

Theorem 6.19. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an admissible domain and $T \in \mathrm{SK}^{-m}_\Omega(\delta)$, where $0 < m < n$ and $0 < \delta < 1$. Then the following conditions are equivalent*

- $T\chi_\Omega, T^*\chi_\Omega \in \dot{f}^{m,2}_\infty(\Omega)$,
- $\partial^\alpha T, \partial^\alpha T^* \in \mathscr{L}(L^2(\Omega))$ *if* $\alpha \in \mathbb{N}^n_0$ *satisfies* $|\alpha| = m$,
- $\partial^\alpha T, \partial^\alpha T^* \in \mathscr{L}(L^p(\Omega))$ *if* $1 < p < \infty$ *and* $\alpha \in \mathbb{N}^n_0$ *satisfies* $|\alpha| = m$.

We also record the following asymmetric endpoint boundedness result, which follows by combining the preparations above with Theorem 3.68 and Theorem 3.76.

Theorem 6.20. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an admissible domain, $0 < m < n$, and $0 < \delta < 1$. Let $T \in \mathrm{SK}^{-m}_\Omega(\delta)$ be such that $\partial^\alpha T \in \mathscr{L}(L^2(\Omega))$ if $|\alpha| = m$. Then*

- $T \in \mathscr{L}(L^\infty(\Omega), \dot{f}^{m,2}_\infty(\Omega))$,
- $T\chi_\Omega = 0$ *in* $\dot{f}^{m,2}_\infty(\Omega)$ *if, and only if,* $T \in \mathscr{L}(\mathrm{BMO}(\Omega), \dot{f}^{m,2}_\infty(\Omega))$.

There are other results based on Corollary 3.117 but we omit their formulation here. We finish our treatment with a simple application related to potential theory.

Example 6.21. Recall the operator $\mathbf{G} \in \mathrm{SK}_B^{-2}(\delta)$ in the ball $B = B(0,1) \subset \mathbb{R}^n$, $n \geq 3$, which is an admissible domain. This operator is defined in (1.4) and its basic properties are also established in that connection. The purpose of this example is to show that the $T1$ theorem combined with the Hölder regularity estimate

$$\mathbf{G}1 = \mathbf{G}\chi_B \in C^{2,\delta}(\bar{B}), \quad \text{if } 0 < \delta < 1, \tag{6.22}$$

which is proven in the Introduction, can be used to deduce certain boundedness properties of $\partial^\alpha \mathbf{G}$ for $|\alpha| = 2$. Estimate (6.22), combined with Corollary 5.34, shows that

$$\mathbf{G}1 = \mathbf{G}\chi_B \in C^{2,\delta}(\bar{B}) \subset \dot{F}^{2,2}_\infty(B).$$

The operator $\mathbf{G}$ is associated with a symmetric and real-valued standard kernel so that we also have $\mathbf{G}^*1 \in \dot{F}^{2,2}_\infty(B)$. Then, using Fubini's theorem and definition (6.15), we see that the weak versions of these inclusions hold true so that $\mathbf{G}1, \mathbf{G}^*1 \in \dot{f}^{2,2}_\infty(B)$. Theorem 6.19 and Theorem 6.20 imply the following boundedness properties

- $\{\partial^\alpha \mathbf{G} : |\alpha| = 2\} \subset \mathscr{L}(L^p(B))$ if $1 < p < \infty$,
- $\mathbf{G} \in \mathscr{L}(L^\infty(B), \dot{f}^{2,2}_\infty(B))$.

Appendix A. Notation

- $\mathbb{N} = \{1, 2, \ldots\}$, the natural numbers,
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,
- $a \wedge b = \min\{a, b\}$,
- $a \vee b = \max\{a, b\}$,
- $B(x, r)$, the open ball in $\mathbb{R}^n$ with radius $r > 0$ and center at $x \in \mathbb{R}^n$,
- $\bar{B}(x, r)$, the closed ball in $\mathbb{R}^n$,
- $\mathcal{E} = \{0, 1\}^n \setminus \{(0, 0, \ldots, 0)\}$, indices associated with wavelets,
- $\ell(Q)$, the side-length of a cube $Q \subset \mathbb{R}^n$,
- $Q(x, r)$, the open cube centered at $x \in \mathbb{R}^n$ and with $\ell(Q(x, r)) = 2r > 0$,
- $Q_{\nu k} = \{x \in \mathbb{R}^n : k_i \leq 2^\nu x_i < k_i + 1 \text{ if } i = 1, \ldots, n\}$ a dyadic cube indexed by $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$,
- x^Q, the centerpoint of a cube $Q \subset \mathbb{R}^n$,
- x_Q, the lower left-corner $2^{-\nu}k$ of a dyadic cube $Q = Q_{\nu k}$,
- Q^s, reflection of a dyadic cube $Q = Q_{\nu k}$ in a Whitney coplump domain,
- $\mathcal{D} = \big\{Q_{\nu k} : \nu \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^n\big\}$, the family of all dyadic cubes in $\mathbb{R}^n$,
- $\mathcal{D}_\nu = \{Q_{\nu k} : k \in \mathbb{Z}^n\}$, the family of dyadic cubes $Q \in \mathcal{D}$ satisfying $\ell(Q) = 2^{-\nu}$,
- $\mathcal{D}_I(\alpha, \Omega) = \{Q \in \mathcal{D} : Q \subset \Omega \text{ and } \operatorname{dist}(Q, \partial\Omega) \geq \alpha \operatorname{diam}(Q)\}$, α-interior cubes,
- $\mathcal{D}_I^m(\Omega) = \mathcal{D}_I(C_{m+1}, \Omega)$, where $C_{m+1} > 0$ is the constant defined in Appendix B for which B4)–B5) holds true in the case of $(m+1)$-regular wavelets $\{\psi_Q^\varepsilon\}$,
- $\mathcal{D}_B(\alpha, \Omega) = \{Q \in \mathcal{D} : Q \cap \overline{\Omega} \neq \emptyset\} \setminus \mathcal{D}_I(\alpha, \Omega)$, α-boundary cubes,
- $\mathcal{D}_E(\Omega) = \mathcal{D} \setminus (\mathcal{D}_I(\alpha, \Omega) \cup \mathcal{D}_B(\alpha, \Omega)) = \{Q \in \mathcal{D} : Q \cap \overline{\Omega} = \emptyset\}$, exterior cubes,
- $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, partial differential operator associated with the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ of order $|\alpha| = \sum_{j=1}^n \alpha_j$,
- $\operatorname{supp} f$, the support of a function $f : \mathbb{R}^n \to \mathbb{C}$,
- $f_\nu(x) = 2^{\nu n} f(2^\nu x)$, the L^1-normalization of a function $f : \mathbb{R}^n \to \mathbb{C}$,
- τ_h, the h-translation operator acting on functions $f : \mathbb{R}^n \to \mathbb{C}$ so that $\tau_h f(x) = f(x - h)$. We also denote $\tau_h(x) = \tau_h \operatorname{id}(x) = x - h$,
- $\Delta_h^m(f, \cdot) = (\tau_{-h} - \operatorname{id})^m f = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} f(\cdot + kh)$, the difference operator of order m in $\mathbb{R}^n$. If $f \in C^m(\mathbb{R}^n)$ and $x, h \in \mathbb{R}^n$, then we have the representation

$$
\begin{aligned}
\Delta_h^m(f, x) = &\sum_{j_1=1}^n \cdots \sum_{j_m=1}^n h_{j_1} \cdots h_{j_m} \times \\
&\times \int_0^1 \cdots \int_0^1 (\partial_{j_1} \cdots \partial_{j_m} f)(x + (\theta_1 + \cdots + \theta_m)h)\, d\theta_1 \cdots d\theta_m,
\end{aligned}
\tag{A.1}
$$

- $\Delta_h^m(f, \Omega, \cdot)$, the difference operator of order m in a domain $\Omega \subset \mathbb{R}^n$ is acting on functions $f : \Omega \to \mathbb{C}$ so that, if $x \in \mathbb{R}^n$,

$$
\Delta_h^m(f, \Omega, x) = \begin{cases} \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} f(x + kh), & \text{if } \{x, x+h, \ldots, x+mh\} \subset \Omega, \\ 0, & \text{otherwise.} \end{cases}
$$

  Notice that $\Delta_h^m(f, \mathbb{R}^n, \cdot) = \Delta_h^m(f, \cdot)$,

- $\mathcal{P}_m(\mathbb{R}^n) = \big\{ \sum_{|\alpha|\le m} c_\alpha x^\alpha \,:\, c_\alpha \in \mathbb{C}\big\}$, the vector space of multivariate polynomials of total degree at most $m \in \mathbb{N}_0$. We also denote $\mathcal{P}_{-1} = \{0\}$,
- $P_\alpha f(\cdot, y)$, the Taylor approximation of $\partial^\alpha f$ with basepoint y,
- $R_\alpha f(\cdot, y)$, the error in Taylor approximation of $\partial^\alpha f$ with basepoint y,
- $\langle \Lambda, \varphi\rangle$, the linear form defined by $\int_{\mathbb{R}^n} \Lambda(x)\varphi(x)dx$ if $\Lambda, \varphi \in L^2(\mathbb{R}^n)$,
- $\langle \Lambda \mid \varphi\rangle$, the sesquilinear form defined by $\int_{\mathbb{R}^n} \Lambda(x)\overline{\varphi(x)}dx$ if $\Lambda, \varphi \in L^2(\mathbb{R}^n)$,
- $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, the Schwartz class of test functions equipped with the usual locally convex vector space topology,
- $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions, that is, the vector space of linear and continuous functionals $\Lambda : \mathcal{S} \to \mathbb{C}$ equipped with the weak*-topology,
- $\mathcal{S}_k = \mathcal{S}_k(\mathbb{R}^n)$, the closed subspace of the Schwartz class consisting of functions $\varphi \in \mathcal{S}$ satisfying $\int_{\mathbb{R}^n} x^\alpha \varphi(x)dx = 0$ if $|\alpha| \le k \in \mathbb{N}_0 \cup \{\infty\}$,
- $\mathcal{S}'/\mathcal{P} = \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$, the tempered distributions modulo polynomials, that is, the topological dual space of $\mathcal{S}_\infty$ equipped with the weak*-topology,
- $\dot{B}^{\alpha,q}_p(\mathbb{R}^n)$, the homogeneous Besov space in $\mathbb{R}^n$,
- $H^1(\mathbb{R}^n)$, the real Hardy space in $\mathbb{R}^n$,
- $L^p(\Omega)$, the space of p-integrable, $1 \le p \le \infty$, functions in a domain $\Omega \subset \mathbb{R}^n$ equipped with the norm $||f||_{L^p(\Omega)} = (\int_\Omega |f(x)|^p dx)^{1/p}$ (modification if $p = \infty$),
- $\mathrm{BMO}(\Omega)$, the space of bounded mean oscillation in a domain $\Omega \subset \mathbb{R}^n$, consisting of those $f \in L^1_{\mathrm{loc}}(\Omega)$ that satisfy
$$\sup_{Q\subset\subset\Omega} \left\{ \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \right\} < \infty, \text{ where } f_Q = \frac{1}{|Q|}\int_Q f(x)dx,$$
- $\dot{F}^{m,2}_\infty(\Omega)$, the homogeneous BMO-type Sobolev space on a domain $\Omega \subset \mathbb{R}^n$,
- $\dot{f}^{m,2}_\infty(\Omega)$, the space of sequences $\tau : \mathcal{D} \times \mathcal{E} \to \mathbb{C}$ satisfying
$$||\tau||^2_{\dot{f}^{m,2}_\infty(\Omega)} = \sup_{P\in\mathcal{D}^m_I(\Omega)} \left\{ \frac{1}{|P|} \sum_{Q\subset P} \sum_{\varepsilon\in\mathcal{E}} |Q|^{-2m/n} |\tau(Q,\varepsilon)|^2 \right\} < \infty,$$
- $C^m(\Omega)$, the space of functions $f : \Omega \to \mathbb{C}$ which have continuous (partial) derivatives $\partial^\alpha f : \Omega \to \mathbb{C}$ for $|\alpha| \le m \in \mathbb{N}_0 \cup \{\infty\}$,
- $C(\Omega) = C^0(\Omega)$, the space of continuous functions in Ω,
- $C^m(\overline{\Omega})$, the space of continuous functions $f : \overline{\Omega} \to \mathbb{C}$ such that $f|\Omega \in C^m(\Omega)$ and the derivatives $\partial^\alpha(f|\Omega)$ extend to continuous functions $\overline{\Omega} \to \mathbb{C}$ for $|\alpha| \le m$,
- $C^\infty_{\mathcal{P}}(\mathbb{R}^n)$, the space of polynomially bounded smooth functions, that is, functions $f \in C^\infty(\mathbb{R}^n)$ satisfying $|f| \le C(1 + |\cdot|)^N$ for some $C, N > 0$,
- $C^m_0(\Omega)$, the space of functions $f \in C^m(\Omega)$ with compact support contained in Ω,
- $C_0(\Omega) = C^0_0(\Omega)$, the space of continuous compactly supported functions in Ω,
- $(C^m_0(\Omega))^*$, the algebraic dual of $C^m_0(\Omega)$ consisting of conjugate-linear functionals $\Lambda : C^m_0(\Omega) \to \mathbb{C} : \varphi \mapsto \Lambda(\varphi) = \langle \Lambda \mid \varphi\rangle$,
- $C^{m,\delta}(\overline{\Omega})$, the space of functions $f \in C^m(\overline{\Omega})$ satisfying $\sum_{|\alpha|\le m} ||\partial^\alpha f||_{L^\infty(\Omega)} < \infty$ and whose order m derivatives satisfy a Hölder estimate in Ω with exponent $\delta \in (0, 1)$,

- $C^{m,\delta}_{\rm loc}(\Omega)$, the space of functions $f\in C^m(\Omega)$ satisfying $\sum_{|\alpha|\le m}||\partial^\alpha f||_{L^\infty(\Omega)}<\infty$ and whose order m derivatives satisfy a uniform Hölder estimate in every open cube $Q\subset\subset\Omega$ with exponent $\delta\in(0,1)$,
- $\mathrm{Lip}_{m+\delta}(\overline{\Omega})$, the space of $(m+\delta)$-Lipschitz functions in $\overline{\Omega}$,
- $E_m(f,Q)=\inf_{P\in\mathcal{P}_m(\mathbb{R}^n)}\big\{\int_Q|f(x)-P(x)|dx\big\}$ if $f\in L^1(Q)$ and $Q\subset\mathbb{R}^n$ is a cube,
- $\mathscr{C}^{m+\delta}_\infty(\Omega)$, the local smoothness space consisting of $f\in L^1_{\rm loc}(\Omega)$ for which
$$||f||_{L^\infty(\Omega)}+\sup\big\{|Q|^{-(1+(m+\delta)/n)}E_m(f,Q)\big\}<\infty,$$
where the supremum is taken over all cubes $Q\subset\subset\Omega$,
- $\mathscr{D}(\delta)=\big\{b\in L^1_{\rm loc}(\mathbb{R}^n)\,:\,\int_{\mathbb{R}^n}|b(x)|(1+\ell(Q)^{-1}|x-x_Q|)^{-n-\delta}dx<\infty \text{ if } Q\in\mathcal{D}\big\}$,
- $\mathrm{SK}^{-m}_\Omega(\delta)$, the space of weakly singular integral operators of order $-m$, consisting of linear integral operators T of the form
$$Tf(x)=\int_\Omega K(x,y)f(y)dy,\quad x\in\mathbb{R}^n \text{ and } f\in C_0(\Omega),$$
where $K\in\mathrm{K}^{-m}_\Omega(\delta)$,
- $\mathrm{K}^{-m}_\Omega(\delta)$, the space of standard kernels $K\in C(\Omega\times\Omega\setminus\{(x,x)\})$ which satisfy the estimates $|K(x,y)|\le C_K|x-y|^{m-n}$, if $x,y\in\Omega$, and
$$\sup_{|h|\le \mathrm{diam}(Q)}\left\{\frac{1}{|Q|^{1+(m+\delta)/n}}\int_Q|\Delta_h^{m+1}(K(x,\cdot),Q,y)|dy\right\}\le C_K|x-x^Q|^{-n-\delta},$$
if $x\in\Omega$ and $Q\subset\subset\Omega$ is a cube, $C_K\,\mathrm{diam}(Q)\le|x-x^Q|$. The integral estimate is also assumed with $K(x,\cdot)$ replaced by $K(\cdot,x)$,
- $k^{m+\delta}_{\rm loc}(\Omega)$, the space of Hölder–Zygmund kernels $K\in C(\Omega\times\Omega\setminus\{(x,x)\})$ which satisfy $|K(x,y)|\le C_K|x-y|^{m-n}$, if $x,y\in\Omega$, and
$$|\Delta_h^{m+1}(K(x,\cdot),Q,y)|\le C_K|h|^{m+\delta}|x-y|^{-n-\delta},$$
if $x,y\in\Omega$, $Q\subset\subset\Omega$ is a cube, and $2(m+1)|h|\le|x-y|$. The estimate on differences is also assumed also with $K(x,\cdot)$ replaced by $K(\cdot,x)$,
- $\mathcal{K}^{-m}_\Omega(\delta)$, the space of smooth kernels $K\in C^m(\Omega\times\Omega\setminus\{(x,x)\})$ which satisfy
$$|\partial_x^\alpha\partial_y^\beta K(x,y)|\le C_K|x-y|^{m-n-|\alpha|-|\beta|},$$
if $x,y\in\Omega$ and $|\alpha|+|\beta|\le m$, and
$$|\partial_x^\alpha\partial_y^\beta K(x+h,y)-\partial_x^\alpha\partial_y^\beta K(x,y)|\le C_K|h|^\delta|x-y|^{-n-\delta}$$
if $|\alpha|+|\beta|=m$ and $x,y,x+h\in\Omega$ satisfy $2|h|\le|x-y|$. This is also assumed with h-difference placed to the y-variable and $x,y,y+h\in\Omega$ satisfying $2|h|\le|x-y|$,
- $\mathscr{L}(X,Y)$, the space of bounded linear operators from X to Y, where X and Y are normed vector spaces,
- $\mathscr{L}(X)=\mathscr{L}(X,X)$.

Appendix B. Compactly supported wavelets

We describe the relevant properties of compactly supported wavelets of regularity $r \in \mathbb{N}$. Here we follow [Mey92] but other useful references are [Dau92, Woj97].

A *compactly supported r-regular basic wavelet* is a function $\psi : \mathbb{R} \to \mathbb{C}$ satisfying the conditions 1)–3) below

1) $\psi \in C_0^r(\mathbb{R})$ so that ψ is compactly supported and it has continuous derivatives up to order r,
2) $\int_{\mathbb{R}} x^k \psi(x)dx = 0$ if $0 \le k \le r$,
3) the set $\{2^{j/2}\psi(2^j \cdot -k) \,:\, j,\, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

We assume that ψ is associated with a so called r-regular multiresolution analysis. The details are not important here but this implies that there is an *r-regular scaling function*, that is, a function $\varphi : \mathbb{R} \to \mathbb{C}$ satisfying 1) above, with ψ replaced with φ, and also 3') below

3') the set
$$\{\varphi(\cdot - k) \,:\, k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j \cdot -k) \,:\, j \in \mathbb{N}_0,\, k \in \mathbb{Z}\}$$
is an orthonormal basis of $L^2(\mathbb{R})$.

Multi-dimensional compactly supported wavelets are obtained from the functions above via tensor products. Let $\mathcal{E}$ denote the set of $2^n - 1$ sequences
$$\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{0,1\}^n \setminus \{0,0,\dots,0\}.$$
Then, if $x \in \mathbb{R}^n$ and $\varepsilon \in \mathcal{E}$, define
$$\psi^\varepsilon(x) = \psi^{\varepsilon_1}(x_1)\cdots\psi^{\varepsilon_n}(x_n) = (\psi^{\varepsilon_1} \otimes \cdots \otimes \psi^{\varepsilon_n})(x),$$
where $\psi^0 = \varphi$ and $\psi^1 = \psi$ are the one-dimensional r-regular scaling function and basic wavelet. By scaling and translating these tensor products we obtain r-regular compactly supported wavelets in $\mathbb{R}^n$. For this purpose we denote
$$\psi_Q^\varepsilon(x) = (\psi^\varepsilon)_Q(x) = |Q|^{-1/2}\psi^\varepsilon(\ell(Q)^{-1}(x - x_Q)) = 2^{jn/2}\psi^\varepsilon(2^j x - k)$$
if $x \in \mathbb{R}^n$ and $(Q,\varepsilon) \in \mathcal{D}_{jk} \times \mathcal{E}$. These functions satisfy B1)–B5) and the properties described in Lemma B.2 below. The constant $C_r > 0$ occuring in B4)–B5) is chosen so that Lemma B.2 holds also true with the same constant.

B1) the set $\{\psi_Q^\varepsilon \,:\, (Q,\varepsilon) \in \mathcal{D} \times \mathcal{E}\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. If $f \in L^2(\mathbb{R}^n)$, then

(B.1)
$$f = \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} \langle f \mid \psi_Q^\varepsilon\rangle\psi_Q^\varepsilon, \quad ||f||^2_{L^2(\mathbb{R}^n)} = \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} |\langle f \mid \psi_Q^\varepsilon\rangle|^2,$$

where the first series converges unconditionally in $L^2(\mathbb{R}^n)$,

B2) if $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then
$$f = \sum_{Q\in\mathcal{D}}\sum_{\varepsilon\in\mathcal{E}} \langle f \mid \psi_Q^\varepsilon\rangle\psi_Q^\varepsilon$$
so that the series converges unconditionally in $L^p(\mathbb{R}^n)$ [Woj97, p. 196],

B3) ψ_Q^ε's have vanishing moments,

$$\int_{\mathbb{R}^n} x^\alpha \psi_Q^\varepsilon(x)dx = 0$$

if $(Q,\varepsilon) \in \mathcal{D}\times\mathcal{E}$ and $\alpha \in \mathbb{N}_0^n$ satisfies $|\alpha| \le r$,

B4) $\operatorname{supp}\psi_Q^\varepsilon \subset C_r Q$, where $C_r Q$ is the cube with the same midpoint x^Q as Q but whose sidelength is $C_r\ell(Q)$,

B5) $\psi_Q^\varepsilon \in C_0^r(\mathbb{R}^n)$ and $||\partial^\alpha \psi_Q^\varepsilon||_{L^\infty(\mathbb{R}^n)} \le C_r|Q|^{-1/2-|\alpha|/n}$ if $\alpha\in\mathbb{N}_0^n$, $|\alpha|\le r$.

We also record the following lifting property which is useful in many occasions.

Lemma B.2. *Let $\varepsilon \in \mathcal{E}$ and $0 \le m \le r$, $m \in \mathbb{N}_0$. There exists a canonical multi-index $\alpha = \alpha(\varepsilon, m) \in \mathbb{N}_0^n$, $|\alpha| = m$, and a function $\psi^{\varepsilon,m} : \mathbb{R}^n \to \mathbb{C}$, depending on $\psi^\varepsilon, \varepsilon, m$, so that $\psi^\varepsilon = \partial^\alpha(\psi^{\varepsilon,m})$. Furthermore, if $Q \in \mathcal{D}$, then $\psi_Q^{\varepsilon,m} = (\psi^{\varepsilon,m})_Q$ satisfies the moment conditions*

$$\int_{\mathbb{R}^n} x^\alpha \psi_Q^{\varepsilon,m}(x)dx = 0, \quad |\alpha| \le r - m,$$

and also the properties B4)–B5) above with ψ_Q^ε replaced by $\psi_Q^{\varepsilon,m}$.

Proof. Consider the case $n = 1$. First we define functions $\psi^{1,k} : \mathbb{R} \to \mathbb{C}$, $k = 0, 1, \ldots, m$, inductively so that $\psi^{1,0} = \psi^1 = \psi$ and

$$\psi^{1,k}(x) = \int_{-\infty}^x \psi^{1,k-1}(y)dy, \quad k = 1, 2, \ldots, m.$$

Now $(\psi^{1,k})' = \psi^{1,k-1}$ and $(\psi^{1,k})^{(k)} = \psi$, $k = 1, 2, \ldots, m$. Induct on $k \in \{0, 1, \ldots, m\}$ and apply 2) with integration by parts at each step to show that the support of $\psi^{1,k}$ is contained in the convex hull of $\operatorname{supp}\psi$ and that

$$\int_{-\infty}^\infty x^\ell \psi^{1,k}(x)dx = 0, \quad k = 0, 1, \ldots, m \text{ and } \ell = 0, 1, \ldots, r - k.$$

The one-dimensional result follows from these considerations. Then assume that $n > 1$. Due to the definition of $\mathcal{E}$ there is the smallest $j \in \{1, 2, \ldots, n\}$ so that $\varepsilon_j = 1$. Choose $\alpha = me_j$, where e_j is the j'th base vector in $\mathbb{R}^n$. Then the function

$$\psi^{\varepsilon,m} = \psi^{\varepsilon_1} \otimes \cdots \otimes \psi^{\varepsilon_j,m} \otimes \cdots \otimes \psi^{\varepsilon_n}$$

satisfies the desired properties, where $\psi^{\varepsilon_j,m} = \psi^{1,m}$ is placed in the j'th position. □

Appendix C. Whitney decomposition

We collect the basic properties of the so called *Whitney decomposition* of an open set $\emptyset \neq \Omega \subsetneq \mathbb{R}^n$. In addition to this, we invoke the partition of unity that is related to this decomposition. We follow the treatment in [Ste70, 167–171].

There is a a Whitney decomposition of Ω, that is, a family $\mathcal{F}$ of closed dyadic cubes in $\mathbb{R}^n$ satisfying C1)–C3) below

C1) $\bigcup_{Q\in\mathcal{F}} Q = \Omega$,
C2) the family of open cubes $\{\operatorname{int} Q : Q \in \mathcal{F}\}$ is disjoint,
C3) $\operatorname{diam}(Q) \leq \operatorname{dist}(Q, \partial\Omega) \leq 4 \operatorname{diam}(Q)$ if $Q \in \mathcal{F}$.

We expand the cubes in $\mathcal{F}$ twice. We will construct a partition of unity which is subordinate to the first family $\{Q^*\}$ of expanded cubes. The second family of larger expanded cubes $\{Q^{**}\}$ will serve as a starting point for certain reproducing functions. Let $Q \in \mathcal{F}$ and fix $\varepsilon > 0$ be such that $1+\varepsilon < (1+\varepsilon)^2 < 5/4$. Denote by Q^* the cube which has the same center as Q and side-length $(1+\varepsilon)\ell(Q)$,

$$Q^* = (1+\varepsilon)(Q - x^Q) + x^Q$$

where x^Q is the center of the cube Q. Also, denote by Q^{**} the cube with the center x^Q and side-length $(1+\varepsilon)^2\ell(Q)$. In C4)–C8) below we collect the properties of these expansions of a cube $Q \in \mathcal{F}$

C4) $Q \subset \operatorname{int} Q^* \subset Q^* \subset \operatorname{int} Q^{**} \subset Q^{**}$,
C5) $\operatorname{diam}(Q^{**}) \leq 5 \operatorname{diam}(Q)/4$,
C6) $3 \operatorname{diam}(Q^{**})/5 \leq \operatorname{dist}(Q^{**}, \partial\Omega) \leq 5 \operatorname{diam}(Q^{**})$,
C7) $\sup_{x\in Q^{**}} \operatorname{dist}(x, \partial\Omega) \leq 10 \operatorname{dist}(Q^{**}, \partial\Omega)$,
C8) for every $x \in \Omega$ there is at most C_n cubes $R \in \mathcal{F}$ such that $x \in R^{**}$.

Next we construct a partition of unity. Let $Q_0 = [-1/2, 1/2]^n$. Fix $\Phi \in C^\infty(\mathbb{R}^n)$ so that $0 \leq \Phi \leq 1$, $\Phi(x) = 1$ if $x \in Q_0$, and $\operatorname{supp} \Phi \subset \operatorname{int}(Q_0^*)$. Let Φ_Q denote the function Φ but adjusted to the cube $Q \in \mathcal{F}$ so that

$$\Phi_Q(x) = \Phi\left(\frac{x - x^Q}{\ell(Q)}\right), \quad x \in \mathbb{R}^n.$$

Notice that $\Phi_Q(x) = 1$, if $x \in Q$, and $\operatorname{supp} \Phi_Q \subset \operatorname{int}(Q^*)$. We also have

$$||\partial^\alpha \Phi_Q||_\infty \leq C_{n,\alpha} \operatorname{diam}(Q)^{-|\alpha|}, \quad \alpha \in \mathbb{N}_0^n.$$

Define $\varphi_Q \in C_0^\infty(\mathbb{R}^n)$, $Q \in \mathcal{F}$, by

$$\varphi_Q(x) = \begin{cases} \Phi_Q(x)\big(\sum_{Q\in\mathcal{F}} \Phi_Q(x)\big)^{-1}, & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then $||\partial^\alpha \varphi_Q||_\infty \leq C_{n,\alpha} \operatorname{diam}(Q)^{-|\alpha|}$, $\operatorname{supp} \varphi_Q \subset \operatorname{int}(Q^*)$. The identity

$$\sum_{Q\in\mathcal{F}} \varphi_Q(x) = 1, \quad x \in \Omega,$$

is a justification for that $\{\varphi_Q\}$ is a partition of unity subordinate to $\{Q^*\}$.

The properties of this partition of unity are collected in C9)–C12) below

C9) $0 \leq \varphi_Q \leq 1$ if $Q \in \mathcal{F}$,

C10) $\sum_{Q\in\mathcal{F}} \varphi_Q(x) = 1$ if $x \in \Omega$,

C11) $\operatorname{supp} \varphi_Q \subset \operatorname{int}(Q^*)$ if $Q \in \mathcal{F}$,

C12) $||\partial^\alpha \varphi_Q||_\infty \leq C_{n,\alpha} \operatorname{diam}(Q)^{-|\alpha|}$ if $\alpha \in \mathbb{N}_0^n$.

We also use the following reproducing functions. Fix a function $\psi \in C_0^\infty(\mathbb{R}^n)$ so that $0 \leq \psi \leq 1$, $\psi(x) = 1$ if $x \in Q_0^*$, and $\operatorname{supp} \psi \subset \operatorname{int}(Q_0^{**})$. Let $\psi_Q \in C_0^\infty(\mathbb{R}^n)$ denote the function ψ but adjusted to the cube $Q \in \mathcal{F}$ so that

$$\psi_Q(x) = \psi\left(\frac{x - x^Q}{\ell(Q)}\right), \quad x \in \mathbb{R}^n.$$

Then, if $Q \in \mathcal{F}$, we have C13)–C14) below

C13) $\psi_Q(x) = 1$ if $x \in Q^*$ and $\operatorname{supp} \psi_Q \subset \operatorname{int}(Q^{**})$,

C14) $||\partial^\alpha \psi_Q|| \leq C_{n,\alpha} \operatorname{diam}(Q)^{-|\alpha|}$ if $\alpha \in \mathbb{N}_0^n$.

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